Exercice I. **Lyndon-Schützenberger Theorem.** Let $m, n, p \geq 2$, prove that for x, y, z words on an alphabet $\mathcal{A}$, if $x^m = y^n z^p$ then x, y and, z are powers of a common word w. *You should start with the easier case $m = n = p = 2$.*

Exercice II. **1.** Let $\mathcal{A} = \{a, b\}$, check that in the free group $F_{\mathcal{A}}$, $[a, b] = a^{-1}b^{-1}ab$ is a produced of three squares.
2. Prove that all groups of exponent 2 are commutative.
3. Prove that in a free group every commutator can be written as a reduced product $X^{-1}Y^{-1}Z^{-1}XYZ$, with X, Y and, Z three reduced words in the free group (possibly empty).
4. Check that for any elements x, y of any group

$$[x^y, x] = ((xy^{-1})^y)^3(y^2x^{-1})^3((y^{-1})^x)^3$$

Conclude that any group of exponent 3 is nilpotent of class 3.
5. Do you know **a.** A non-abelian group of exponent 3 ? **b.** how to write $[[[a, b], c], d]$ as a product of cubes in the free group on $\{a, b, c, d\}$?

Exercice III. **1.** What is the universal property of free abelian groups ?
2. Prove that any subgroup of a free abelian group is free abelian.
3. Let M be a finitely generated abelian group.
a. Prove that M decomposes as $M = T \oplus L$ with T a finite abelian group and L a free abelian group of finite rank.
b. Recall that T decomposes as

$$T = \oplus_i \mathbb{Z}/n_i\mathbb{Z} = \oplus_{i,j} \mathbb{Z}/{p_i}^{\alpha_{ij}}\mathbb{Z},$$

with $n_1 | n_2 | \cdots | n_r$ and with p_i primes.
4. Give a an abelian group with all its finitely generated subgroups are free abelian. But which is not itself free *(you know this group pretty well).*
5. *Set theory.* For which set I, the group Z^I is free abelian ?

Exercice IV. **1.** Let $A_0 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ and $B_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
a. Compute A_0B_0 et B_0A_0.
b. Check that ${A_0}^6 = {B_0}^4 = {A_0}^3{B_0}^2 = I_2$.
c. Is $\mathrm{SL}_2(\mathbb{Z})$ a free group ?
2. Let $H =< A_0, B_0 >$ the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by A_0 and B_0. We aim to prove that $H = \mathrm{SL}_2(\mathbb{Z})$. Ad absurdo, let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ a matrix in $\mathrm{SL}_2(\mathbb{Z}) - H$ such that $|a| + |c|$ is minimal.
a. Multiply on the left by $(A_0B_0)^{\pm 1}$ or $(B_0A_0)^{\pm 1}$ and use minimality to get $a = 0$ ou $c = 0$.
b. Prove that if $c = 0$, $X = \pm(A_0B_0)^{\pm b}$.
c. Prove that $a = 0$, B_0X is as above.
d. Conclude : $H = \mathrm{SL}_2(\mathbb{Z})$.

Exercice V. **Hyperbolic picture of the Cayley-graph of a free group.**
We proved that the group generated by the Möbius transforms $\alpha : z \mapsto z + 2$ and $\beta : z \mapsto \frac{z}{2z+1}$ is a free group. Plot the orbit of i (the complex root of -1) under the group generated by α and β. Starts with $\alpha(i)$, $\alpha^2(i)$, $\alpha^{-1}(i)$, $\beta(i)$, $\beta^2(i)$, $\beta^{-1}(i)$. Add the images by the words $\alpha\beta$, $\alpha\beta^{-1}$, $\alpha\beta\alpha$, etc.
Plot the edges as geodesics between the corresponding vertices. *you should program that, use Sagemath or Geogebra.*

Exercice VI. Let $\phi : F_{\{a,b\}} \to S_4$ defined by $a \mapsto (1\ 2)$ and $b \mapsto (1\ 2\ 3\ 4)$. Give a basis of the subgroup H made of elements h such that $\phi(h)(1) = 1$

Exercice VII. Give a basis of the subgroup of $F_{\{a,b\}}$:

$$H =< b^2ab^{-1}a^{-2},\ a^{-1}b^{-1}ab^{-1}a^{-2},\ b^3a^4b^3a,\ a^5b^{-1}ab^{-1}a > .$$

Exercice VIII. Prove that for all automorphism ϕ of $F_{\{a,b\}}$, $\phi([a, b])$ is conjugated to $[a, b]$ or $[b, a] = [a, b]^{-1}$.

Exercice IX. **1.** What is the inverse of the automorphism of $F_{\{a,b,c\}}$, $\varphi : a \mapsto ab, b \mapsto ac, c \mapsto a$.
2. Is the morphism $F_{\{a,b,c\}}$ such that $a \mapsto bc^{-1}aca$, $b \mapsto a^{-1}c^{-1}a^{-1}a^{-1}c^{-1}$ and $c \mapsto ca$ injective ? onto ? an automorphism ?