Exercice I. Let T be a tree and G a group acting on T. Recall that for $g \in G$, the translation length $\|g\|_T$ is the minimal displacement :

$$\|g\|_T = \min_{v \in V(t)} d(v, gv),$$

the axis of g is the subtree with vertices the minimally displaced vertices.

1. For an element $g \in G$ and a vertex v, prove that v is in the axis of g if and only if gv is a vertex of the segment $[v, g^2v]$.

2. For two loxodromic elements g and h, prove that if the intersection of there axes is a segment of length greater than $\|g\|_T + \|h\|_T$ then g and h commutes and the axis are equal.

3. Let g and h be two loxodromix elements with disjoint axes. Prove that $\|gh\|_T = \|g\|_T + \|h\|_T + 2d(\mathrm{Ax}(g), \mathrm{Ax}(h))$ is loxodromic.

4. Let g and h be two elements with disjoint axes (if an element is elliptic its axis is its fix points set). Prove that gh is loxodromic.