**Exercice I. Hanna Neumann inequality**

**1.** Let $\Gamma$ with a connected graph with all vertices of valence 2 or 3. $Br(\Gamma)$ denotes the set of branch points of $\Gamma$ : the set of vertices of valence 3 (in general of valence strictly greater than 2). Prove that $\mathrm{rank}(\pi_1(\Gamma)) = \#Br(\Gamma)/2 + 1$.

**2.** Let $H$ and $H'$ two finitely generated subgroups of a free group $F$. Let $R$ be a connected graph with all vertices of valence 3 and base point $v_0$ such that $\pi_1(R, v_0) = F$. Let $\Gamma$ and $\Gamma'$ be two connected finite graphs that encode $H$ and $H'$ : $\Gamma$ and $\Gamma'$ are given with base points $v_0$ and $v_0'$ and immersions $f$ and $f'$ to $R$ inducing injective group morphisms $f^* : \pi_1(\Gamma, v_1) \hookrightarrow \pi_1(R, v_0)$ and $f'^* : \pi_1(\Gamma', v_1') \hookrightarrow \pi_1(R, v_0)$ with images $H$ and $H'$. Note that we assume $f(v_1) = f'(v_1') = v_0$.
Let $\Gamma''$ be the graph with vertex set and edge set :

$$V(\Gamma'') = \{(v, v') \in V(\Gamma) \times V(\Gamma') \mid f(v) = f'(v')\}, \quad E(\Gamma'') = \{(e, e') \in E(\Gamma) \times E(\Gamma') \mid f(e) = f'(e')\}.$$

The edge $(e, e')$ going from $(i(e), i(e'))$ towards $(t(e), t(e'))$. We define the graph morphism $f'' : \Gamma'' \to R$ by $f''(v, v') = f(v) = f'(v')$ and $f''(e, e') = f(e) = f'(e')$.
Let $\Gamma''_0$ be the connected component of $(v_1, v_1')$

**a.** Prove that $f''$ is an immersion.

**b.** Prove that $f''^*(\pi_1(\Gamma'', (v_1, v_1'))) = H \cap H'$.

**c.** Using these graphs and morphisms, prove that the rank of $H \cap H'$ is bounded above by $2(\mathrm{rank}(H) - 1)(\mathrm{rank}(H') - 1) + 1$

**Exercice II.** Compute the inverse automorphism of $\varphi : a \mapsto a^{-1}b, b \mapsto c, c \mapsto ca$ and $\psi : a \mapsto abc, b \mapsto bcabc, c \mapsto cbcabc$.

**Exercice III. Gromov product**

We define the Gromov product of three points in a metric space as $(y \cdot z)_x = \frac{1}{2}(d(x, y) + d(x, z) - d(y, z))$.

**1.** Check that the definition of Gromov's four points condition given in my course, namely :

$$\forall x, y, z, t, \quad d(x, y) + d(z, t) \leq \max\{d(x, z) + d(y, t), d(x, t) + d(y, z)\}$$

is equivalent to

$$(y \cdot z)_x \geq \min\{(z \cdot t)_x, (y \cdot t)_x\}.$$