Free product, profinite topology and finitely generated subgroups.

Thierry Coulbois

Abstract

We consider the following property for a group G : *(RZ_n) if $H_1, \dots, H_n$ are finitely generated subgroups of G then the set $H_1 H_2 \cdots H_n = \{h_1 \cdots h_n | h_1 \in H_1, \dots, h_n \in H_n\}$ is closed with respect to the profinite topology of G.* It is obvious that finite groups and finitely generated commutative groups have the property (RZ_n). L. Ribes and P. Zalesskiĭ proved that any free group has (RZ_n). We show that the property (RZ_n) is stable under the free product operation. We use techniques developped by B. Herwig and D. Lascar on the one hand, R. Gitik on the other hand.

M. Hall [5] defined the **profinite topology** on a group G as being the coarsest topology such that every morphism from G into a finite discrete group is continuous. A group whose profinite topology is Hausdorff is called **residually finite (RF)**. It is known that a free group is RF. K. Gruenberg [3] proved that the free product of two RF groups is also RF. A group whose all finitely generated subgroups are closed with respect to the profinite topology is called **locally extended residually finite (LERF)**. M. Hall [4] proved that a free group is LERF, N. Romanovskiĭ [11] and R. Burns [1] proved independently that the free product of two LERF groups is itself LERF.

We are interested, here, in a stronger property introduced by J.-E. Pin and C. Reutenauer [9] : a group is said to have the **Ribes-Zalesskiĭ property at rank n ($\mathbf{RZ}_n$)** where n is an integer, if for all finitely generated subgroups $H_1, \dots, H_n$ the set $H_1 H_2 \cdots H_n = \{h_1 \cdots h_n \mid h_1 \in H_1, \dots, h_n \in H_n\}$ is closed for the profinite topology. Property **RZ** is the conjunction of properties RZ_n for all n. RZ_1 is simply the property of being LERF. L. Ribes and P.A. Zalesskiĭ [10] proved that free groups satisfy RZ.

For the rest of this article we fix an integer n.

The aim of this article is to prove the following theorem :

Theorem 1 *If G_1 and G_2 are two groups with RZ_n then the free product $G_1 * G_2$ also has RZ_n.*

To prove this result we use methods developed on the one hand by B. Herwig and D. Lascar [6][7] who gave another proof of L. Ribes and P. Zalesskiĭ's result and on the other hand by R. Gitik [2] who gave other proofs of Gruenberg's, Romanovskiĭ's and Burns' results.

I am really indebted to B. Herwig and D. Lascar for showing me and explaining to me their works. Moreover I will present here part of their work (section 1 and definitions and propositions 10, 11 and 12 of section 4) in order to have a self-contained article.

From theorem 1 one easily deduces L. Ribes and P. Zalesskiĭ's result. Indeed, $\mathbb{Z}$ has RZ and a finitely generated free group is a free product of copies of $\mathbb{Z}$.

1 n-partitioned structures

B. Herwig and D. Lascar introduced a class of structures, that will be called n-partitioned, which are deeply linked with property RZ_n.

Definition *[7] An n-***partitioned structure** *is a relational structure $\mathfrak{M}$ in the language $\{U_1, \ldots, U_n, \longrightarrow\}$ such that the U_i are unary predicates that define a partition of $\mathfrak{M}$, $\longrightarrow$ is a binary relation that goes from U_i to U_{i+1} $(i = 1, \ldots, n-1)$.*

In an n-partitioned structure $\longrightarrow^*$ is the transitive closure of $\longrightarrow$.

We will always denote a first order structure and its underlying set by the same symbol.

If on a set, we have several n-partitioned structures we will write $\longrightarrow_{\mathfrak{M}}$ and $\longrightarrow^*_{\mathfrak{M}}$ to prevent ambiguities.

$U_i(\mathfrak{M})$ or simply U_i stands for the set : $\{m \in \mathfrak{M} \mid \mathfrak{M} \models U_i(m)\}$.

We denote by $\not\twoheadrightarrow$ the "complement" of $\longrightarrow$, that is to say the relation from U_i to U_{i+1} $(i = 1, \ldots, n-1)$ such that $s \not\twoheadrightarrow t \iff \neg(s \longrightarrow t)$. $\not\twoheadrightarrow^*$ denotes, in the same way the complement of $\longrightarrow^*$ (and not the transitive closure of $\not\twoheadrightarrow$).

The link between n-partitioned structures and RZ_n property is illustrated by the following definition and proposition.

Definition *[7] Let G be a group and let $H_1, \ldots, H_n$ be subgroups of G. We define $\mathcal{G}(G, H_1, \ldots, H_n)$ as the n-partitioned structure such that for all $i = 1, \ldots, n$ $U_i = \{xH_i \mid x \in G\}$ and for all $i = 1, \ldots, n-1$ and for all $x, y \in G$ $xH_i \longrightarrow yH_{i+1}$ if and only if $xH_i \cap yH_{i+1} \neq \emptyset$.*

Observe that G acts on $\mathcal{G}(G, H_1, \ldots, H_n)$ by left multiplication.

Proposition 2 *[7] Let G be a group and $H_1, \ldots, H_n$ be subgroups of G and let $\mathcal{G}$ be the associated n-partitioned structure. Then for all $x \in G$ we have :*

$$x \in H_1 \cdots H_n \iff H_1 \longrightarrow^*_{\mathcal{G}} xH_n$$

Proof : Let $x \in H_1 \cdots H_n$, there exists $h_1 \in H_1, \ldots, h_n \in H_n$ such that $x = h_1 \cdots h_n$. We easily see that in $\mathcal{G}$, $H_1 \longrightarrow h_1H_2 \longrightarrow h_1h_2H_3 \longrightarrow \cdots \longrightarrow h_1h_2 \cdots h_{n-1}H_n = xH_n$.

Conversely, let $x_2, \ldots, x_{n-1}$ be elements of G such that $H_1 \longrightarrow x_2H_2 \longrightarrow \cdots \longrightarrow xH_n$. Using the definition of $\mathcal{G}$, $H_1 \cap x_2H_2 \neq \emptyset$, thus there exists $h_1 \in H_1$ such that $x_2H_2 = h_1H_2$. Similarly there exists $h_2 \in H_2, \ldots, h_n \in H_n$ such that $x_kH_k = h_1 \cdots h_{k-1}H_k$. This proves the proposition. □

In fact we could prove a slightly stronger equivalence. Let i, j be two integers $1 \leq i < j \leq n$ and $x, y \in G$ then :

$$xH_i \longrightarrow^*_{\mathcal{G}} yH_j \iff x^{-1}y \in H_i \cdots H_j$$

Let $\mathfrak{M} \subset \mathfrak{N}$ be two n-partitioned structures. We will say that $\mathfrak{M}$ is a **substructure** of $\mathfrak{N}$ (or that $\mathfrak{N}$ is an extension of $\mathfrak{M}$) if the relations $U_1, \ldots, U_n, \longrightarrow$ on $\mathfrak{M}$ are the restrictions of those relations on $\mathfrak{N}$.

2 Pre-actions

We define a quite general notion for first order relational structures, but we will only use it in the context of n-partitioned structures.

Definition *Let $\mathfrak{M}$ be a relational structure, a* **partial isomorphism** *of $\mathfrak{M}$ is an isomorphism between two substructures of $\mathfrak{M}$. We will consider the empty application as a partial isomorphism.*

$\mathrm{PI}(\mathfrak{M})$ *is the set of partial isomorphisms of $\mathfrak{M}$.*

For every partial isomorphism g of $\mathfrak{M}$ and every element $m \in \mathfrak{M}$ the notation $g(m)$ supposes that m is in the domain of g. $\mathrm{PI}(\mathfrak{M})$ is equipped with a composition operation, an inverse and a partial order :

$$\forall g, h \in \mathrm{PI}(\mathfrak{M}),\ \forall m, m' \in \mathfrak{M}\ \ (g \circ h)(m) = m' \iff \exists m'' \in \mathfrak{M},\ h(m) = m'' \wedge g(m'') = m'$$

$$\forall g \in \mathrm{PI}(\mathfrak{M}),\ \forall m, m' \in \mathfrak{M}\ \ g^{-1}(m) = m' \iff g(m') = m$$

$$\forall g, h \in \mathrm{PI}(\mathfrak{M})\ g \subset h \iff \forall m, m' \in \mathfrak{M}\ \ (g(m) = m' \Rightarrow h(m) = m')$$

It is obvious that the group of automorphisms of $\mathfrak{M}$, $\mathrm{Aut}(\mathfrak{M})$ is a sub-monoid of $\mathrm{PI}(\mathfrak{M})$.

Let $g \in \mathrm{PI}(\mathfrak{M})$ and $\mathfrak{A}$ be a substructure of $\mathfrak{M}$. The restriction of g to $\mathfrak{A}$ is the partial isomorphism $g\lceil\mathfrak{A}$ defined by :

$$\forall m, m' \in \mathfrak{M}\ \ (g\lceil\mathfrak{A})(m) = m' \iff m, m' \in \mathfrak{A} \wedge g(m) = m'$$

Our work is based on the following definition of pre-action. We will say that a subset, S, of a group G is **symmetric** if it is stable by inverse and if it contains the unit element of G.

Definition *Let $\mathfrak{M}$ be a relational structure, let G be a group, S a finite symmetric subset of G. Let $\mathfrak{N}$ be an extension of $\mathfrak{M}$ and $\bar{\varphi}$ an action of G on $\mathfrak{N}$. An application $\varphi : G \longrightarrow \mathrm{PI}(\mathfrak{M})$ is the* **pre-action** *induced by S and $\bar{\varphi}$ if and only if :*

- $\forall s \in S\ \ \varphi(s) = \bar{\varphi}(s)\lceil\mathfrak{M}$
- $\forall g \in G, \forall m, m' \in \mathfrak{M}\ \varphi(g)(m) = m' \iff \exists s_1, \ldots, s_\ell \in S\ g = s_1 \cdots s_\ell\ \wedge\ \varphi(s_1) \circ \cdots \circ \varphi(s_\ell)(m) = m'$

Remark 1 *In the conditions of the previous definition :*

- *φ is uniquely determined by S and $\bar{\varphi}$;*
- *$\forall g \in G,\ \varphi(g) \subset \bar{\varphi}(g)$;*
- *$\forall g, h \in G,\ \varphi(g) \circ \varphi(h) \subset \varphi(gh)$;*
- *$\forall g \in G,\ \varphi(g)^{-1} = \varphi(g^{-1})$;*
- *$\varphi(e) = Id_{\mathfrak{M}}$.*

We leave the proof of this remark to the reader.

As for an action we have a notion of stabilizer :

Definition *Let φ be a pre-action of a group G on a structure $\mathfrak{M}$, then for every element $m \in \mathfrak{M}$, the* **stabilizer** *of m is the subset of G : $\mathrm{Stab}(m) = \{g \in G \mid \varphi(g)(m) = m\}$.*

It is clear that with these definitions a stabilizer is always a subgroup of G.

Proposition 3 *Let φ be a pre-action of G on a finite structure $\mathfrak{M}$, induced by a finite symmetric subset S and an action $\bar{\varphi}$ of G on an extension $\mathfrak{N}$ of $\mathfrak{M}$, then for all element $m \in \mathfrak{M}$, $\mathrm{Stab}(m)$ is a finitely generated subgroup of G. Moreover if φ is an action of G on $\mathfrak{M}$ then every stabilizer is a subgroup of finite index in G.*

<u>Proof</u> : Let $t_1, \ldots, t_r$ be the elements of S, let $m \in \mathfrak{M}$. We will show that $\mathrm{Stab}(m)$ is generated by those of its elements that can be written as $w(t_1, \ldots, t_r)$, where $w(x_1, \ldots, x_r)$ is a word in the free group on generators $x_1, \ldots, x_r$ of length smaller than twice the cardinality of $\mathfrak{M}$ plus two. Indeed, let $g \in \mathrm{Stab}(m)$, there exists $s_1, \ldots, s_\ell \in S$ such that $g = s_1 \cdots s_\ell$ and $\varphi(s_1) \circ \cdots \circ \varphi(s_\ell)(m) = m$. So for each i, $1 \le i \le \ell$, $\varphi(s_i) \circ \cdots \circ \varphi(s_\ell)(m)$ is defined as an element of $\mathfrak{M}$. If $\ell > 2\mathrm{card}(\mathfrak{M}) + 1$ then there are i, j $1 \le i < j \le \mathrm{card}(\mathfrak{M}) + 1$ such that $\varphi(s_i \cdots s_\ell)(m) = \varphi(s_j \cdots s_\ell)(m)$. Let $h = s_1 \cdots s_{j-1} s_{i-1}^{-1} \cdots s_1^{-1}$ and $k = s_1 \cdots s_{i-1} s_j \cdots s_\ell$. Then $h, k \in \mathrm{Stab}(m)$, $g = hk$ and h and k can be written as products of elements of S of length strictly smaller than ℓ.

The second part of the proposition is classical. □

Let φ be a pre-action of a group G on $\mathfrak{M}$. An **orbit** over φ is a minimal subset of $\mathfrak{M}$ stable under the pre-action.

3 A characterization of groups with RZ_n

In [2] a characterization of residually finite groups and of LERF groups was given using partial isomorphisms of finite graphs. This method can be generalized to RZ_n thanks to the n-partionned structures that were introduced by B. Herwig and D. Lascar in [7]. To prove that the free groups satisfy RZ, they established a weaker version of the part (ii)⇒(i) of the next proposition.

Proposition 4 *Let G be a group. The following statements are equivalent :*

(i) G satisfies RZ_n;

(ii) For every n-partitioned finite structure $\mathfrak{A}$ and for every pre-action φ of G on $\mathfrak{A}$ that is induced by a finite symmetric subset S and the action $\bar{\varphi}$ of G on an extension $\mathfrak{M}$ of $\mathfrak{A}$, there exists a finite extension $\mathfrak{B}$ of $\mathfrak{A}$ and an action $\tilde{\varphi}$ of G on $\mathfrak{B}$ that extends φ such that $(\longrightarrow^*_{\mathfrak{M}})\lceil\mathfrak{A} = (\longrightarrow^*_{\mathfrak{B}})\lceil\mathfrak{A}$.

Proof :

(ii)⇒ (i) : Let $H_1, \ldots, H_n$ be finitely generated subgroups of G and $c \notin H_1 \cdots H_n$. Let $\mathcal{G}(G, H_1, \ldots, H_n)$ be the associated n-partitionned structure. Let $\mathfrak{A}$ be the finite substructure of $\mathcal{G}(G, H_1, \ldots, H_n)$ that contains $H_1, \ldots, H_n$ and cH_n. Let S be a symmetric finite subset of G that contains c and a system of generators for each of the H_i. Such an S exists because $H_1, \ldots, H_n$ are finitely generated subgroups. G acts on $\mathcal{G}(G, H_1, \ldots, H_n)$ by left multiplication, and we consider the pre-action φ of G on $\mathfrak{A}$ induced by this action and S.

Using (ii) there exists a finite n-partitioned structure $\mathfrak{B}$ that extends $\mathfrak{A}$ and an action $\tilde{\varphi}$ of G on $\mathfrak{B}$, that extends φ. Let $K_i = \mathrm{Stab}_{\mathfrak{B}}(H_i) = \{g \in G \mid \tilde{\varphi}(g)(H_i) = H_i\}$, these are subgroups of G of finite index by proposition 3. Also we have $H_i \leq K_i$ because S contains a system of generators of H_i.

Let $\mathcal{G}' = \mathcal{G}(G, K_1, \ldots, K_n)$ be the n-partitioned structure associated to $G, K_1, \ldots, K_n$. Looking carefully at the embedding of $\mathcal{G}'$ in $\mathfrak{B}$ defined by $xK_i \longmapsto \tilde{\varphi}(x)(H_i)$, we can view $\mathcal{G}'$ as a subset of $\mathfrak{B}$. It is not precisely a substructure although the U_i agree and the following relation holds : for all $s, t \in \mathcal{G}'$ if $s \longrightarrow_{\mathcal{G}'} t$ then $s \longrightarrow_{\mathfrak{B}} t$. Indeed the restriction of the action $\tilde{\varphi}$ agrees with the left multiplication on $\mathcal{G}'$, thus if $g, h \in G$ and $u \in gK_i \cap hK_{i+1}$ then $gK_i = \tilde{\varphi}(u)(H_i)$ and $hK_{i+1} = \tilde{\varphi}(u)(H_{i+1})$. But we know that $H_i \longrightarrow_{\mathfrak{B}} H_{i+1}$. So $\tilde{\varphi}(u)(H_i) \longrightarrow_{\mathfrak{B}} \tilde{\varphi}(u)(H_{i+1})$.

Condition (ii) implies that $(\longrightarrow^*_{\mathcal{G}})\lceil\mathfrak{A} = (\longrightarrow^*_{\mathfrak{B}})\lceil\mathfrak{A}$. But proposition 2 shows that $H_1 \twoheadrightarrow\!\!\!\!\leftarrow^*_{\mathcal{G}} cH_n$, so $H_1 \twoheadrightarrow\!\!\!\!\leftarrow^*_{\mathfrak{B}} cH_n$ and as a result $K_1 \twoheadrightarrow\!\!\!\!\leftarrow^*_{\mathcal{G}'} cK_n$. Using proposition 2 again we get $c \notin K_1 \cdots K_n$. K_1 is a clopen subgroup of G, and $K_1 \cdots K_n$ is a finite union of cosets of K_1 and thus a clopen subset of G.

For each element c of G that is not in $H_1 \cdots H_n$ we found a closed neighborhood $K_1 \cdots K_n$ of $H_1 \cdots H_n$ that does not contain c. Hence $H_1 \cdots H_n$ is closed with respect to the profinite topology of G.

(i)⇒ (ii) : This part is much more technical, and the reader should be familiar with the previous paragraph before going further.

Let $\mathfrak{A}$, $\mathfrak{M}$, φ, $\bar{\varphi}$ be as in the hypothesis of (ii). Let $\mathfrak{A}'$ be a finite substructure of $\mathfrak{M}$ that contains $\mathfrak{A}$ such that $(\longrightarrow^*_{\mathfrak{M}})\lceil\mathfrak{A} = (\longrightarrow^*_{\mathfrak{A}'})\lceil\mathfrak{A}$. $\mathfrak{A}'$ is easily obtained from $\mathfrak{A}$ by adding a finite number of elements of $\mathfrak{M}$.

Let Ω be the set of orbits of $\mathfrak{A}'$ under the pre-action φ. Note that each orbit is a subset of one of the U_i and thus the partition of $\mathfrak{A}'$ induces a partition $\Omega = \Omega_1 \uplus \cdots \uplus \Omega_n$ ($\uplus$ denotes the disjoint union). For each $C \in \Omega$ we choose a base point $\tau(C) \in C$. We choose also an application $\sigma : \mathfrak{A}' \longmapsto G$ such that for all $s \in \mathfrak{A}'$, $\varphi(\sigma(s))(\tau(C)) = s$. At last we define for every $C \in \Omega$, $H(C) = \mathrm{Stab}(\tau(C))$ which is a finitely generated subgroup of G.

Lemma 5 *Let $C_1 \in \Omega_1, \ldots, C_n \in \Omega_n$, let $H_i = H(C_i)$. The following three conditions hold for $i = 1, \ldots, n$ and $1 \leq i < j \leq n$:*

(i) *for all* $s, s' \in C_i$*, if* $s \neq s'$ *then* $1 \notin \sigma(s)H_i\sigma(s')^{-1}$

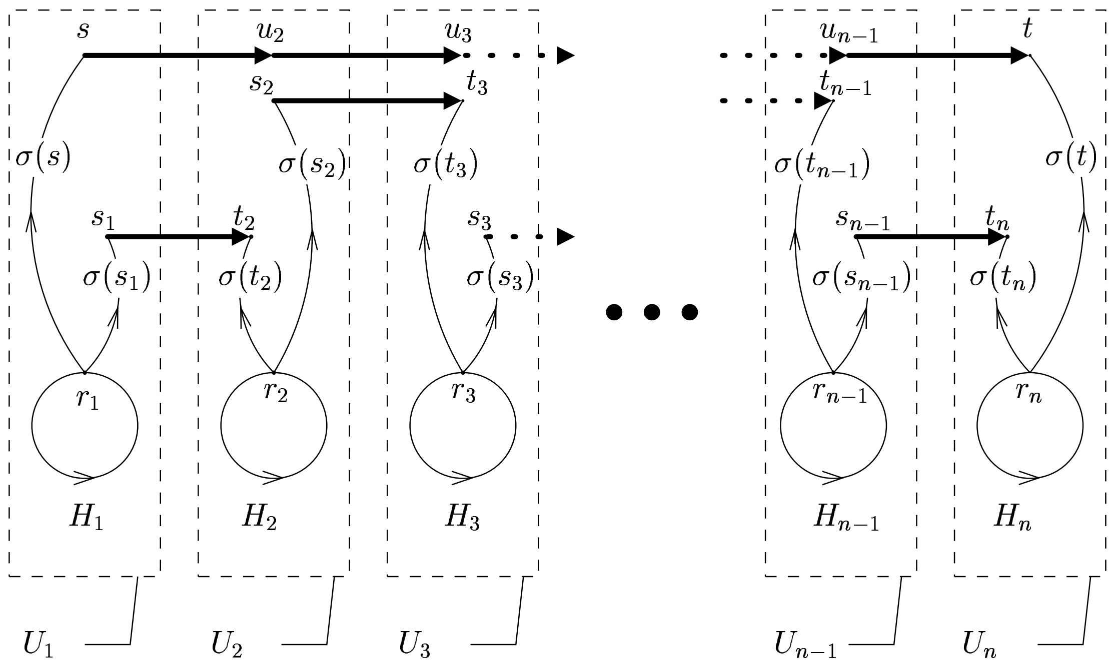
(ii) *for all* $s_1, s_2 \in C_i$*, for all* $t_1, t_2 \in C_{i+1}$ *if* $s_1 \longrightarrow t_1$ *and* $s_2 \not\longrightarrow t_2$ *then*

$$1 \notin \sigma(s_2)H_i\sigma(s_1)^{-1}\sigma(t_1)H_{i+1}\sigma(t_2)^{-1}$$

(iii) $\forall s \in C_i, \forall t \in C_j$*, if* $s \not\longrightarrow^*_{\mathfrak{M}} t$, $\forall s_i \in C_i, \forall s_{i+1}, t_{i+1} \in C_{i+1} \ldots, \forall s_{j-1}, t_{j-1} \in C_{j-1}, \forall t_j \in C_j$ *such that* $s_i \longrightarrow t_{i+1}$, $s_{i+1} \longrightarrow t_{i+2}, \ldots, s_{j-1} \longrightarrow t_j$ *then*

$$1 \notin \sigma(s)H_i\sigma(s_i)^{-1}\sigma(t_{i+1})H_{i+1}\sigma(s_{i+1})^{-1} \cdots \sigma(t_{j-1})H_{j-1}\sigma(s_{j-1})^{-1}\sigma(t_j)H_j\sigma(t)^{-1}$$

We are aware of the awkwardness of these conditions, but we fear that they cannot be avoided. Let us make a picture for the previous lemma :



Situation of the n-partitioned structure $\mathfrak{M}$ and of pre-action φ in hypothesis of condition (iii), with $i = 1$ and $j = n$.
Thick and horizontal arrows represent relation $\longrightarrow_{\mathfrak{M}}$,
Thin and curve arrows represent the pre-action of G on $\mathfrak{M}$.

<u>Proof</u> : In the hypothesis of (i) suppose that $1 = \sigma(s)h_i\sigma(s')^{-1}$ with $h_i \in H_i$. Then $\varphi(1)(s') = s'$ but also

$$\varphi(1)(s') = \varphi(\sigma(s)h_i\sigma(s')^{-1})(s') = \varphi(\sigma(s)h_i)(\tau(C_i)) = \varphi(\sigma(s))(\tau(C_i)) = s$$

a contradiction.

It is clear that condition (ii) is a particular case of condition (iii). Nevertheless it would be useful for the reader to prove that (ii) holds before tackling condition (iii).

In the hypothesis of condition (iii) suppose now that

$$1 = \sigma(s)h_i\sigma(s_i)^{-1}\sigma(t_{i+1})h_{i+1}\sigma(s_{i+1})^{-1} \cdots \sigma(t_{j-1})h_{j-1}\sigma(s_{j-1})^{-1}\sigma(t_j)h_j\sigma(t)^{-1}$$

with $h_i \in H_i, h_{i+1} \in H_{i+1}, \ldots, h_j \in H_j$, then we consider the following elements of $\mathfrak{M}$:

$$\begin{aligned}
u_{i+1} &= \bar{\varphi}(\sigma(s)h_i\sigma(s_i)^{-1})(t_{i+1}) \\
&= \bar{\varphi}(\sigma(s)h_i\sigma(s_i)^{-1}\sigma(t_{i+1})h_{i+1}\sigma(s_{i+1})^{-1})(s_{i+1}) \\
u_{i+2} &= \bar{\varphi}(\sigma(s)h_i\sigma(s_i)^{-1}\sigma(t_{i+1})h_{i+1}\sigma(s_{i+1})^{-1})(t_{i+2}) \\
&= \bar{\varphi}(\sigma(s)h_i\sigma(s_i)^{-1}\sigma(t_{i+1})h_{i+1}\sigma(s_{i+1})^{-1}\sigma(t_{i+2})h_{i+2}\sigma(s_{i+2})^{-1})(s_{i+2}) \\
\ldots & \\
u_{j-1} &= \bar{\varphi}(\sigma(s)h_i\sigma(s_i)^{-1}\cdots\sigma(t_{j-2})h_{j-2}\sigma(s_{j-2})^{-1})(t_{j-1}) \\
&= \bar{\varphi}(\sigma(s)h_i\sigma(s_i)^{-1}\cdots\sigma(t_{j-2})h_{j-2}\sigma(s_{j-2})^{-1}\sigma(t_{j-1})h_{j-1}\sigma(s_{j-1})^{-1})(s_{j-1})
\end{aligned}$$

Let $u_i = s$. For each $i \leq k < j-1$, we have $u_k = \bar{\varphi}(x_k)(s_k)$ and $u_{k+1} = \bar{\varphi}(x_k)(t_{k+1})$ for some x_k. So $u_k \longrightarrow_{\mathfrak{M}} u_{k+1}$ that is

$$s \longrightarrow_{\mathfrak{M}} u_{i+1} \longrightarrow_{\mathfrak{M}} u_{i+2} \longrightarrow_{\mathfrak{M}} \cdots \longrightarrow_{\mathfrak{M}} u_{j-1}$$

Moreover

$$t = \bar{\varphi}(\sigma(t)h_j^{-1}\sigma(t_j)^{-1})(t_j) = \bar{\varphi}(\sigma(s)h_i\sigma(s_i)^{-1}\cdots\sigma(t_{j-1})h_{j-1}\sigma(s_{j-1})^{-1})(t_j)$$

and thus $u_{j-1} \longrightarrow_{\mathfrak{M}} t$. As a result $s \longrightarrow^*_{\mathfrak{M}} t$ which is a contradiction. □

Let us point out that any of the subsets of G we considered in the previous lemma ($\sigma(s)H_i\sigma(s')^{-1}$, $\sigma(s_2)H_i\sigma(s_1)^{-1}\sigma(t_1)H_{i+1}\sigma(t_2)^{-1}$,etc...) is closed with respect to the profinite topology of G. Indeed G satisfies RZ_n and these subsets are translated (say by left multiplication) of products of finitely generated subgroups of G.

For every possible choice of $i = 1, \ldots, n$, $C \in \Omega$ and $s, s' \in C$, $s \neq s'$ we can find a finite image G/N of G (N is a normal subgroup of finite index in G) such that $1 \notin \sigma(s)H_i\sigma(s')^{-1} \pmod N$. As there are only finitely many choices for i, C, s and s', we can find a finite image G/N that satisfies all these conditions simultaneously. This is the very place where we use that $\mathfrak{A}'$ is finite.

In the same way, there are only finitely many conditions of type (iii), thus we can find a finite image G/M of G such that :

$$\forall i,j, 1 \leq i < j \leq n, \forall C_i \in \Omega_i, \ldots, \forall C_j \in \Omega_j, \forall s, s_i \in C_i, \forall s_{i+1}, t_{i+1} \in C_{i+1}, \ldots, \forall t_j, t \in C_j,$$

$$\text{if } s \not\longrightarrow^*_{\mathfrak{M}} t \text{ and } s_i \longrightarrow t_{i+1}, \ldots, s_{j-1} \longrightarrow t_j \text{ then}$$

$$1 \notin \sigma(s)H_i\sigma(s_i)^{-1}\sigma(t_{i+1})H_{i+1}\sigma(s_{i+1})^{-1}\cdots\sigma(t_{j-1})H_{j-1}\sigma(s_{j-1})^{-1}\sigma(t_j)H_j\sigma(t)^{-1} \pmod M.$$

Taking $K(C) = (M \cap N)H(C)$ for each $C \in \Omega$, $K(C)$ is a subgroup of finite index of G and we can state the following lemma :

Lemma 6 *Let $C_1 \in \Omega_1, \ldots, C_n \in \Omega_n$, let $K_i = K(C_i)$. The following four conditions hold for $i = 1, \ldots, n$ and $1 \leq i < j \leq n$:*

(i) $H_i \subset K_i$

(ii) for all $s, s' \in C_i$, if $s \neq s'$ then $1 \notin \sigma(s)K_i\sigma(s')^{-1}$

(iii) for all $s_1, s_2 \in C_i$, for all $t_1, t_2 \in C_{i+1}$ if $s_1 \longrightarrow t_1$ and $s_2 \not\longrightarrow t_2$ then

$$1 \notin \sigma(s_2)K_i\sigma(s_1)^{-1}\sigma(t_1)K_{i+1}\sigma(t_2)^{-1}$$

(iv) $\forall s \in C_i$, $\forall t \in C_j$, if $s \not\longrightarrow^*_{\mathfrak{M}} t$, $\forall s_i \in C_i$, $\forall s_{i+1}, t_{i+1} \in C_{i+1} \ldots, \forall s_{j-1}, t_{j-1} \in C_{j-1}, \forall t_j \in C_j$ such that $s_i \longrightarrow t_{i+1}$, $s_{i+1} \longrightarrow t_{i+1}, \ldots, s_{j-1} \longrightarrow t_j$ then

$$1 \notin \sigma(s)K_i\sigma(s_i)^{-1}\sigma(t_{i+1})K_{i+1}\sigma(s_{i+1})^{-1}\cdots\sigma(t_{j-1})K_{j-1}\sigma(s_{j-1})^{-1}\sigma(t_j)K_j\sigma(t)^{-1}$$

□

We are now going to define an extension $\mathfrak{B}$ of $\mathfrak{A}'$ (and therefore of $\mathfrak{A}$) satisfying the conclusion of proposition 4 (ii). For each orbit $C \in \Omega$, let $\tilde{C} = \{xK(C) | x \in G\}$. $\tilde{C}$ is finite because $K(C)$ is a subgroup of G of finite index.

Lemma 7 *Each C embeds naturally in $\tilde{C}$ and the action of G on $\tilde{C}$ (by left multiplication) extends the pre-action of G on C.*

Proof : The "natural" embedding is : $\left\{ \begin{array}{lcl} C & \longrightarrow & \tilde{C} \\ s & \longmapsto & \sigma(s)K(C) \end{array} \right.$. It is injective because of lemma 6 (ii). For each $s \in C$, and for each $g \in S$ one has $\sigma(\varphi(g)(s))H(C) = g\sigma(s)H(C)$. Therefore, as $H(C) \subset K(C)$ the action of G on $\tilde{C}$ extends the pre-action φ. □

Let us now define $\tilde{U}_i = \biguplus_{C\in\Omega_i} \tilde{C}$ and $\mathfrak{B} = \biguplus_i \tilde{U}_i$. G acts on $\mathfrak{B}$ by left multiplication in a way that preserves each $\tilde{C}$. We call this action $\tilde{\varphi}$. Through the previous lemma we can see $\mathfrak{A}'$ as a subset of $\mathfrak{B}$, and the action $\tilde{\varphi}$ of G on $\mathfrak{B}$ is an extension of the pre-action φ of G on $\mathfrak{A}'$. We define on $\mathfrak{B}$ the smallest n-partitioned structure compatible with the structure of $\mathfrak{A}'$ and with the action of G. That is to say :

$$\forall (s,t) \in \tilde{U}_i \times \tilde{U}_{i+1},\ s \longrightarrow_{\mathfrak{B}} t \overset{\text{def}}{\iff} \exists (s',t') \in U_i(\mathfrak{A}') \times U_{i+1}(\mathfrak{A}'), \exists u \in G \left\{ \begin{array}{l} s' \longrightarrow_{\mathfrak{A}'} t' \\ \wedge\ s = \tilde{\varphi}(u)(s') \\ \wedge\ t = \tilde{\varphi}(u)(t') \end{array} \right.$$

Lemma 8 *$\mathfrak{A}'$ is a substructure of $\mathfrak{B}$.*

Proof : Of course if $s,t \in \mathfrak{A}'$ and $s \longrightarrow_{\mathfrak{A}'} t$ then $s \longrightarrow_{\mathfrak{B}} t$. Conversely if $s,t \in \mathfrak{A}'$ and $s \longrightarrow_{\mathfrak{B}} t$ there exists $s',t' \in \mathfrak{A}'$ and $u \in G$ such that $s = \tilde{\varphi}(u)(s')$, $t = \tilde{\varphi}(u)(t')$ and $s' \longrightarrow_{\mathfrak{A}'} t'$. From the definition of $\mathfrak{B}$ and $\tilde{\varphi}$ we see that s and s' are in the same orbit under φ, say C. Similarly t and t' are in the same orbit $C' \in \Omega$. One has $\sigma(s')^{-1}u\sigma(s) \in K(C)$ and $\sigma(t')^{-1}u\sigma(t) \in K(C')$. Thus $\sigma(s')^{-1}\sigma(t') \in K(C)\sigma(s)^{-1}\sigma(t)K(C')$. From Lemma 6 (iii) we deduce that $s \longrightarrow_{\mathfrak{A}'} t$. □

Lemma 9 $(\longrightarrow^*_{\mathfrak{B}})\lceil\mathfrak{A} = (\longrightarrow^*_{\mathfrak{M}})\lceil\mathfrak{A}$.

Proof : From the previous lemma and the definition of $\mathfrak{A}'$, we have $(\longrightarrow^*_{\mathfrak{M}})\lceil\mathfrak{A} \subseteq (\longrightarrow^*_{\mathfrak{B}})\lceil\mathfrak{A}$. Conversely let $s,t \in \mathfrak{A}$ such that $s \longrightarrow^*_{\mathfrak{B}} t$. Suppose $s \in U_1(\mathfrak{A})$ and $t \in U_n(\mathfrak{A})$ to simplify notations. There exist $s_1, s_2, t_2, \ldots, s_{n-1}, t_{n-1}, t_n \in \mathfrak{A}'$, $r_2, \ldots, r_{n-1} \in \mathfrak{B}$ and $w_1, \ldots w_{n-1} \in G$ such that : $\tilde{\varphi}(w_1)(s_1) = s$, $r_2 = \tilde{\varphi}(w_1)(t_2) = \tilde{\varphi}(w_2)(s_2)$,…,$r_{n-1} = \tilde{\varphi}(w_{n-2})(t_{n-1}) = \tilde{\varphi}(w_{n-1})(s_{n-1})$ and $t = \tilde{\varphi}(w_{n-1})(t_n)$. Let C_1 be the orbit of s and s_1 in $\mathfrak{A}'$, C_2 the orbit of s_2, t_2,..., C_n the orbit of t_n and t. From the equalities we get that

$$1 \in \sigma(s)K(C_1)\sigma(s_1)^{-1}\sigma(t_2)K(C_2)\sigma(s_2)^{-1}\cdots\sigma(t_{n-1})K(C_{n-1})\sigma(s_{n-1})^{-1}\sigma(t_n)K(C_n)\sigma(t)^{-1}$$

and lemma 6 (iv) shows that $s \longrightarrow^*_{\mathfrak{M}} t$. □

This construction of $\mathfrak{B}$ completes the proof of proposition 4.

4 Coloring n-partitioned structures

In this section the definitions and propositions 10, 11 and 12 are of B. Herwig and D. Lascar. As their work is not in print we are going to state all definitions, results and proofs which we will need in the last section.

Definition *[7] Let $\mathfrak{M}$ be an n-partitioned structure and V be a subset of $\mathfrak{M}$.*

V is **closed** *in $\mathfrak{M}$ if and only if for all $s,t \in \mathfrak{M}$ if $s \longrightarrow t$ and $s \in V$ then $t \in V$.*

The **closure** *of V is the smallest closed subset of $\mathfrak{M}$ that contains V.*

A **root** *of V is an element s of V such that for all t in the closure of V $t \not\twoheadrightarrow s$.*

For all $k = 1, \dots, n$ **$\mathbf{dim}_k(V)$** *is the number of roots of V that are in U_k.*
The **dimension** *of V is the tuple $(dim_1(V), \dots, dim_n(V))$.*

Dimensions are ordered lexicographically. The reader should prove that if V is a closed subset, W a subset and $V \subsetneq W \subset \mathfrak{M}$ then $\mathrm{dim}V < \mathrm{dim}W$.

Proposition 10 *[7] Let $\mathfrak{M}$ be a finite n-partitioned structure, then there exists a finite set, P, of unary predicates on $\mathfrak{M}$ (the so-called "colors") such that :*

(i) for all Q in P, $Q(\mathfrak{M})$ is closed in $\mathfrak{M}$;

(ii) for all $m \in \mathfrak{M}$, there exists $Q_m \in P$ such that $Q_m(\mathfrak{M})$ is the closure of $\{m\}$ in $\mathfrak{M}$;

(iii) for each closed subset V of $\mathfrak{M}$, $\mathrm{card}(\{Q \in P \mid V \subseteq Q(\mathfrak{M})\})$ only depends on $\mathrm{dim}V$.

Proof : Let $\{d_1, \dots, d_r\} = \{\mathrm{dim}V \mid V \subset \mathfrak{M}\}$, $d_1 > \cdots > d_r$, be the set of all dimensions of subsets of $\mathfrak{M}$. We will decide by downward induction on $\mathrm{dim}V$, the number of predicates we want in P such that $V = Q(\mathfrak{M})$.

Let P_0 be a finite set of predicates on $\mathfrak{M}$ that satisfy properties (i) and (ii) above. Let $r_1 = \max\{\mathrm{card}\{Q \in P_0 \mid V \subset Q(\mathfrak{M})\} \mid V \subset \mathfrak{M}, \mathrm{dim}V = d_1\}$.

For every closed subset, V, of $\mathfrak{M}$ of dimension d_1, we add new predicates Q such that $Q(\mathfrak{M}) = V$ to P_0 to get P_1 such that $\mathrm{card}\{Q \in P_1 \mid V \subset Q(\mathfrak{M})\} = r_1$.

Now, suppose we have constructed P_{k-1} and r_{k-1} such that for every closed subset V of $\mathfrak{M}$ of dimension d_i greater than d_{k-1}, $\mathrm{card}\{Q \in P_{k-1} \mid V \subset Q(\mathfrak{M})\} = r_i$. Then we define P_k and r_k.

Let $r_k = \max\{\mathrm{card}\{Q \in P_{k-1} \mid V \subset Q(\mathfrak{M})\} \mid V \subset \mathfrak{M}, \mathrm{dim}V = d_k\}$. For every closed subset V of $\mathfrak{M}$, of dimension d_k we add new predicates Q such that $Q(\mathfrak{M}) = V$ to P_{k-1} to get a set P_k such that $\mathrm{card}\{Q \in P_k \mid V \subset Q(\mathfrak{M})\} = r_k$.

The set of predicates P_r constructed in this way obviously satisfies conditions (i) and (ii) of the proposition. Let V be a closed subset of $\mathfrak{M}$, $d_k = \mathrm{dim}V$ and $Q \in P_r$. If $V \subset Q(\mathfrak{M})$ then $\mathrm{dim}Q(\mathfrak{M}) \geq \mathrm{dim}V$ and thus $Q \in P_k$. We have constructed the set P_k in such a way that $\mathrm{card}\{Q \in P_k \mid V \subset Q(\mathfrak{M})\} = r_k$. This proves that the set of predicates P_r satisfies condition (iii). □

This proposition will be very useful, because of the following construction. If P is a set and r an integer, $P^{[r]}$ is the collection of subsets of P of cardinality r.

Definition *[7] Let P be a set (the set of colors). And let $\bar{r} = (r_1, \dots, r_n)$ be a tuple of positive integers. The* **homogeneous n-partitioned structure associated to P and $\bar{r}$** *is the structure $\mathfrak{M}$ such that $U_i(\mathfrak{M}) = P^{[r_i]}$ and for all $i = 1, \dots, n-1$ and for all $s \in P^{[r_i]}$ and $t \in P^{[r_{i+1}]}$, $s \longrightarrow t$ if and only if $s \subset t$.*

Proposition 11 *[7] Let $\mathfrak{M}$ be a finite n-partitioned structure, P a set of colors on $\mathfrak{M}$ that satisfies condition (i), (ii) and (iii) of proposition 10. For $i = 1, \dots, n$ and $m \in U_i(\mathfrak{M})$, let $r_i = \mathrm{card}\{Q \in P \mid m \in Q(\mathfrak{M})\}$. From condition (iii), r_i does not depend on the choice of $m \in U_i$. Let $\bar{r} = (r_1 \dots, r_n)$, then there is an injection from $\mathfrak{M}$ into the homogeneous n-partitioned structure associated to P and $\bar{r}$, say $\mathfrak{N}$, which is given by :*

$$\begin{array}{rccl} f: & \mathfrak{M} & \longrightarrow & \mathfrak{N} \\ & m & \longmapsto & f(m) = \{Q \in P \mid m \in Q(\mathfrak{M})\} \end{array}.$$

This injection f is a morphism of n-partitioned structures such that for all $m, n \in \mathfrak{M}$, $m \longrightarrow^*_{\mathfrak{M}} n$ if and only if $f(m) \longrightarrow^*_{\mathfrak{N}} f(n)$.

Proof : From condition (iii) of proposition 10 we know that for every $m \in U_i$, its image under f is in $P^{[r_i]}$. From proposition 10 (ii) f is an injection. For all $m, n \in \mathfrak{M}$, we have from proposition 10 (i) $m \longrightarrow^*_{\mathfrak{M}} n$ implies that $f(m) \subseteq f(n)$ and by definition of $\mathfrak{N}$, $f(m) \longrightarrow^*_{\mathfrak{N}} f(n)$. Conversely, from proposition 10 (i) and (ii) if $f(m) \subseteq f(n)$ then $n \in Q_m(\mathfrak{M})$. This proves that $m \longrightarrow^*_{\mathfrak{M}} n$. □

To prove that the free groups satisfy RZ, B. Herwig and D. Lascar prove that they satisfy property (ii) of proposition 4. For this purpose they use the very important following proposition:

Proposition 12 *[7] In the conditions of the previous proposition, every partial isomorphism p of $\mathfrak{M}$, such that*

$$\forall m, n \in \mathfrak{M}, m \longrightarrow^*_{\mathfrak{M}} n \iff p(m) \longrightarrow^*_{\mathfrak{M}} p(n),$$

extends to an automorphism of $\mathfrak{N}$.

We will not give the proof of this result here because we will not need it. But it explains why $\mathfrak{N}$ is called homogeneous, and why coloring structures is a natural operation when dealing with partial isomorphisms.

Proposition 13 *In the hypothesis of proposition 11, if φ is an action of a group G on $\mathfrak{M}$ then there exists an action ψ of G on $\mathfrak{N}$ that extends φ.*

Proof : Let $\mathcal{W}$ be the set of all closed subsets of $\mathfrak{M}$. The action φ induces an action $\tilde{\varphi}$ of G on $\mathcal{W}$: $\tilde{\varphi}(g)(V) = \{\varphi(g)(m) \mid m \in V\}$. To check that this definition is correct, one has to check that for every closed subset V of $\mathfrak{M}$, $\tilde{\varphi}(g)(V)$ is also a closed subset of $\mathfrak{M}$. Moreover this action preserves dimension.

Lemma 14 *for all $g \in G$, for every subset $V \in \mathcal{W}$:*

$$\mathrm{card}\{Q \in P \mid V = Q(\mathfrak{M})\} = \mathrm{card}\{Q \in P \mid \tilde{\varphi}(g)(V) = Q(\mathfrak{M})\}$$

Proof : We prove this lemma by downward induction on $\dim V$.

If V is of maximal dimension then $\{Q \in P \mid V = Q(\mathfrak{M})\} = \{Q \in P \mid V \subset Q(\mathfrak{M})\}$ and the lemma results from condition (iii) of proposition 10. In the general case, let $V \in \mathcal{W}$ be a closed subset of $\mathfrak{M}$, and let $\mathcal{V}$ be the set of closed subsets of $\mathfrak{M}$ that strictly contain V. B. Herwig noticed the following formula :

$$\mathrm{card}\{Q \in P \mid V = Q(\mathfrak{M})\} = \mathrm{card}\{Q \in P \mid V \subset Q(\mathfrak{M})\} - \sum_{W \in \mathcal{V}} \mathrm{card}\{Q \in P \mid W = Q(\mathfrak{M})\}$$

Applying the automorphism $\tilde{\varphi}(g)$ we get :

$$\begin{aligned} \mathrm{card}\{Q \in P \mid \tilde{\varphi}(g)(V) = Q(\mathfrak{M})\} &= \mathrm{card}\{Q \in P \mid \tilde{\varphi}(g)(V) \subset Q(\mathfrak{M})\} \\ &\quad - \sum_{W \in \mathcal{V}} \mathrm{card}\{Q \in P \mid \tilde{\varphi}(g)(W) = Q(\mathfrak{M})\} \end{aligned}$$

Thanks to condition (iii) of proposition 10, taking in account that for all $W \in \mathcal{V}$, $\dim V < \dim W$ and the induction hypothesis we can conclude. □

Consider the partition $(P_V)_{V \in \mathcal{W}}$ of P, where $P_V = \{Q \in P \mid V = Q(\mathfrak{M})\}$. The cardinality of P_V only depends on the orbit of V under the action $\tilde{\varphi}$. For each orbit C let α_C denotes the corresponding cardinal of P_V. For each $V \in \mathcal{W}$ we fix a one-to-one correspondence σ_V from $\{1, 2, \ldots, \alpha_C\}$ to P_V, where C is the orbit of V under the action $\tilde{\varphi}$ of G on $\mathcal{W}$. We can now define an action ψ of G on P :

$$\forall g \in G, \forall V \in \mathcal{W}, \forall Q \in P_V, \psi(g)(Q) = \sigma_{\tilde{\varphi}(g)(V)} \circ \sigma_V^{-1}(Q)$$

This clearly defines an application from G into $\mathrm{Aut}(P)$. We have to show that it is a morphism of groups. Let $g, h \in G$, $V \in \mathcal{W}$ and $Q \in P_V$ then $\psi(gh)(Q) = \sigma_{\tilde{\varphi}(gh)(V)} \circ {\sigma_V}^{-1}(Q)$ and $\psi(g) \circ \psi(h)(Q) = \psi(g)(\sigma_{\tilde{\varphi}(h)(V)} \circ {\sigma_V}^{-1}(Q))$. Now, $\sigma_{\tilde{\varphi}(h)(V)} \circ {\sigma_V}^{-1} \in P_{\tilde{\varphi}(h)(v)}$ and thus $\psi(g) \circ \psi(h)(Q) = \sigma_{\tilde{\varphi}(g)(V)} \circ \sigma^{-1}_{\tilde{\varphi}(h)(V)} \circ \sigma_{\tilde{\varphi}(h)(V)} \circ {\sigma_V}^{-1}(Q)$. This proves that ψ is a morphism.

This action ψ induces an action, that we will also call ψ on $\mathfrak{N}$:

$$\forall g \in G, \forall n \in \mathfrak{N}, \psi(g)(n) = \{\psi(g)(Q) \mid Q \in n\}$$

From the construction and the definition of the injection from $\mathfrak{M}$ into $\mathfrak{N}$, it is obvious that for all $g \in G$, $\psi(g)$ extends $\varphi(g)$. □

5 Proof of the main theorem

Let G_1 and G_2 be two groups which satisfy RZ_n. Let $G = G_1 * G_2$ be their free product. K. Gruenberg [3], R. Burns [1] and N. Romanovskiĭ [11] have shown that G is residually finite and LERF. Gitik [2] gave an another proof of this result using finite graphs and pre-actions. We are going to mimic her argument and use proposition 4 to prove that G satisfies RZ_n.

Let $\mathfrak{A}$ be a finite n-partitioned structure, S a finite symmetric subset of G and φ a pre-action of G on $\mathfrak{A}$ which is induced by S and by an action $\bar{\varphi}$ of G on an extension $\mathfrak{M}$ of $\mathfrak{A}$. We are going to prove that there exists a finite extension $\mathfrak{C}$ of $\mathfrak{A}$ and an action ψ of G on $\mathfrak{C}$ that extends φ such that $(\longrightarrow^*_{\mathfrak{M}})\lceil\mathfrak{A} = (\longrightarrow^*_{\mathfrak{C}})\lceil\mathfrak{A}$.

Let $S_1 \subset G_1$ and $S_2 \subset G_2$ be two finite symmetric subsets such that $S \subset \langle S_1, S_2\rangle$. To be convinced that such S_1 and S_2 exist just write the elements of S in normal form with respect to the free product. Only finitely many elements of G_1 and G_2 occur in these normal forms, giving rise to suitable sets S_1 and S_2 respectively. Let $\mathfrak{A}'$ be a finite subset of $\mathfrak{M}$ that contains $\mathfrak{A}$ and such that the pre-action φ is a restriction of the pre-action φ' of G on $\mathfrak{A}'$ induced by $\bar{\varphi}$ and $S_1 \cup S_2$. Such a finite set $\mathfrak{A}'$ exists, for it can be constructed from $\mathfrak{A}$ by adding a finite number of elements of $\mathfrak{M}$: for each $s \in S$ and for all $m, n \in \mathfrak{A}$ such that $\varphi(s)(m) = n$, we add the elements $\bar{\varphi}(s_\ell)(m), \ldots, \bar{\varphi}(s_2 \cdots s_\ell)(m)$, where $s = s_1 s_2 \cdots s_\ell$ is the normal form of s in the free product $G = G_1 * G_2$.

It is enough to prove that there exists $\mathfrak{C}$ a finite extension of $\mathfrak{A}'$ and an action ψ of G on $\mathfrak{C}$ such that $(\longrightarrow^*_{\mathfrak{M}})\lceil\mathfrak{A}' = (\longrightarrow^*_{\mathfrak{C}})\lceil\mathfrak{A}'$. Indeed in these conditions $\mathfrak{C}$ is a finite extension of $\mathfrak{A}$, ψ extends φ and $(\longrightarrow^*_{\mathfrak{M}})\lceil\mathfrak{A} = (\longrightarrow^*_{\mathfrak{C}})\lceil\mathfrak{A}$.

We now assume that $S = S_1 \cup S_2$ and $\mathfrak{A} = \mathfrak{A}'$ We define pre-actions of G_1 and G_2 on $\mathfrak{A}$, respectively φ_1 and φ_2, that are induced by S_1 and S_2 and $\bar{\varphi}$.

G_1 and G_2 satisfy RZ_n and thus using proposition 4 there exist finite structures $\mathfrak{B}_1$ and $\mathfrak{B}_2$ which extend $\mathfrak{A}$, there exist actions $\tilde{\varphi_1}$ and $\tilde{\varphi_2}$ of G_1 and G_2 which extend φ_1 and φ_2 respectively with the following property :

$$(\longrightarrow^*_{\mathfrak{B}_1})\lceil\mathfrak{A} = (\longrightarrow^*_{\mathfrak{M}})\lceil\mathfrak{A} = (\longrightarrow^*_{\mathfrak{B}_2})\lceil\mathfrak{A}$$

Let $\mathfrak{B}$ be the **amalgamated sum** of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ over $\mathfrak{A}$: the underlying set of $\mathfrak{B}$ is the disjoint union of $\mathfrak{B}_1\backslash\mathfrak{A}$, $\mathfrak{B}_2\backslash\mathfrak{A}$ and $\mathfrak{A}$, the predicates U_i are defined naturally and the relation $\longrightarrow_{\mathfrak{B}}$ is the union of $\longrightarrow_{\mathfrak{B}_1}$ and $\longrightarrow_{\mathfrak{B}_2}$. As there are no $\longrightarrow_{\mathfrak{B}}$ between elements of $\mathfrak{B}_1\backslash\mathfrak{A}$ and elements of $\mathfrak{B}_2\backslash\mathfrak{A}$ the previous equality leads to :

$$(\longrightarrow^*_{\mathfrak{B}})\lceil\mathfrak{A} = (\longrightarrow^*_{\mathfrak{M}})\lceil\mathfrak{A} \quad \text{and} \quad \longrightarrow^*_{\mathfrak{B}_\mathrm{i}} = (\longrightarrow^*_{\mathfrak{B}})\lceil\mathfrak{B}_\mathrm{i} \quad (\mathrm{i} = 1, 2).$$

Moreover in an n-partitioned structure the dimension of a subset only depends on the restriction of $\longrightarrow^*$ to this subset. Therefore if V is a subset of $\mathfrak{B}_i$ $(i = 1, 2)$ then $\dim_{\mathfrak{B}_i} V = \dim_{\mathfrak{B}} V$.

Let $\mathfrak{C}$ be an homogeneous n-partitioned structure built from $\mathfrak{B}$ as in proposition 10 and 11. Then we can see $\mathfrak{B}$ as a substructure of $\mathfrak{C}$ and from proposition 11 we have $(\longrightarrow^*_{\mathfrak{C}})\lceil\mathfrak{B} = \longrightarrow^*_{\mathfrak{B}}$. Let P be the set of predicates used to build $\mathfrak{C}$. Let V be a closed subset of $\mathfrak{B}_i$ $(i = 1, 2)$. The number of predicates $Q \in P$ such that $Q(\mathfrak{B}_i) = V$ only depends on $\dim_{\mathfrak{B}} V = \dim_{\mathfrak{B}_i} V$. Therefore $\mathfrak{C}$ is also an homogeneous n-partitioned structure over $\mathfrak{B}_1$ and $\mathfrak{B}_2$. We can use proposition 13. There exist ψ_1 and ψ_2 actions of G_1 and G_2, respectively, on $\mathfrak{C}$ that extends the actions $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$.

From the universal property of free products we can define an action ψ of G on $\mathfrak{C}$ that extends ψ_1 and ψ_2. We defined the ψ_i such that ψ extends the pre-action of G on $\mathfrak{A}$ and thus the pre-action φ of G of A.

For any finite n-partitioned structure $\mathfrak{A}$ and any pre-action of G on $\mathfrak{A}$ we have constructed a finite extension C of $\mathfrak{A}$ that fulfills the condition (ii) of proposition 4. This shows that G has property RZ_n and ends the proof of our theorem.

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References

[1] R Burns, *On finitely generated subgroups of free products*, J. of the Australian Math. Soc. **12** (1971) 358-364

[2] R Gitik, *Graphs and separability properties of groups*, J. of Algebra, **188** (1997) 125-143

[3] K Gruenberg, *Residual properties of infinite soluble groups*, Proc. London Math. Soc. **7** (1957) 29-62

[4] M Hall, *Coset representation in free groups*, Trans. Am. Math. Soc. **67** (1949) 421-432

[5] M Hall, *A topology for free groups and related groups*, Annals of Math. **52** (1950) 127-139

[6] B Herwig, *Extending partial isomorphisms of finite structure*, Combinatorica **15** (1995) 365-371

[7] B Herwig, D Lascar, *Extending partial automorphisms and the profinite topology on free groups*, To Appear in Trans. Amer. Math. Soc.

[8] E Hrushovski, *Extending partial automorphisms of graphs*, Combinatorica **12** (1992) 411-416

[9] J-E Pin, C Reuteunauer, *A conjecture on the Hall topology for the free group*, Bull. London Math. Soc. **23** (1991) 356-362

[10] L Ribes, P Zalesskií, *On the profinite topology on a free group*, Bull. London Math. Soc. **25** (1993) 37-43

[11] N Romanovskiĭ , *On the residual finitness of free products with respect to subgroups*, Izv. Akad. Nauk. SSSR Ser. Mat. **33** (1969) 1324-1329

Author :
Thierry Coulbois
Équipe de Logique Mathématique
Université Paris 7
2 place Jussieu
75251 Paris Cedex 05
FRANCE
tel : 33 (1) 44 27 56 14
e-mail : coulbois@logique.jussieu.fr

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