The Tree-Bonacci

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with Xavier Bressaud, Arnaud Hilion, Martin Lustig

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From the Boundary of Outer Space: the Map Q

Theorem (Levitt-Lustig)

T: $\mathbb{R}$-tree with very small, minimal action of F_N by isometries (i.e. $T \in \overline{CV_N}$) with dense orbits.

$$\exists!\ Q:\ \partial F_N \to \widehat{T}\ (= \overline{T} \cup \partial T)\ \textit{equivariant}$$

$$\left.\begin{array}{l} \forall u_n \in F_N,\ s.t.\ (u_n)_{n\in\mathbb{N}} \to X \in \partial F_N \\ \forall P \in T,\ s.t.\ (u_n P)\ converges \end{array}\right\} \Rightarrow Q(X) = \lim u_n P.$$

Other Definition of Q

$$\forall (u_n)_{n\in\mathbb{N}} \to X,\ \forall P \in T,\ [P, Q(X)] = \overline{\bigcup_{n>0} \bigcap_{k>n} [P, u_k P]}$$

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The Attractive Real Tree of an Iwip Automorphism

Tribonacci

$$\varphi: \begin{array}{ccc} a & \mapsto & ab \\ b & \mapsto & ac \\ c & \mapsto & a \end{array}$$

- $\varphi \in \mathrm{Aut}(F_3)$ iwip
- φ acts on Outer Space
- North-South Dynamic on the Closure of Projectivized Outer Space.

- $[T_\varphi]$ repulsive fix point of φ (attractive of φ^{-1})
- T_φ has a very small minimal action of F_3 by isometries with dense orbits.

Constructing the Repulsive Tree T_φ

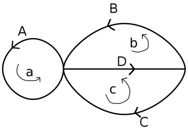
- Find a train-track representative for φ^{-1}:

φ^{-1}

$$\begin{aligned} a &\mapsto c \\ b &\mapsto c^{-1}a \\ c &\mapsto c^{-1}b \end{aligned}$$

f

$$\begin{aligned} A &\mapsto DC \\ B &\mapsto D^{-1}A \\ C &\mapsto B \\ D &\mapsto C^{-1} \end{aligned}$$



- $\tilde{\Gamma}$ universal cover of Γ, $\tilde{f}$ a cover map of f
- $d_n(x,y) = \frac{d(\tilde{f}^n(x),\tilde{f}^n(y))}{\lambda^n}$
- $d_\infty = \lim d_n$
- $T_\varphi = \tilde{\Gamma}/d_\infty$, $(H = \tilde{f}/d_\infty)$

Summing Up

- T_φ has a very small minimal, minimal action of F_3 by isometries with dense orbits.
- $Q : \partial F_3 \to \widehat{T}_\varphi$ almost continuous.

The Tribonacci Substitution and its Attractive Fix Point

Tribonacci

$$\varphi: \begin{array}{ccc} a & \mapsto & ab \\ b & \mapsto & ac \\ c & \mapsto & a \end{array}$$

$$\varphi(a) = ab$$

Attracive Fix Point $X_\varphi = \varphi^\infty(a) \in \partial F_3$

$$X_\varphi = abacabaabacababacabaabacabacabaabacababacabaabac\cdots$$

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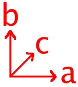

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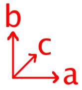

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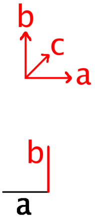

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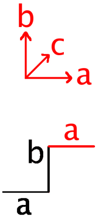

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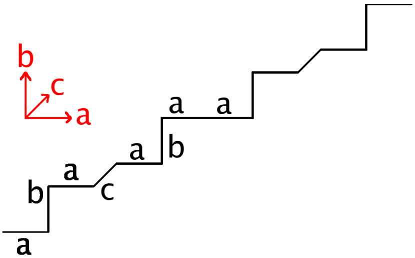

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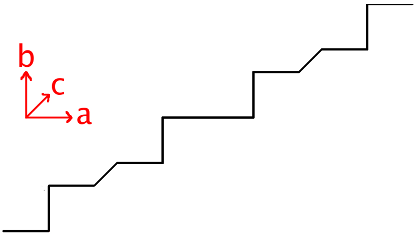

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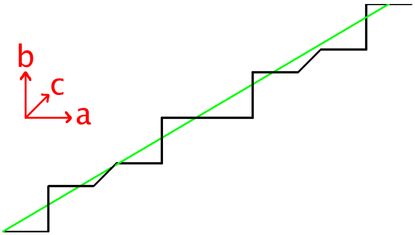
$$\lambda^3 = \lambda^2 + \lambda + 1$$



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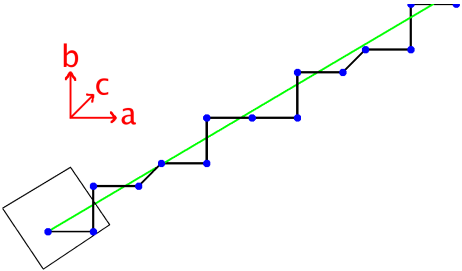

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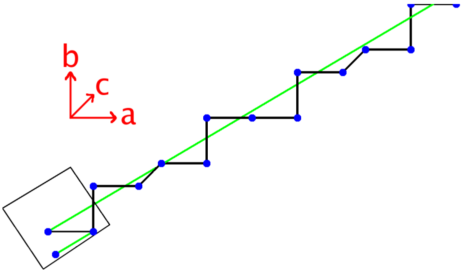

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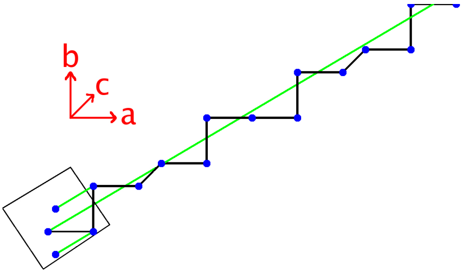

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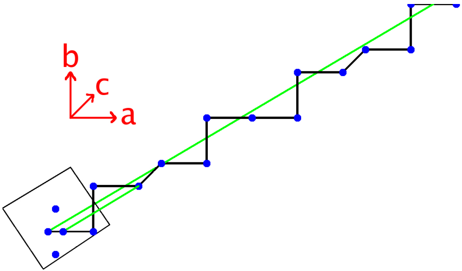

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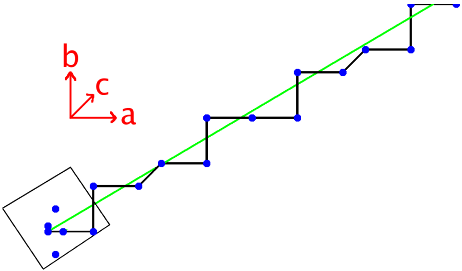

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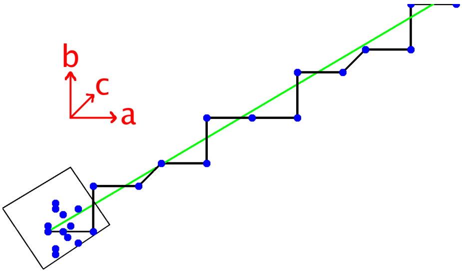

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The Tribonacci Substitution and its Attractive Fix Point

Tribonacci

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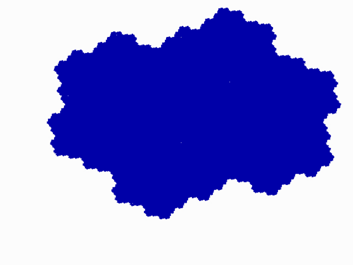

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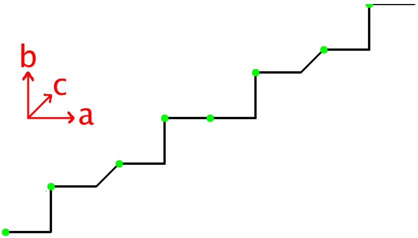

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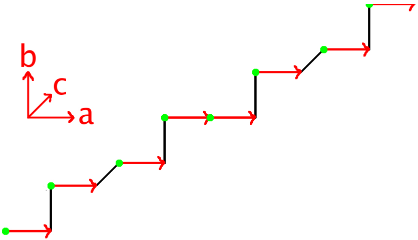

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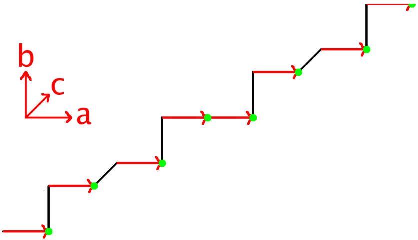

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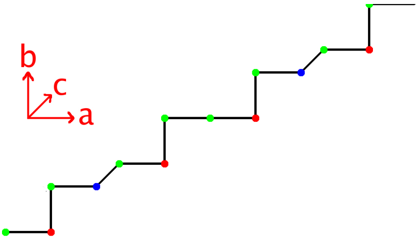

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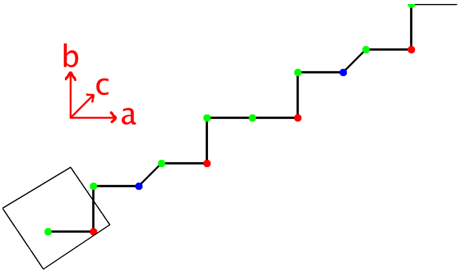

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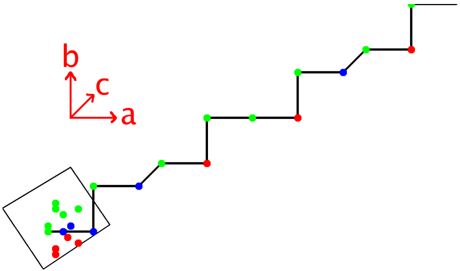

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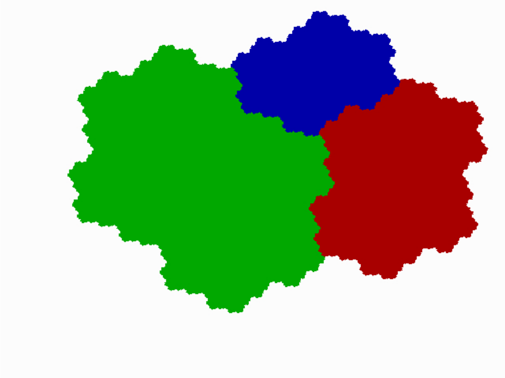
The Rauzy Fractal and the Piecewise Exchange



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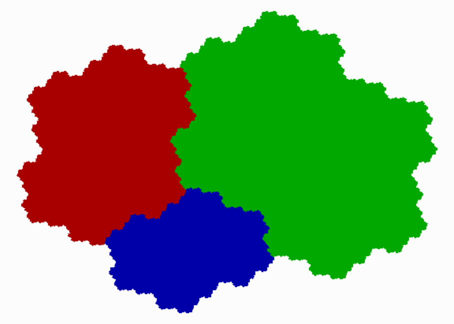

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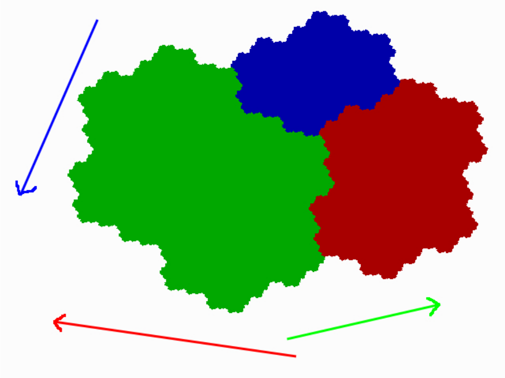

The Rauzy Fractal and the Piecewise Exchange

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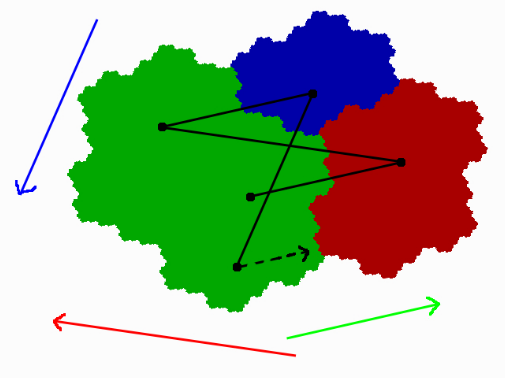

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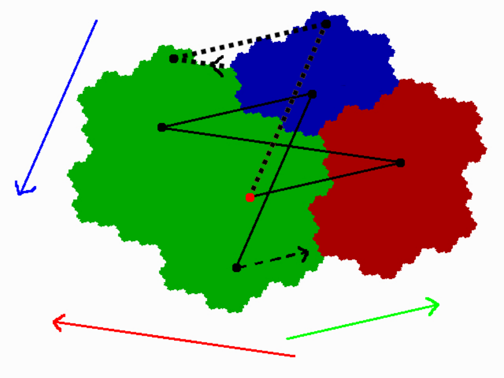

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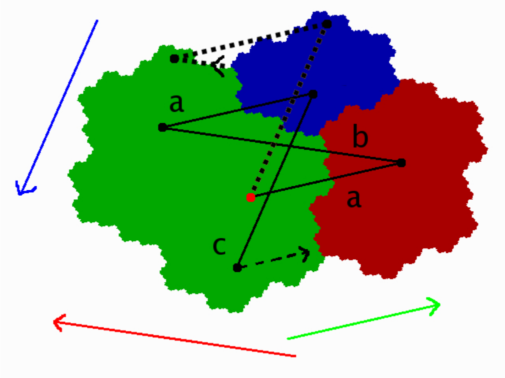

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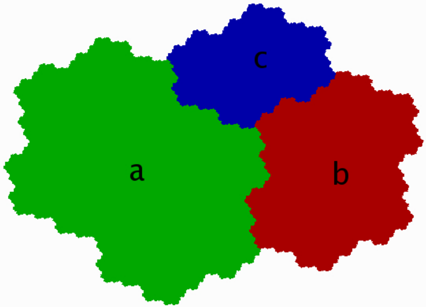

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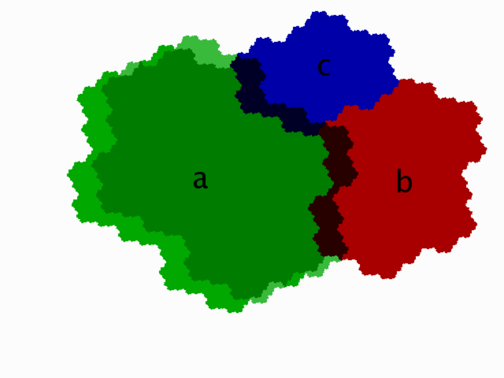

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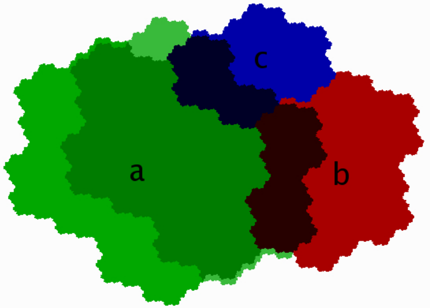

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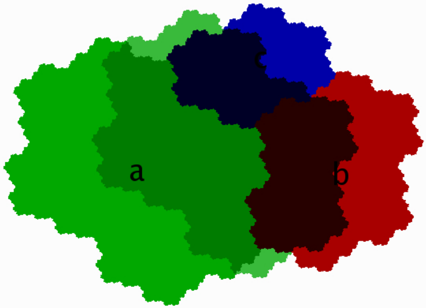

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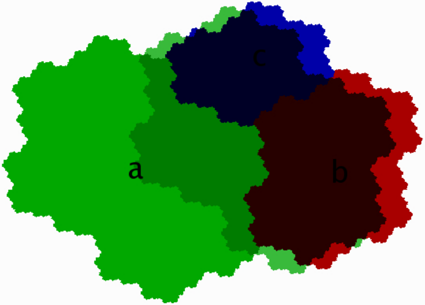

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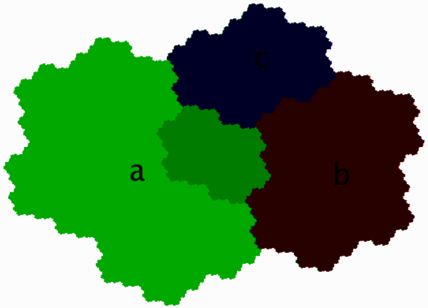

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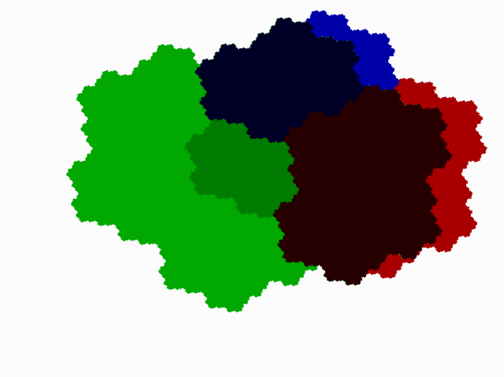

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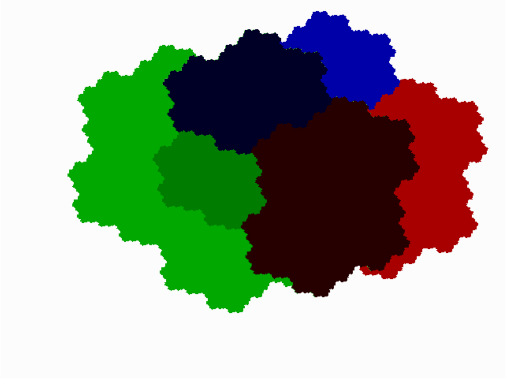

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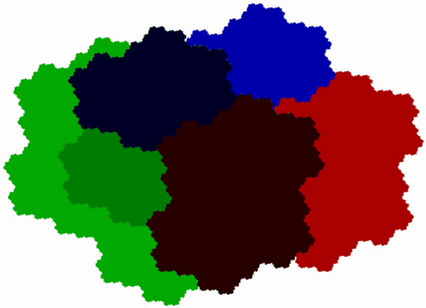

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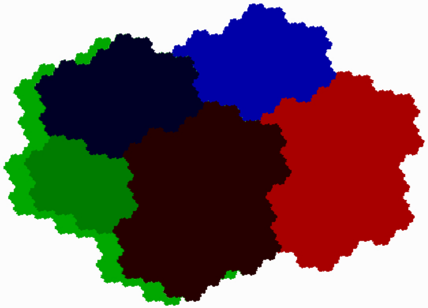

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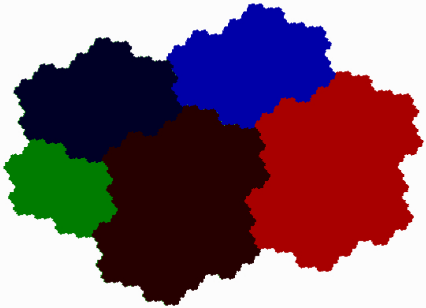

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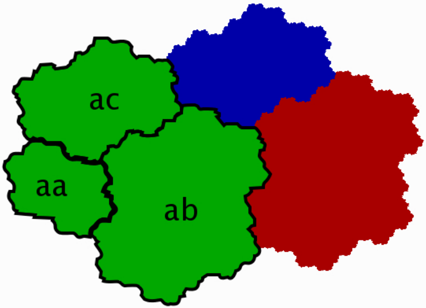

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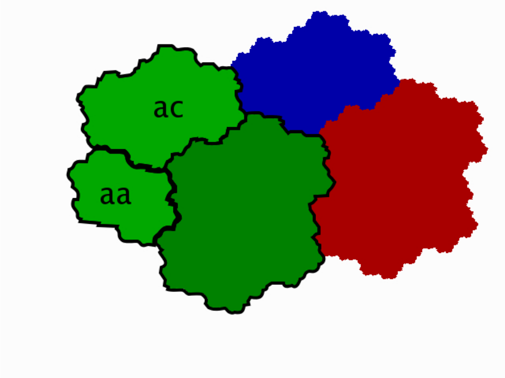

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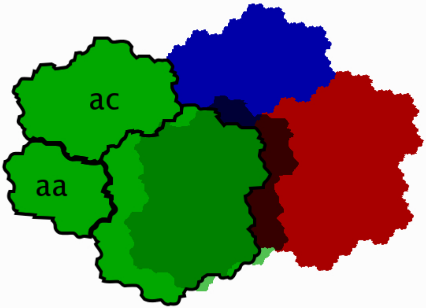

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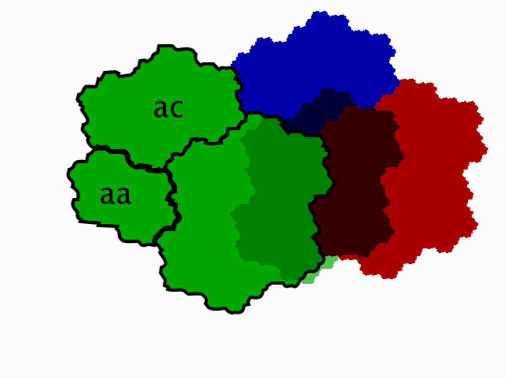

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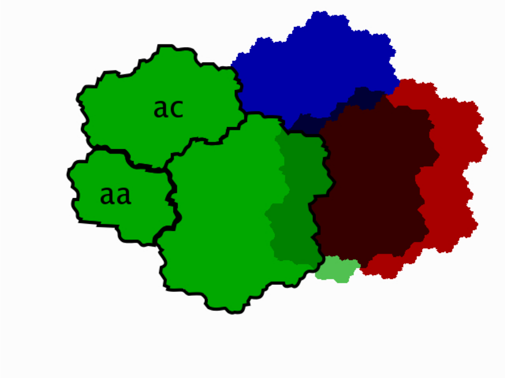

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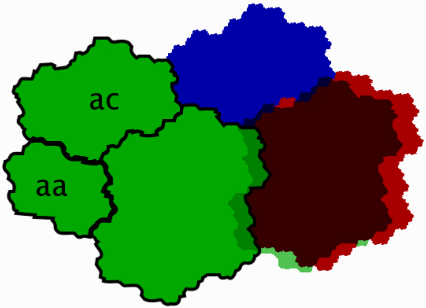

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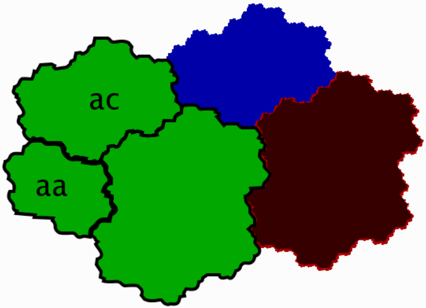

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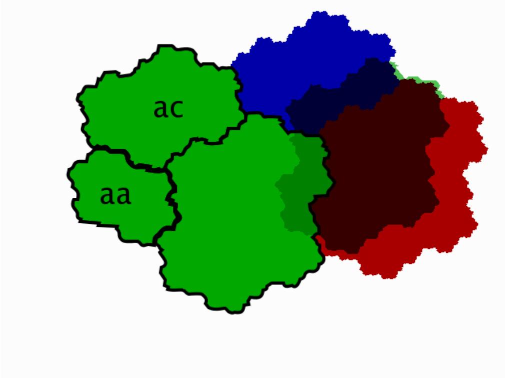

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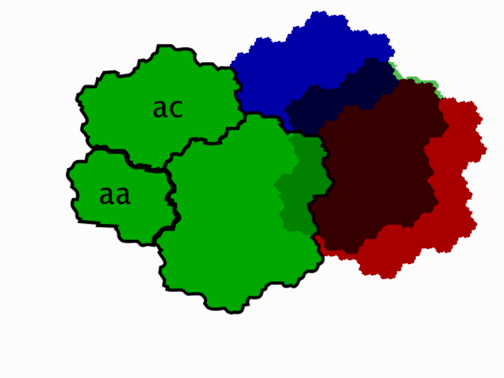

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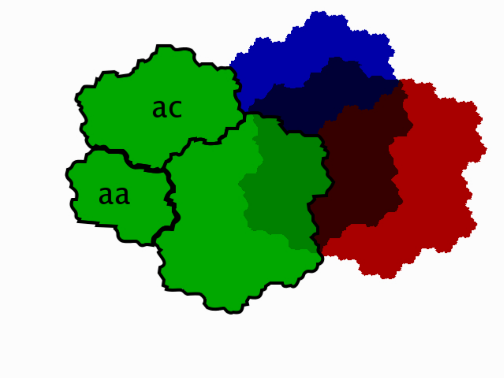

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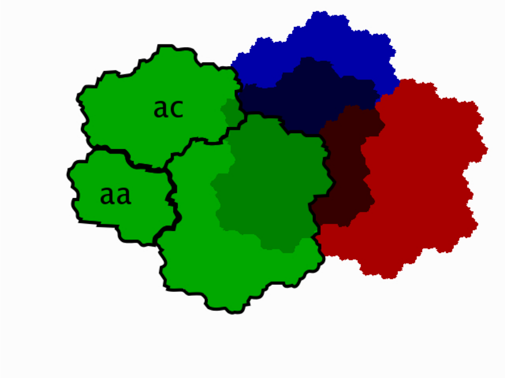

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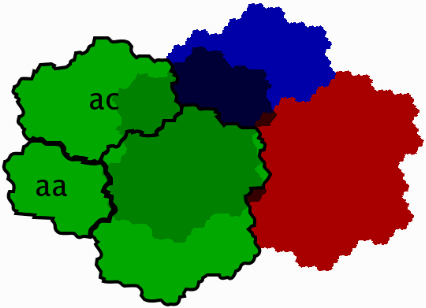

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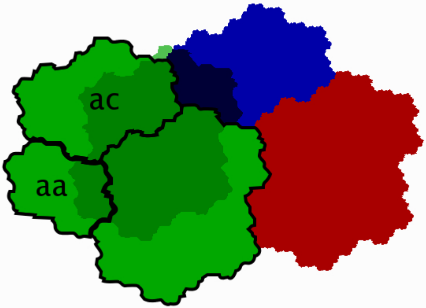

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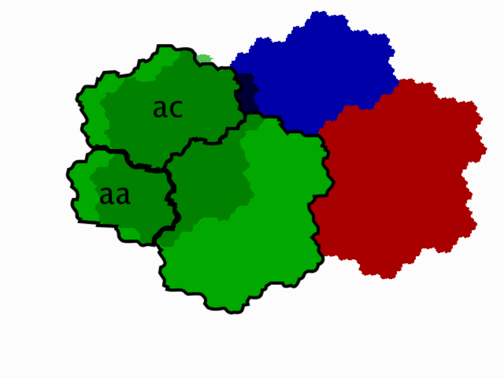

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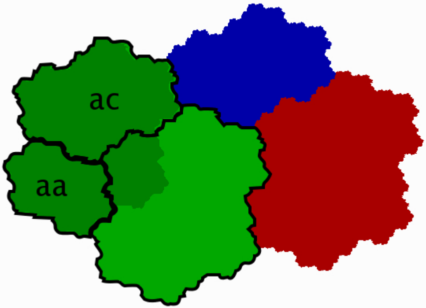

Attracive Fix Point $X_\varphi \in \partial F_3$

$X_\varphi = abacabaabacababacabaabacabacabaabacababacabaabac\cdots$

The Rauzy Fractal and the Piecewise Exchange

- Each point has a futur trajectory and a past
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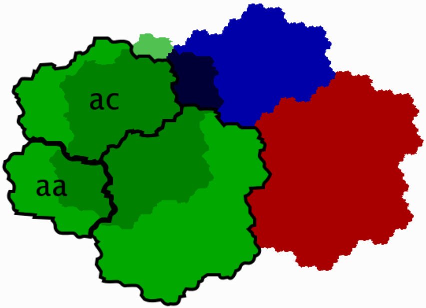

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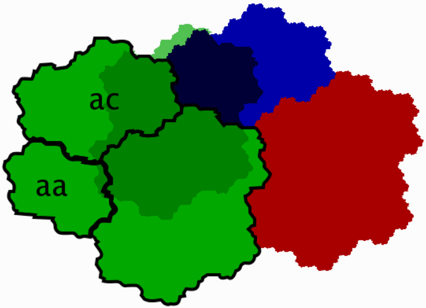

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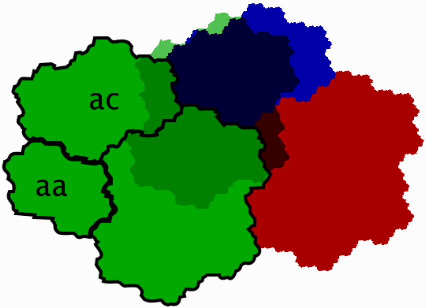

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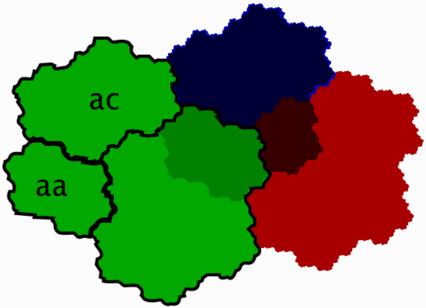

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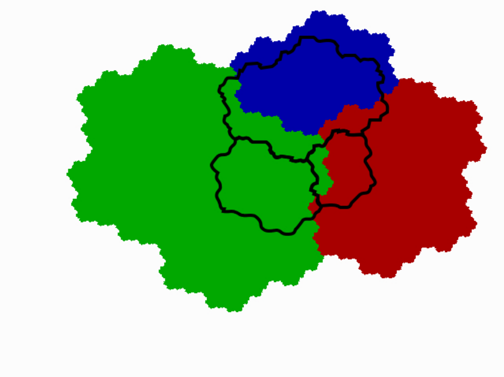

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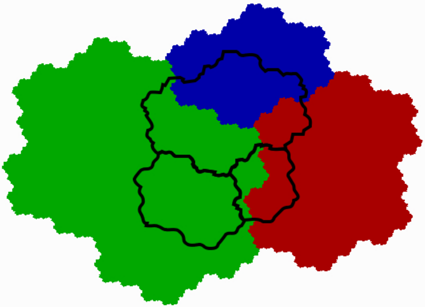

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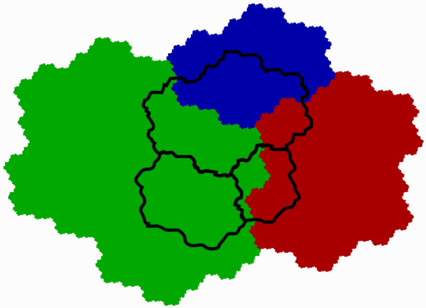

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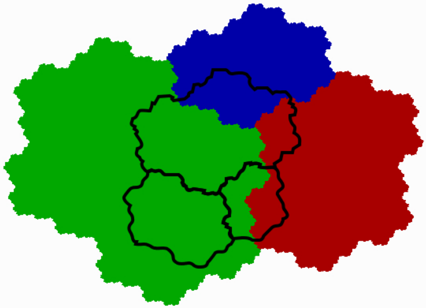

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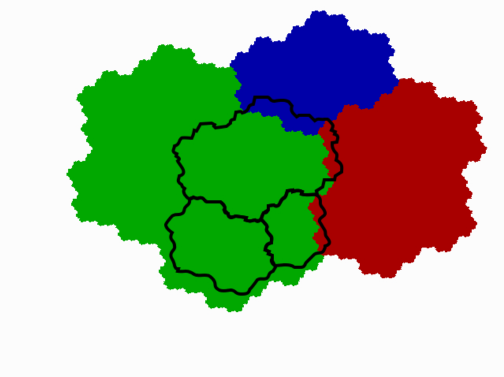

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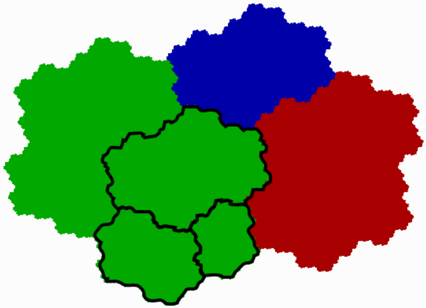

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The Rauzy Fractal and the Piecewise Exchange

- Each point has a futur trajectory and a past
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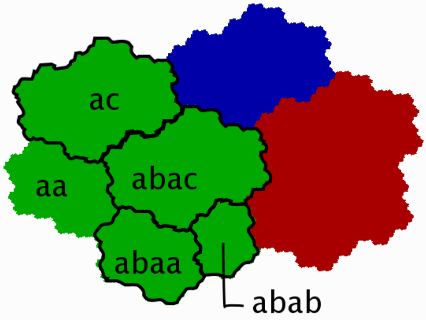
- L: set of possible bi-infinite trajectories
- $Q : L \to R$ is continuous



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Summing Up (2)

Tribonacci

$$\varphi : \begin{array}{lcl} a & \mapsto & ab \\ b & \mapsto & ac \\ c & \mapsto & a \end{array}$$

φ is an Automorphism

- T_φ
- $Q : \partial F_3 \to \widehat{T}_\varphi$ almost continuous

φ is a Substitution

- Rauzy Fractal R
- $Q : L \to R$ continuous

Goal

Make the two maps Q coincide

Attractive Lamination

L set of bi-infinite possible trajectories in the Rauzy Fractal

- finite subwords of $Z \in L$ are subwords of X_φ.
- L is a Cantor Set.

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L is the Attractive Lamination of φ.

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Both Q are defined on L

$\forall Z \in L$, write $Z = X^{-1}Y$ then $$Q(X) = Q(Y) \in \overline{T}_\varphi.$$

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Theorem (Coulbois-Hilion-Lustig)

$Q : L \to \overline{T_\varphi}$ is continuous.

Compact Core of T_φ

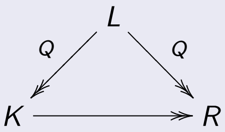
1. L (Attractive Lamination) is compact
2. $Q : L \longrightarrow \overline{T_\varphi}$ is continuous

Compact Core of T_φ

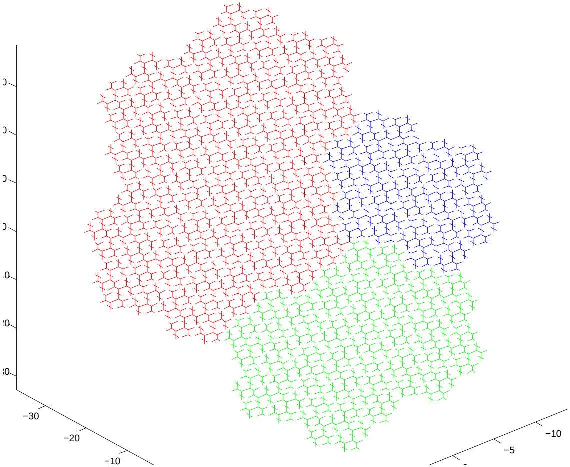
$K = Q(L)$ is a compact subtree of $\overline{T_\varphi}$.

Theorem (Bressaud)

The Core K of the tree T_φ embeds continuously and surjectively to the Rauzy Fractal.



The Peano Curve in the Rauzy Fractal



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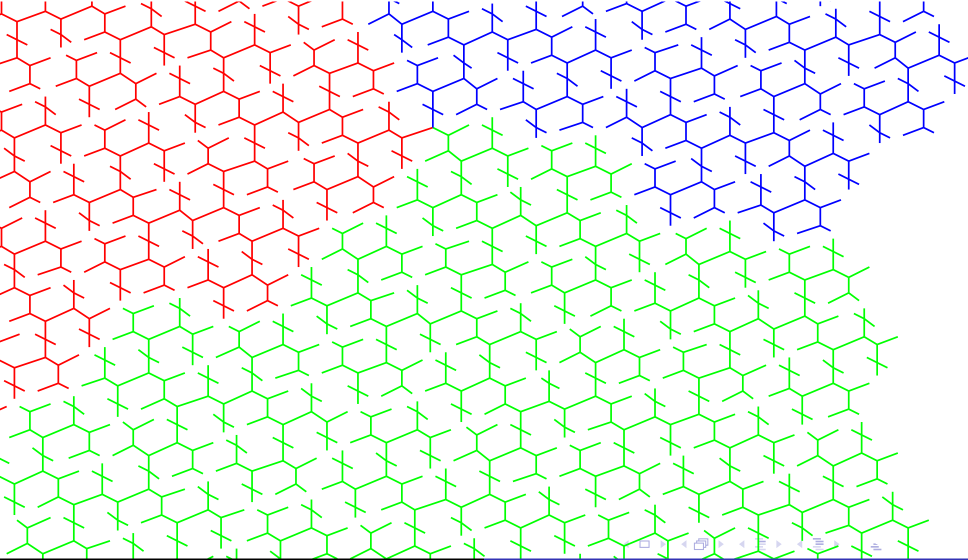
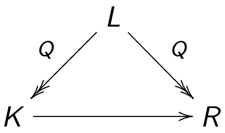


Proof



1. $Q(Z) = Q(Z') \in \overline{T_\varphi} \Rightarrow Q(Z) = Q(Z') \in R.$
2. Thus the bottom arrow exist.
3. **Theorem (Coulbois-Hilion-Lustig)**

   The topology on K is completely determined by L.
4. Thus the bottom arrow is continuous.

Partial Isometries

Definition

Action of a, b and c on $\overline{T_\varphi}$ restrict to partial isometries of K.

$L^1(K)$ set of infinite (on the right) possible trajectories.

Definition of Q

$$Q: \begin{array}{ccc} L^1(K) & \rightarrow & K \\ X & \mapsto & \text{source of } X \end{array}$$

Q is continuous.

$L(K)$: possible bi-infinite trajectories.

Theorem (Coulbois-Hilion-Lustig)

$L(K) = L$

General Results

T an $\mathbb{R}$ tree with a very small minimal action of F_N with dense orbits.

[Levitt-Lustig]

$Q : \partial F_N \to \widehat{T}(= \overline{T} \cup \partial T)$, equivariant, continuous for the observers' topology on $\widehat{T}$.

[Coulbois-Hilion-Lustig]

- Dual lamination $L(T) = \{Z \text{ bi-infinite} \mid Z = X^{-1}Y,\ Q(X) = Q(Y)\}$
- $L(T)$ completely determines $\widehat{T}$ (but may be not the metric)

General Results (2)

[Coulbois-Hilion-Lustig]

- Limit set of T for a basis $\mathcal{A}$: $\Omega_{\mathcal{A}} = Q(L(T))$.
- Compact Core $K_{\mathcal{A}}$ of T: convex hull of $\Omega_{\mathcal{A}}$.
- System of Isometries $(K_{\mathcal{A}}, \mathcal{A})$ complety determines T.