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Measure Data and Numerical Schemes for Elliptic Problems

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Dedicated to H. Brezis in the occasion of his 60th birthday

Abstract. In order to show existence of solutions for linear elliptic problems with measure data, a first classical method, due to Stampacchia, is to use a duality argument (and a regularity result for elliptic problems). Another classical method is to pass to the limit on approximate solutions obtained with regular data (converging towards the measure data). A third method is presented. It consists to pass to the limit on approximate solutions obtained with numerical schemes such that Finite Element schemes or Finite Volume schemes. This method also works for convection-diffusion problems which lead to non coercive elliptic problems with measure data. Thanks to a uniqueness result, the convergence of the approximate solutions as the mesh size vanishes is also achieved.

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1. Introduction

The first result of existence and uniqueness of solutions for the Dirichlet problem for a linear elliptic equation (with possibly discontinuous coefficients and) with measure data is probably due to G. Stampacchia in his paper of 1965, see [1]. In this paper, G. Stampacchia use a duality method. A regularity result on a primal problem leads to an existence and uniqueness result on the dual problem. It is interesting to notice that the solution obtained by this method satisfies the equation with a stronger sense than the classical weak sense (such as (2.10) below) as it is shown by the counterexample given in Prignet [2], which is an adaptation of Serrin [3].

In the seventies, H. Brezis studied some semilinear elliptic equations such as:

$$\begin{aligned} -\Delta u + g(u) &= \mu \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

with a nondecreasing function $g \in C(\mathbb{R}, \mathbb{R})$. The case $\mu \in L^1(\Omega)$ is solved in the well-known papers of Brezis-Strauss [4], for the case where Ω is a bounded open subset of $\mathbb{R}^N$ with a smooth boundary, and of Bénilan-Brezis-Crandall [5] for the case $\Omega = \mathbb{R}^N$ (in this latter case, one assumes $g(0) = 0$ and the boundary condition "$u = 0$" has to be changed in a convenient condition). A well-known result of Bénilan-Brezis is devoted to the case of the Thomas-Fermi equation where μ is a measure on Ω, see [6] and the recent paper [7]. In fact, in the case of (1.1), the function g makes very different the cases "$\mu \in L^1(\Omega)$" and "μ measure on Ω". Indeed, if Ω is a bounded open subset of $\mathbb{R}^N$ with a smooth boundary and if $g \in C(\mathbb{R}, \mathbb{R})$ is such that $g(s)s \geq 0$ for all $s \in \mathbb{R}$, then, the problem (1.1) has a unique solution for all $\mu \in L^1(\Omega)$. But, the existence part of this result is not always true if μ is a measure on Ω. For instance, let $p \in]1, \infty[$, $g(s) = |s|^{p-1}s$ and μ be a measure on Ω. Then, (1.1) has a solution if and only if $\mu \in L^1(\Omega) + W^{-2,p}(\Omega)$. This latter condition is equivalent to say that $|\mu|(A) = 0$ for for every borelian subset of Ω whose $W^{2,p'}$-capacity is zero, see Gallouët-Morel [8] and Baras-Pierre [9].

Following the works of H. Brezis, the case of quasilinear equations with the classical Leray-Lions conditions may be studied:

$$-\mathrm{div}(a(\cdot, u, \nabla u)) = \mu \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega. \tag{1.2}$$

Here also, one obtains, for all measure μ on Ω, the existence of a solution to (1.2), see Boccardo-Gallouët [10] and [11].

In order to obtain these existence results (for (1.1) or (1.2)), a classical method is to consider approximate solutions obtained with a sequence of regular functions $(\mu_n)_{n\in\mathbb{N}}$, bounded in $L^1(\Omega)$ and $\star$-weakly converging to μ (with also some approximations of the function g in the case of (1.1)) and then to obtain some estimates on this sequence of approximate solutions and to pass to the limit as $n \to \infty$ (it is for this last step that some difference occurs between "L^1" and "measure" in the case of (1.1)).

In this paper, we will present a third method to obtain existence of solutions for elliptic problems with measure data. It consists to pass to the limit on the solution obtained with a discretization of the equation by a numerical scheme (such as a Finite Element scheme). This method has a double interest since it gives the existence of a solution for the problem considered and it gives a way to compute an approximation of this solution (especially if one has also a uniqueness result). In some cases, it is also possible to have some error estimates. This question of computation of the solution of an elliptic problem with measure data is crucial for some engineering problems. An example is given by the reservoir simulation in petroleum engineering. In this example, measure data have to be considered since the diameter of a well (about 10 cm) is very small with respect to a typical mesh size (about 100 m). It leads to source terms in the equations which are measures supported on some points (for some 2d models) or some lines (for 3d models), see Fabrie-Gallouët [12] for instance.

In Section 2, a model example is considered which is generalized in Section 3.

2. A model example

This section presents a result given in Gallouët-Herbin [13].

Let Ω be a polygonal open subset of $\mathbb{R}^2$ and $\mu \in M_b(\Omega)$, where $M_b(\Omega)$ denotes the set of bounded measures on Ω, that is the set of σ-additives applications from the borelian subsets of Ω to $\mathbb{R}$. An element $\mu \in M_b(\Omega)$ may be considered as an element of $(C(\overline{\Omega}))'$, setting $\mu(\varphi) = \int_\Omega \varphi d\mu$ if $\varphi \in C(\overline{\Omega})$. In the sequel, $C(\overline{\Omega})$ is endowed with its usual "sup-norm" and $\|\mu\|_{M_b}$ denotes the norm of μ in the dual space $(C(\overline{\Omega}))'$. One considers the Dirichlet problem with μ as datum:

$$\begin{aligned} -\Delta u &= \mu \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

In order to prove the existence of a (weak) solution to (2.1), the method developed in [10] considers a sequence $(\mu_n)_{n\in\mathbb{N}}$ of regular functions such that $\mu_n \to \mu$ for the $\star$-weak topology of $C(\overline{\Omega})'$ and the sequence $(u_n)_{n\in\mathbb{N}} \subset H_0^1(\Omega)$ of (weak) solutions of (2.1) with μ_n instead of μ, that is

$$\begin{aligned} -\Delta u_n &= \mu_n \quad \text{in } \Omega, \\ u_n &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.2}$$

The method developed in this paper is to consider a sequence of solutions of a numerical scheme as the mesh size goes to 0. Roughly speaking, it consists to "regularize the operator" (the discretized problem is a linear system in a finite-dimensional space) instead of "regularize the datum".

Let $\mathcal{M}$ be a Finite Element triangular mesh of Ω (see, e.g., Ciarlet [14]). One chooses the piecewise Finite Element approximation of (2.1). One sets $H = \{u \in C(\overline{\Omega});\, u_{|_K} \in P^1$ for all $K \in \mathcal{M}\}$, where P^1 denotes the set of affine functions, and $H_0 = \{u \in H;\, u = 0$ on $\partial\Omega\}$. The Finite Element approximation of (2.1) leads to the following problem:

$$\begin{aligned} & u_{\mathcal{M}} \in H_0, \\ & \int_\Omega \nabla u_{\mathcal{M}} \cdot \nabla v dx = \int_\Omega v d\mu, \ \forall v \in H_0. \end{aligned} \tag{2.3}$$

It is classical that (2.3) has a unique solution. The aim is to proves the convergence of $u_{\mathcal{M}}$ to some u, as the mesh size goes to zero, and that u is the unique solution of (2.1) in a convenient sense. The main difficulty is to obtain some estimates on $u_{\mathcal{M}}$.

In order to obtain these estimates, one recalls the way to obtain some estimates on the solution u_n of (2.2) (the method of [10]). Since $(\mu_n)_{n\in\mathbb{N}} \subset L^1(\Omega)$ and $\mu_n \to \mu$ for the $\star$-weak topology of $C(\overline{\Omega})'$, the sequence $(\mu_n)_{n\in\mathbb{N}}$ is bounded in $L^1(\Omega)$. Indeed, in order to simplify, one may assume that $\|\mu_n\|_{L^1} \le \|\mu\|_{M_b}$ for all n. Then, let $\theta > 1$ and define:

$$\varphi(s) = \int_0^s \frac{1}{(1+|t|)^\theta} dt;\ s \in \mathbb{R}.$$

Taking $\varphi(u_n)$ as test function in the weak formulation of (2.2) (note that $\varphi(u_n) \in H_0^1(\Omega)$) leads to:

$$\int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^\theta} dx \le C_\theta \|\mu\|_{M_b}, \tag{2.4}$$

where $C_\theta = \int_0^\infty \frac{1}{(1+|t|)^\theta} dt < \infty$ (and $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$, for any $d \ge 1$).

Using Hölder Inequality, Sobolev embedding and the fact that θ can be chosen arbitrarily close to 1, one deduces from (2.4) the existence, for all $q < 2$ (if Ω is a bounded open of $\mathbb{R}^d$, $d \ge 2$, the bound on q is $q < \frac{d}{d-1}$), of C_q, only depending on Ω, q and $\|\mu\|_{M_b}$ such that:

$$\int_\Omega |\nabla u_n|^q dx \le C_q.$$

A quite similar method can be used in order to obtain some estimates on the solution $u_\mathcal{M}$ of (2.3). The first difficulty is that $\varphi(u_\mathcal{M})$ does not belong to H_0, then it is not possible to take $v = \varphi(u_\mathcal{M})$ in (2.3). But, we can take for v the interpolate of $\varphi(u_\mathcal{M})$. Indeed, let $\mathcal{V}$ the set of vertices of $\mathcal{M}$ and ϕ_K the Finite Element basis function associated to $K \in \mathcal{V}$ (that is $\phi_K \in H$, $\phi_K(K) = 1$ and $\phi_K(L) = 0$ if $L \in \mathcal{V}$, $L \ne K$). One has, with $u_K = u_\mathcal{M}(K)$ for all $K \in \mathcal{V}$:

$$u_\mathcal{M} = \sum_{K \in \mathcal{V}} u_K \phi_K.$$

Taking $v = \sum_{K \in \mathcal{V}} \varphi(u_K)\phi_K$ in (2.3) leads to:

$$\sum_{(K,L)\in(\mathcal{V})^2} T_{K,L}(u_K - u_L)(\varphi(u_K) - \varphi(u_L)) \le C_\theta \|\mu\|_{M_b}, \tag{2.5}$$

where $T_{K,L} = -\int_\Omega \nabla\phi_K \cdot \nabla\phi_L dx$ and noting that $\sum_{L\in\mathcal{V}} T_{K,L} = 0$, for all $K \in \mathcal{V}$ since $\sum_{L\in\mathcal{V}} \phi_L(x) = 1$ for all $x \in \Omega$.

In order to deduce from (2.5) a $W_0^{1,q}$-estimate on $u_\mathcal{M}$ (for $1 \le q < 2$), an additional hypothesis is assumed. It is supposed that, the mesh $\mathcal{M}$ satisfies, for some positive ζ, the following Delaunay and non degeneracy conditions:

(i) For any interior edge of $\mathcal{M}$, the sum of the angles facing that edge is less or equal to $\pi - \zeta$,
(ii) For any edge lying on the boundary, the facing angle is less or equal to $\frac{\pi}{2} - \zeta$, (2.6)
(iii) For any angle θ of any triangle T of the mesh $\mathcal{M}$, $\theta \ge \zeta$.

Under this hypothesis, it follows from (2.5) the existence, for all $q < 2$, of C_q, only depending on Ω, q, $\|\mu\|_{M_b}$ and ζ such that:

$$\|u_\mathcal{M}\|_{W_0^{1,q}(\Omega)} \le C_q. \tag{2.7}$$

A way to prove (2.7), using (2.5), can be done with similar results using Finite Volume schemes, see Gallouët-Herbin [15] or Droniou-Gallouët-Herbin [16]. Indeed,

$u_{\mathcal{M}} = \sum_{K \in \mathcal{V}} u_K \phi_K$ is solution of (2.3) if and only if the family $(u_K)_{K \in \mathcal{V}}$ is solution of

$$\begin{aligned} &\sum_{L \in \mathcal{V}} T_{K,L}(u_K - u_L) = \int_\Omega \phi_K d\mu, \ \forall K \in \mathcal{V}, \\ &u_K = 0, \ \forall K \in \mathcal{V} \cap \partial\Omega. \end{aligned} \tag{2.8}$$

The left-hand side of the first equation of (2.8) is the same than the left-hand side obtained with the classical Finite Volume scheme on the Voronoï mesh associated to the set $\mathcal{V}$. The control volume (of this Voronoï mesh) associated to $K \in \mathcal{V}$ is the set of points of Ω whose distance to K is less than its distance to any other element of $\mathcal{V}$. Thanks to Condition (2.6), the control volumes of the Voronoï mesh are also defined by the orthogonal bisectors of the edges of $\mathcal{M}$, see Figure 1. The fact that the schemes (Finite Element on $\mathcal{M}$ and Finite Volume on the Voronoï mesh associated to $\mathcal{V}$) differ only by the right-hand sides is due to the following computation for any $T \in \mathcal{M}$:

$$-\int_T \nabla\phi_K \cdot \nabla\phi_L dx = \frac{1}{2}\text{cotan}(\theta_{K,L}),$$

where $\theta_{K,L}$ is the angle of T facing the edge with vertices K and L. Hence, for $K, L \in \mathcal{V}$, $K \neq L$:

$$T_{K,L} = \frac{m_{K,L}}{d_{K,L}},$$

where $m_{K,L}$ denotes the distance between the points intersecting the orthogonal bisectors in each of the triangles with vertices K and L (except for the case $K \in \mathcal{V} \cap \partial\Omega$ and $L \in \mathcal{V} \cap \partial\Omega$ which has no importance), and $d_{K,L}$ denotes the distance between K and L.

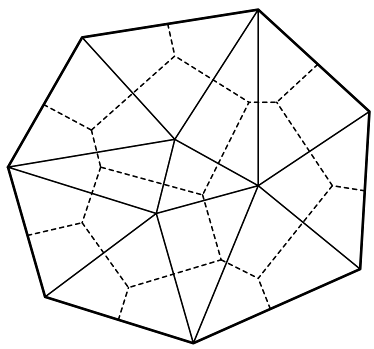


FIGURE 1. Continuous line: Finite Element mesh. Dashed line: Voronoï mesh associated to the vertices of the Finite Element mesh.

It is now possible possible to use the results of [15] (or [16]) which use the Hölder Inequality and a discrete version of the Sobolev embedding. It gives, for

$1 \le q < 2$, the existence for $\overline{C}_q$, only depending on Ω, q and $\|\mu\|_{M_b}$ such that:

$$\sum_{(K,L)\in(\mathcal{V})^2} m_{K,L} d_{K,L} \left(\frac{u_K - u_L}{d_{K,L}}\right)^q \le \overline{C}_q,$$

from which follows (2.7) for some C_q, only depending on Ω, q, $\|\mu\|_{M_b}$ and ζ.

Thanks to these $W_0^{1,q}$-estimates on $u_{\mathcal{M}}$, it is now possible to pass to the limit as size($\mathcal{M}$) goes to zero, where size($\mathcal{M}$) is the supremum of the diameters of the elements of $\mathcal{M}$.

Assuming $u_{\mathcal{M}} \to u$ for the weak topology of $W_0^{1,q}$, for all $1 \le q < 2$, as size($\mathcal{M}$)$\to 0$ (indeed, it is not possible, up to now, to assume such a convergence, one has to consider subsequences of sequences of meshes satisfying (2.6)), let $\psi \in C_c^\infty(\Omega)$ (a regular function with compact support). Taking $v = \psi_{\mathcal{M}} = \sum_{K\in\mathcal{V}} \psi(K)\phi_K$ in (2.3) (this is possible since $\psi_{\mathcal{M}} \in H_0$) gives:

$$\int_\Omega \nabla u_{\mathcal{M}} \cdot \nabla \psi_{\mathcal{M}} dx = \int_\Omega \psi_{\mathcal{M}} d\mu. \tag{2.9}$$

Since $\psi_{\mathcal{M}} \to \psi$, $\nabla\psi_{\mathcal{M}} \to \nabla\psi$ uniformly on Ω and $u_{\mathcal{M}} \to u$ for the weak topology of $W_0^{1,q}$, as size($\mathcal{M}$) $\to 0$, (2.9) gives that u satisfies:

$$\int_\Omega \nabla u \cdot \nabla\psi dx = \int_\Omega \psi d\mu.$$

Then, since $u \in W_0^{1,q}(\Omega)$ for all $1 \le q < 2$ and since $W_0^{1,r}(\Omega) \subset C(\overline{\Omega})$ for all $r > 2$, a density argument gives that u is solution of:

$$\begin{aligned} &u \in \cap_{1\le q<2} W_0^{1,q}(\Omega), \\ &\int_\Omega \nabla u \cdot \nabla\psi dx = \int_\Omega \psi d\mu, \ \forall \psi \in \cup_{r>2} W_0^{1,r}(\Omega). \end{aligned} \tag{2.10}$$

The solution of (2.10) is unique (this is also true for a more general elliptic operator in dimension 2, but not for a general elliptic operator with discontinuous coefficients, in dimension $d \ge 3$, replacing 2 by $\frac{d}{d-1}$ and 2 by d in the two assertions of (2.10), a counterexample is in [2]).

Finally, thanks to this uniqueness result, it is proven that $u_{\mathcal{M}} \to u$ for the weak topology of $W_0^{1,q}$, for all $1 \le q < 2$, as size($\mathcal{M}$) $\to 0$, $\mathcal{M}$ satisfying (2.6) (with a fixed $\zeta > 0$). This gives the following theorem:

Theorem 2.1. *Let Ω be a polygonal open subset of $\mathbb{R}^2$, $\mu \in M_b(\Omega)$ and $\zeta > 0$. For a Finite Element mesh $\mathcal{M}$ of Ω satisfying Condition* (2.6)*, let $u_{\mathcal{M}}$ be the solution of* (2.3)*. Then, $u_{\mathcal{M}} \to u$, unique solution of* (2.10)*, for the weak topology of $W_0^{1,q}(\Omega)$, for all $1 \le q < 2$, as size($\mathcal{M}$) $\to 0$.*

The convergence which is proven in Theorem 2.1 is only a weak convergence in $W_0^{1,q}(\Omega)$ for all $q < 2$. Then, it gives the (strong) convergence in $L^q(\Omega)$ for all $q < \infty$. It is perhaps also possible to prove a strong convergence in $W_0^{1,q}(\Omega)$ for any $q < 2$. In some cases, such that a Dirac measure for μ, it is possible to obtain some error estimates, see Scott [17].

The generalization of this proof of convergence for a Finite Element method in dimension $d = 3$ is not clear. It needs some additional work. In the following section, a generalization is given for a convection-diffusion operator, in dimension $d = 2$ or 3, using a Finite Volume method.

3. Convection-diffusion equations

This section presents a result given in Droniou-Gallouët-Herbin [16] (where more general problems are considered).

Let Ω be a polygonal (for $d = 2$) or polyhedral (for $d = 3$) open subset of $\mathbb{R}^d$ ($d = 2$ or 3). Let $v \in C(\overline{\Omega})^d$ and $\mu \in M_b(\Omega)$, the problem under consideration is:

$$\begin{aligned} -\Delta u + \operatorname{div}(vu) = \mu \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

Such a problem is studied, for instance, in Droniou [18], where an existence and uniqueness result is given using the method of Stampacchia (see [1]), that is a regularity result and a duality argument. The objective, here, is to obtain an existence result, passing to the limit on numerical schemes (and this gives also the convergence of numerical schemes).

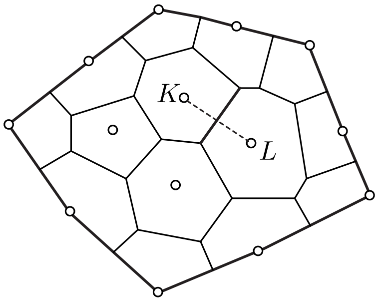
Remark 3.1. For some $v \in (C(\overline{\Omega}))^d$, the problem 3.1 appears to be associated to a noncoercive operator. Let $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ be defined by $Au = -\Delta u + \operatorname{div}(vu)$ for $u \in H_0^1(\Omega)$. Then, it may exist some $u \in H_0^1(\Omega)$, $u \neq 0$, such that $\langle Au, u\rangle_{H^{-1},H_0^1} = 0$, which leads to the noncoercivity of A.

A solution of (3.1) is a function u satisfying (using the fact that $W_0^{1,r}(\Omega) \subset C(\overline{\Omega})$ for $r > d$):

$$\begin{aligned} & u \in \cap_{1 \le q < \frac{d}{d-1}} W_0^{1,q}(\Omega), \\ & \int_\Omega \nabla u \cdot \nabla \psi dx - \int vu \cdot \nabla \psi = \int_\Omega \psi d\mu, \ \forall \psi \in \cup_{r>d} W_0^{1,r}(\Omega). \end{aligned} \tag{3.2}$$

The uniqueness of the solution of (3.2) is quite simple, using a regularity result on the dual problem to (3.2) (see [16] or [18]). In order to prove an existence result, a discretization of (3.1) by a Finite Volume scheme is used.

In [16] a large class of "admissibles" meshes of Ω is considered. Here, in order to simplify, one considers only some particular meshes. Let $\mathcal{T}$ be a mesh of Ω. One assumes that $\mathcal{T}$ is the Voronoï mesh associated to a family $\mathcal{V}$ of points of $\overline{\Omega}$ with the assumption that any point of $\partial\Omega$ belongs to a control volume (or its boundary) associated to an element of $\mathcal{V}$ which is also belonging to $\partial\Omega$ (this is always possible, adding to $\mathcal{V}$ some points on $\partial\Omega$ if necessary). In the sequel, a Voronoï mesh satisfying this property on the points of $\partial\Omega$ will be called a "genuine Voronoï mesh". An example is given in the preceding section. Indeed, the Voronoï mesh associated to the vertices of a Finite Element mesh $\mathcal{M}$ satisfying Condition (2.6) is a genuine Voronoï mesh, see Figure 2. The definition of a Voronoï mesh gives that the element of $\mathcal{T}$ are some open sets. In order to take into account the fact that the measure μ may charge some parts of the edges of $\mathcal{T}$, the elements of $\mathcal{T}$ are slightly modified such that $\mathcal{T}$ is now a (borelian) partition of Ω.



$$T_{K,L} = \frac{m_{K,L}}{d_{K,L}}$$

FIGURE 2. A genuine Voronoï mesh.

Let $K \in \mathcal{V}$. The control volume associated to K is denoted by V_K and $\mu_K = \mu(V_K)$. For $K \in \mathcal{V}$, the set of elements L of $\mathcal{V}$ such that V_K and V_L have a common edge is denoted by $\mathcal{N}_K$. If $L \in \mathcal{N}_K$, the common edge to V_K and V_L is denoted by $\sigma_{K,L}$ and its $(d-1)$-Lebesgue measure is denoted by $m_{K,L}$. The normal unit vector on $\sigma_{K,L}$, outward K, is denoted by $n_{K,L}$ (so that $n_{L,K} = -n_{K,L}$). Furthermore $d_{K,L}$ is the distance between K and L and:

$$T_{K,L} = \frac{m_{K,L}}{d_{K,L}}.$$

The discretization of (3.1) is performed with the classical Finite Volume scheme for the diffusion term and an upwind Finite Volume scheme for the convection term:

$$\begin{aligned}&\sum_{L\in\mathcal{N}_L} T_{K,L}(u_K - u_L) + m_{K,L}v_{K,L}u_{K,L} = \mu_K,\ \forall K \in \mathcal{V}\cap\Omega,\\ &u_K = 0,\ \forall K \in \mathcal{V}\cap\partial\Omega,\end{aligned} \tag{3.3}$$

where $v_{K,L}$ is the mean value of $v \cdot n_{K,L}$ on $\sigma_{K,L}$ and $u_{K,L}$ is equal to u_K or u_L depending on the sign of $v_{K,L}$:

$$\begin{aligned}u_{K,L} &= u_K \quad \text{if } v_{K,L} > 0,\\ u_{K,L} &= u_L \quad \text{if } v_{K,L} < 0.\end{aligned} \tag{3.4}$$

The system (3.3)–(3.4) appears to be a linear system of N unknowns, namely $\{u_K, K \in \mathcal{V}\cap\Omega\}$, and N equations, where N is the number of elements of $\{K \in \mathcal{V}\cap\Omega\}$. Existence and uniqueness of the solution of this system is an easy consequence of the following property of positivity (interesting for its own sake), which is due to the upwind choice of $u_{K,L}$ (that is (3.4)):

$$\left.\begin{array}{l}\{u_K, K \in \mathcal{V}\} \text{ solution of (3.3)–(3.4)}\\ \mu_K \geq 0 \text{ for all } K \in \mathcal{V}\end{array}\right\} \Rightarrow u_K \geq 0 \text{ for all } K \in \mathcal{V}. \tag{3.5}$$

The proof of (3.5) is classical. If M is the matrix which determines the linear system (3.3)–(3.4), after an ordering of the unknowns, the property (3.5) is: $X \in \mathbb{R}^N$, $MX \geq 0 \Rightarrow X \geq 0$, which a consequence of the same property on $M^\star$, namely $X \in \mathbb{R}^N$, $M^\star X \geq 0 \Rightarrow X \geq 0$.

The solution $\{u_K,\ K \in \mathcal{V}\}$ of (3.3)–(3.4) gives an approximate solution of (3.1) $u_\mathcal{V}$ defined by:

$$u_\mathcal{V}(x) = u_K \text{ if } x \in V_K,\ K \in \mathcal{V}. \tag{3.6}$$

The proof that $u_\mathcal{V}$ converges to u, solution of (3.2), as the mesh size goes to 0, is now divided in four steps:

1. Estimates on $u_\mathcal{V}$ for a so-called discrete $W_0^{1,q}$-norm, for $1 \leq q < \frac{d}{d-1}$ (note that $u_\mathcal{V} \notin W_0^{1,q}(\Omega)$ except for some very particular cases !).
2. Relative compactness in $L^q(\Omega)$, for $1 \leq q < \frac{d}{d-2}$, of the family of approximate solutions.
3. Any possible limit of the approximate solutions as the mesh size goes to 0 is belonging to $W_0^{1,q}(\Omega)$ for $1 \leq q < \frac{d}{d-1}$.
4. Any possible limit of the approximate solutions as the mesh size goes to 0 is solution of (3.2).

With this four steps, the uniqueness of the solution of (3.2) gives that $u_\mathcal{V}$ converges to u, solution of (3.2), as the mesh size goes to 0, in $L^q(\Omega)$, for $1 \leq q < \frac{d}{d-2}$.

The main arguments of these four steps are now described.

Step 1. Estimates on $u_\mathcal{V}$. Using the method of Section 2, it is quite easy to obtain some estimates on $u_\mathcal{V}$ in the case where $\mathrm{div}(v) \geq 0$ (which gives some coercivity). But, it is not so easy without this assumption. Indeed, a first step is to control $\mathrm{meas}(\{u_\mathcal{V} \geq k\})$, as $k \to \infty$, uniformly with respect to $\mathcal{V}$. This is possible thanks to an estimate on $\ln(1 + |u_\mathcal{V}|)$. The way to obtain this estimate on $\ln(1 + |u_\mathcal{V}|)$ is described below in the continuous case that for the weak solution $u \in H_0^1(\Omega)$ of (3.1) when $\mu \in H^{-1}(\Omega) \cap L^1(\Omega)$.

Let $\varphi \in C^1(\mathbb{R}, \mathbb{R})$ be the function defined in Section 2 for $\theta = 2$, that is $\varphi(s) = \int_0^s \frac{1}{(1+|s|)^2}$ for $s \in \mathbb{R}$. Taking $\varphi(u)$ as test function in the weak formulation of (3.1) leads to:

$$\begin{aligned} \int_\Omega \frac{|\nabla u|^2}{(1+|u|)^2} dx &\leq C_2 \|\mu\|_{M_b} + \int_\Omega \frac{|v||u||\nabla u|}{(1+|u|)^2} dx \\ &\leq C_2 \|\mu\|_{M_b} + \|v\|_\infty \int_\Omega \frac{|\nabla u|}{1+|u|} dx, \end{aligned} \tag{3.7}$$

with $C_2 = \int_0^\infty \varphi(s) ds = 1$ and $\|v\|_\infty = \sup_{x\in\Omega} |v(x)| < \infty$.

Using Cauchy-Schwarz Inequality, Inequality (3.7) gives a bound on $\nabla \ln(1 + |u|)$ in $L^2(\Omega)$, only depending on v, $\|\mu\|_{M_b}$ and Ω. Then, since $\ln(1+|u|) \in H_0^1(\Omega)$, Poincaré Inequality gives a bound on $\ln(1 + |u|)$ in $L^2(\Omega)$ only depending on v, $\|\mu\|_{M_b}$ and Ω.

A similar estimate holds for $u_\mathcal{V}$, solution of the discretized problem, namely (3.3)–(3.4) and (3.6). The bound on $\ln(1 + |u_\mathcal{V}|)$ in $L^2(\Omega)$ is also only depending on v, $\|\mu\|_{M_b}$ and Ω. The proof of this bound uses the same arguments, with some technical difficulties, and uses the upwind choice of $u_{K,L}$ in (3.4).

The bound on $\ln(1+|u_{\mathcal{V}}|)$ in $L^2(\Omega)$ gives a bound on meas$\{|u_{\mathcal{V}}| \geq k\})$, namely:

$$\text{meas}\{|u_{\mathcal{V}}| \geq k\}) \leq \frac{C}{(\ln(1+k))^2}, \tag{3.8}$$

where C is only depending on v, $\|\mu\|_{M_b}$ and Ω. Using this bound, it is now possible to obtain estimates on the so-called discrete $W_0^{1,q}$-norm of $u_{\mathcal{V}}$ (recall that, generally, $u_{\mathcal{V}} \notin W_0^{1,q}(\Omega)$), for $1 \leq q < \frac{d}{d-1}$. This discrete $W_0^{1,q}$-norm is defined, for $u_{\mathcal{V}}$ satisfying (3.6) and such that $u_K = 0$ if $K \in \mathcal{V} \cap \partial\Omega$, by:

$$\|u_{\mathcal{V}}\|_{1,q,\mathcal{V}}^q = \sum_{(K,L)\in(\mathcal{V})^2} m_{K,L} d_{K,L} \left(\frac{u_K - u_L}{d_{K,L}}\right)^q.$$

A bound on $\|u_{\mathcal{V}}\|_{1,q,\mathcal{V}}$ is obtained, for $1 \leq q < \frac{d}{d-1}$, when $u_{\mathcal{V}}$ is solution of (3.3)–(3.4) and (3.6), using (3.8), the function φ of Section 2, with $\theta > 1$ (close to 1), and the functions T_k and S_k defined by $T_k(s) = \max(-k, \min(s,k))$, $S_k(s) = s - T_k(s)$, for $s \in \mathbb{R}$. It is also used, for proving this estimate on $\|u_{\mathcal{V}}\|_{1,q,\mathcal{V}}$, that, if $L \in \mathcal{N}_K$, the distance from K to $\sigma_{K,L}$ is equal to the distance from L to $\sigma_{K,L}$. The conclusion of this step is that, for $1 \leq q < \frac{d}{d-1}$, there exists C_q, only depending on v, $\|\mu\|_{M_b}$ and Ω, such that:

$$\|u_{\mathcal{V}}\|_{1,q,\mathcal{V}} \leq C_q. \tag{3.9}$$

Step 2. Relative compactness in $L^q(\Omega)$, for $1 \leq q < \frac{d}{d-2}$, of the family of approximate solutions. With the discrete $W_0^{1,q}$-norm and $q < d$, a discrete version of the Sobolev embedding holds. Here also, the fact that, if $L \in \mathcal{N}_K$, the distance from K to $\sigma_{K,L}$ is equal to the distance from L to $\sigma_{K,L}$ is used. There exists S_q, only depending on q, such that, if $u_{\mathcal{V}}$ is defined by (3.6) and $u_K = 0$ for $K \in \mathcal{V} \cap \partial\Omega$:

$$\|u_{\mathcal{V}}\|_{L^{q^\star}(\Omega)} \leq S_q \|u_{\mathcal{V}}\|_{1,q,\mathcal{V}}, \tag{3.10}$$

where $q^\star = \frac{qd}{d-q}$.

Then, if $u_{\mathcal{V}}$ is the solution of (3.3)–(3.4) and (3.6), Estimate (3.9) (where $1 \leq q < \frac{d}{d-1}$) leads, with (3.10), to an estimate on $u_{\mathcal{V}}$ in $L^r(\Omega)$ for $1 \leq r < \frac{d}{d-2}$. This estimate gives the relative weak-compactness in $L^r(\Omega)$ of the family of approximate solutions (that is the family of $u_{\mathcal{V}}$, solution of (3.3)–(3.4) and (3.6), as $\mathcal{V}$ describes all the possible sets of points of Ω leading to a genuine Voronoï mesh). In order to obtain the relative (strong-)compactness of the family of approximate solutions, an equivalent to the Rellich theorem, using the norm $\|\cdot\|_{1,q,\mathcal{V}}$ instead of the $W_0^{1,q}$-norm, is needed. This compactness theorem is, thanks to the Kolmogorov compactness theorem, a consequence of the following inequality, which holds for $q \leq 2$, $h \in \mathbb{R}^d$ and any $u_{\mathcal{V}}$ defined by (3.6) and such that $u_K = 0$ for $K \in \mathcal{V} \cap \partial\Omega$:

$$\int_{\mathbb{R}^d} |u_{\mathcal{V}}(x+h) - u_{\mathcal{V}}(x)|^q \leq |h|(|h| + C\text{size}(\mathcal{V}))^{q-1} \|u_{\mathcal{V}}\|_{1,q,\mathcal{V}}, \tag{3.11}$$

where C is only depending on Ω, size$(\mathcal{V})$ is the supremum of the diameters of the elements of the Voronoï mesh associated to $\mathcal{V}$, and $u_{\mathcal{V}}$ is defined outside Ω by setting $u_{\mathcal{V}}(x) = 0$ if $x \notin \Omega$.

Estimate (3.9) (where $1 \le q < \frac{d}{d-1}$) gives, with (3.10) and (3.11), the relative compactness of the family of approximate solutions in $L^q(\Omega)$ for $1 \le q < 2$, thanks to the Kolmogorov compactness theorem. Then, using the estimate on $u_{\mathcal{V}}$ in $L^r(\Omega)$ for $1 \le r < \frac{d}{d-2}$, the relative compactness of the family of approximate solutions is obtained in $L^q(\Omega)$ for $1 \le q < \frac{d}{d-2}$.

Step 3. Let $u_{\mathcal{V}}$ be the solution of (3.3)–(3.4) and (3.6). Assuming that $u_{\mathcal{V}}$ converges to some u in $L^q(\Omega)$, for all $1 \le q < \frac{d}{d-2}$, as size$(\mathcal{V}) \to 0$, the fact that $u \in W_0^{1,q}(\Omega)$ for all $1 \le q < \frac{d}{d-1}$ is a consequence of Estimate (3.9) and (3.11). Indeed, for $h \in \mathbb{R}^d$, $h \neq 0$, (3.11) gives with (3.9) (recall that $u_{\mathcal{V}}$ is defined outside Ω by setting $u_{\mathcal{V}}(x) = 0$ if $x \notin \Omega$):

$$\int_{\mathbb{R}^d} \frac{|u_{\mathcal{V}}(x+h) - u_{\mathcal{V}}(x)|^q}{|h|^q} \le \frac{|h|(|h| + C\mathrm{size}(\mathcal{V}))^{q-1}}{|h|^q} C_q,$$

which leads, for $1 \le q < \frac{d}{d-1}$, passing to the limit as size$(\mathcal{V}) \to 0$:

$$\int_{\mathbb{R}^d} \frac{|u(x+h) - u(x)|^q}{|h|^q} \le C_q, \tag{3.12}$$

where, here also, u is defined outside Ω by setting $u(x) = 0$ if $x \notin \Omega$. Inequality (3.12) gives $\nabla u \in L^q(\mathbb{R}^d)$ and therefore, since $u = 0$ outside Ω, $u \in W_0^{1,q}(\Omega)$.

Step 4. The proof of this step is easier (at least for a regular v). Indeed, let $u_{\mathcal{V}}$ be the solution of (3.3)–(3.4) and (3.6). Assuming that $u_{\mathcal{V}}$ converges to some u in $L^q(\Omega)$, for all $1 \le q < \frac{d}{d-2}$, as size$(\mathcal{V}) \to 0$, the preceding step gives that $u \in W_0^{1,q}(\Omega)$ for all $1 \le q < \frac{d}{d-1}$. Taking $\psi \in C_0^\infty(\Omega)$, (3.2) is proven, passing to the limit on the numerical scheme (3.3)–(3.4). Then, a density argument gives (3.2) for all $\psi \in \cup_{r>d} W_0^{1,r}(\Omega)$ and this concludes Step 4.

As usual, the steps 3 and 4 hold for "subsequences of sequences of approximate solutions" and it is the uniqueness of the solution of (3.2) which gives, finally, the convergence of all the family, that is the convergence of $u_{\mathcal{V}}$ to u, unique solution of (3.2), in $L^q(\Omega)$, for all $1 \le q < \frac{d}{d-2}$, as size$(\mathcal{V}) \to 0$. Then, the conclusion of this proof is the following theorem:

Theorem 3.2. *Let Ω be a polygonal (for $d = 2$) or polyhedral (for $d = 3$) open subset of $\mathbb{R}^d$ ($d = 2$ or 3). Let $v \in C(\overline{\Omega})^d$ and $\mu \in M_b(\Omega)$. For a genuine Voronoï mesh associated to a set $\mathcal{V}$ of points of Ω, let $u_{\mathcal{V}}$ be the solution of* (3.3)–(3.4) *and* (3.6). *Then, $u_{\mathcal{V}}$ converges to u, unique solution of* (3.2), *in $L^q(\Omega)$, for all $1 \le q < \frac{d}{d-2}$, as size$(\mathcal{V}) \to 0$.*

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