Indices d'anisotropie pour l'analyse de textures Browniennes d'images

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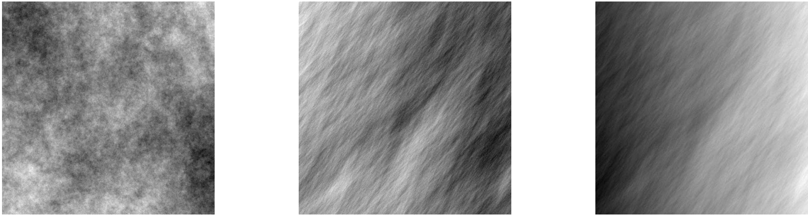
Context and motivations



- Texture : image aspect,
- considered as an effect of image irregularity,
- Texture $=$ high-frequency phenomemon
  $\neq$ low-frequency phenomena (trend).
- Properties of interest : isotropy/anisotropy, homogeneity/heterogeneity.
- Applications : Material Science, Medicine, Arts, etc.

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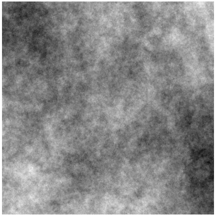 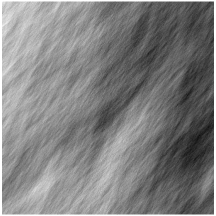 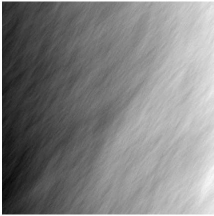

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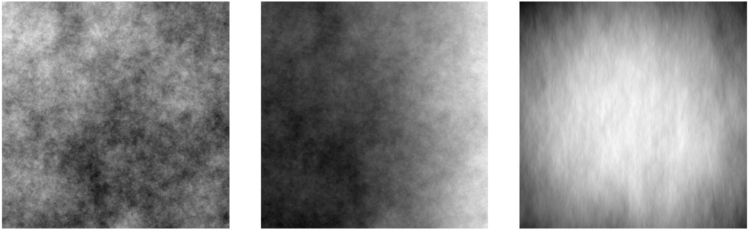

- Statistical analysis of texture anisotropy,
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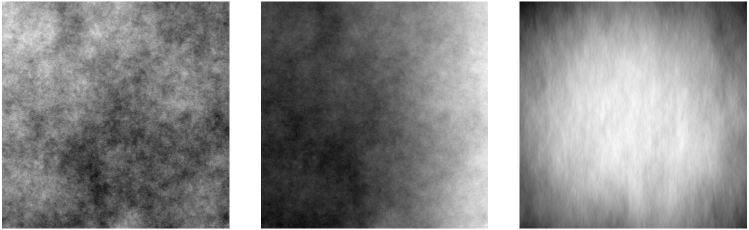

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Outline

- Probabilistic models.
    - Non-stationarity,
    - Irregularity,
    - Anisotropy.
- Statistical Analysis.
- Application to mammograms.

Modeling requirements

- Random field : $\{Z(x), x \in \mathbb{R}^d\}$,
- Required properties :
  - Non-stationarity due to the presence of trends,
  - Irregularity (to be defined),
  - Anisotropy (also to be defined),
  - Heterogeneity (work in progress).

Fields with stationary increments

- Given two positions $x_1, x_2 \in \mathbb{R}^2$, $V_x = Z(x_1) - Z(x_2)$ is a field increment, and

$$\left\{ V_x(y) = Z(x_1 + y) - Z(x_2 + y), y \in \mathbb{R}^2 \right\}$$

  an increment field.

- A field Z has stationary increments if, for any couple of positions $x = (x_1, x_2)$, the increment field $V_x(\cdot)$ is stationary, i.e. for any y and z
  - $\mathbb{E}(V_x(y)) = a$,
  - $\mathbb{E}(V_x(y)V_x(z)) = K_x(y - z)$.

- If Z is square integrable with stationary increments, then
  - $\mathbb{E}(Z(x)) = \langle x, a \rangle + m$,
  - $\mathrm{Cov}(Z(x + h), Z(x)) \leq C|h|$.

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Intrinsic random fields

- An increment $V_x = Z(x_1) - Z(x_2)$ annihilates constants.
- *M*-increment : $Z_{\lambda,x} = \sum_{i=1}^{m} \lambda_i Z(x_i)$

$$\sum_{i=1}^{m} \lambda_i P(x_i) = 0, \forall P, \text{polynomial } d^o P \leq M$$

M-IRF: fields with zero-mean stationary *M*-increment fields, *i.e.* fields $V_{\lambda,x}(y) = \sum_{i=1}^{m} \lambda_i Z(x_i + y)$ satisfy

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Correlation structure of an IRF

- Continuous M-**IRFs** Z are characterized by generalized covariances C that are M-conditionally positive-definite,i.e.

$$\mathbb{E}(Z_{\lambda,x} Z_{\mu,y}) = \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_i \, \mu_j C(x_i - y_j) \geq 0,$$

holds for any M-increments $Z_{\lambda,x}$ and $Z_{\mu,y}$.

- Spectral representation of generalized covariances [Ref. Gelfand & Villenkin, 1964; Matheron 1973].

$$C(h) = \int_{\mathbb{R}^2} \left(\cos(\langle w, h\rangle) - \mathbf{1}_{B(0,\epsilon)}(w) P_M(\langle w, h\rangle)\right) f(w) dw,$$

with $P_M(t) = 1 - \frac{t^2}{2} + \cdots + \frac{(-1)^M}{(2M)!} t^{2M}$, and $\forall \epsilon > 0$,

$$\int_{|w|<\epsilon} |w|^{2M+2} f(w) dw < \infty, \text{ and } \int_{|w|>\epsilon} f(w) dw < \infty.$$

Field irregularity

- $H \in (0,1)$: critical Hölder exponent of a field Z if, on a compact set C and for a positive random variable A

$$|Z(x) - Z(y)| \leq A|x-y|^{\alpha}$$

holds with probability one for any $\alpha < H$, but not for $\alpha > H$.

- Characterization through spectral density high-frequencies [Ref. Bonami and Estrade, 2004; Biermé, 2005].
- A sufficient condition:

$$f(w) = \underset{|w|\to+\infty}{O}(|w|^{-2H-d})$$

and the function $\tau^*(s) = \lim_{\rho\to+\infty} f(s\rho)\rho^{2H+d}$ defined on the unit sphere S is non null on a set of positive measure.

A classical example

- Anisotropic fractional Brownian fields : zero mean 0-IRF with a spectral density of the form

$$f(w) = \tau\left(\frac{w}{|w|}\right)|w|^{-2\beta\left(\frac{w}{|w|}\right)-d}.$$

$(\beta(s) \in (0,1),\ \tau(s) \geq 0)$.

- Hurst function β and topothesy function τ: directional functions characterizing the field anisotropy.
- Holder-irregularity of order $H = \min_{s \in S} \beta(s)$,
- Model without trend.

[Ref. Bonami and Estrade, 2004].

An extended framework

- M-IRF with a spectral density f satisfying

$$|w| > A \Rightarrow 0 \leq f(w) - \tau\left(\frac{w}{|w|}\right)|w|^{-2\beta\left(\frac{w}{|w|}\right)-d} \leq C|w|^{-2H-d-\gamma}$$

  for some $H \in (0,1)$, $A, C, \gamma > 0$.
- M-IRF with Hölder exponent H,
- at low frequency, may have a polynomial trend of order M,
- at high frequency, have same properties as an anisotropic fractional Brownian field.

[Ref. Richard, 2015, 16]

Increments

- Z observed on a lattice: $Z^N[m] = Z(m/N), m \in [\![1, N]\!]^d$.
- K-increments on the lattice:
  $\forall m \in \mathbb{Z}^d, V^N[m] = \sum_{k \in [\![0,L]\!]^d} v[k] Z^N[m-k]$, with an appropriate convolution kernel v.
  [Ref. Chan and Wood 2000, Richard and Bierme 2011, etc.].
- Oriented K-increments

$$\forall\, m \in \mathbb{Z}^d,\ V_u^N[m] = \sum_k v[k] Z^N[m - T_u k],$$

with a transform

$$T_u = |u|^2 \begin{pmatrix} \cos(\arg(u)) & -\sin(\arg(u)) \\ \sin(\arg(u)) & \cos(\arg(u)) \end{pmatrix},$$

[Ref. Richard, 2015, 16]

Asymptotic normality

- Quadratic variations: $W_u^N = \frac{1}{N_e} \sum_m (V_u^N[m])^2$.
- Normalized log-variation vector: $Y^N = (\log(W_{u_k}^N) - \sum_{m \in \mathcal{I}} \lambda_m \log(W_{u_m}^N))_k$.

Theorem [Ref. Richard 2015, 16]:

$$Y_k^N = H \ x_k + \beta(\arg(u_k)) + \epsilon_k^N,$$

where x_k are normalized log-scales and

$$\beta(\arg(u_k)) = \mathcal{C}(\arg(u_k)) - \sum_{m \in \mathcal{I}} \lambda_m \, \mathcal{C}(\arg(u_m)),$$

with

$$\mathcal{C}(\theta) = \log\left(\frac{1}{(2\pi)^2} \int_E \tau(\varphi) \int_{\mathbb{R}^+} \left| \hat{v}\,(\rho \vec{u}(\varphi - \theta)) \right|^2 \rho^{-2H-1} d\rho d\varphi \right),$$

$(N\epsilon_k^N)_{k \in \mathcal{I}}$ is asymptotically Gaussian.

Anisotropy index

- Proposition: $\mathcal{C}$ is constant iff the field is isotropic (τ and β constant).
- Anisotropy index :

$$A_2 = \frac{1}{\pi}\int_0^{\pi}\left(\mathcal{C}(\theta) - \frac{1}{\pi}\int_0^{\pi}\mathcal{C}(\varphi)d\varphi\right)^2 d\theta.$$

- Properties:
  - A_2 vanishes iff the field is isotropic,
  - A_2 is invariant to rotation, rescaling of image and linear transforms of its intensities,
  - A_2 is independent of the choice of the kernel v within a class of mono-directional kernels,
  - closed-form expression for some elementary anisotropic fields.

Estimation

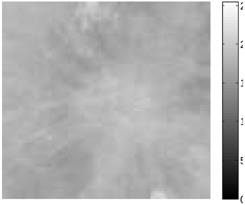
Anisotropy index A_2 estimated with

$$\widehat{A_2} = \sqrt{\sum_m \lambda_m \hat{\beta}^2(\arg(u_m))},$$

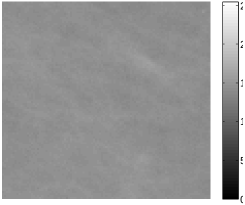
- with some weigths λ_m which are suitable for integral approximation,
- and estimates $\hat{\beta}$ of β obtained by linear regression in the model

$$Y_m^N = H \ x_m + \beta(\arg(u_m)) + \epsilon_m^N.$$

Context



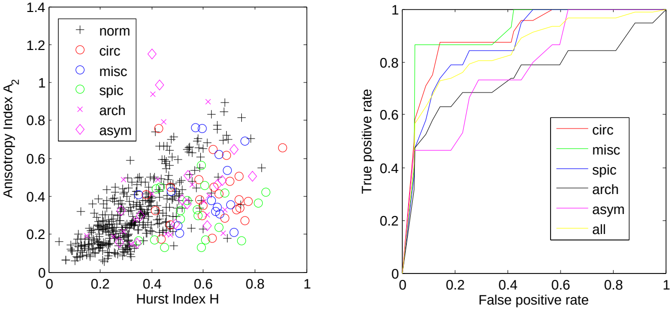
stellate lesion
(isotropic, irregular)



normal
(isotropic, irregular)

normal
(anisotropic, smoother)

Results



Lesion type	circ	misc	spic	arch	asym	all
Nb	19	19	24	15	15	92
AUC	0.897	0.916	0.869	0.743	0.776	0.843

Outline of the approach [PhD thesis of Huong Vu]

- Fields with a non-homogeneous density satisfying

$$0 \leq f_y(w) - \tau_y\left(\frac{w}{|w|}\right)|w|^{-2\beta_y\left(\frac{w}{|w|}\right)-d} \leq C_y|w|^{-2H_y-d-\gamma}$$

- Increments centered at position y :

$$\forall\, m \in \mathbb{Z}^d,\ V_{y,u}^N[m] = \sum_k v[k] Z^N[m - T_u k - p_y],$$

- Localized variations:

$$W_{y,u}^N = \frac{1}{N_e}\sum_{m\in\mathcal{V}_N}(V_u^N[m])^2.$$

- Theorem [Vu, Richard,16]: under mild assumptions,

$$Y_{y,k}^N = \log(W_{y,u_k}^N) = H_y\ x_k + \beta(y, \arg(u_k)) + \epsilon_{y,k}^N,$$

$(N\epsilon_{y,k}^N)_{y,k}$ is asymptotically Gaussian.

Conclusion

- In brief: analysis of texture anisotropy with a generic texture model allowing the presence of trends.
- A limitation:
  - Anisotropic index is dependent on the Hurst index.
- Work in progress:
  - Estimation of topothesy function,
  - Analysis of heterogeneity.

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References

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