Inférence statistique des valeurs extrêmes spatiales

Modèle asymptotique classique ou modèle tendant vers l'indépendance asymptotique?

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Why do we need statistics of spatial extremes ?

- extreme events
  - **rare by definition, but often high impact**
  - physical dynamics and stochastic behavior often different under stress
  - spatial processes : take into account **spatial dependence**

  $\leadsto$ use models with a sound foundation in **extreme value theory**, avoid plain Gaussian models appropriate only for central tendencies
- emphasis is on the tail of the distribution $\leadsto$ **extrapolation**
- **objective : model**, **quantify, predict** extreme risks

Goals of this talk

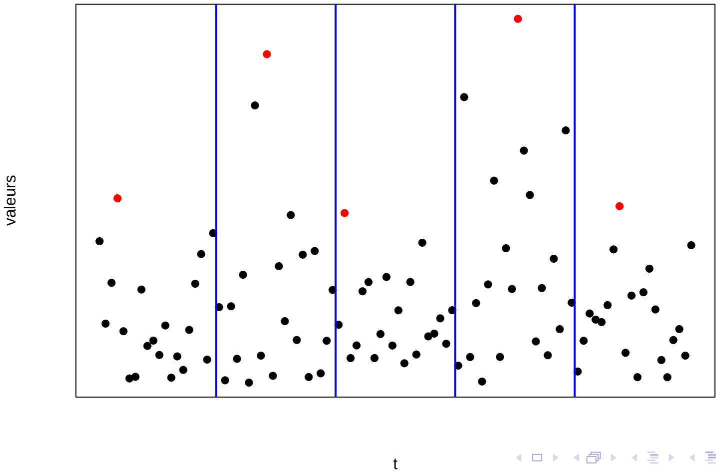
1. present asymptotic theory for **spatial extremes** and the resulting **asymptotic models**, well suited to capture **asymptotic dependence**

2. show that many real data have a tendency towards **asymptotic independence**

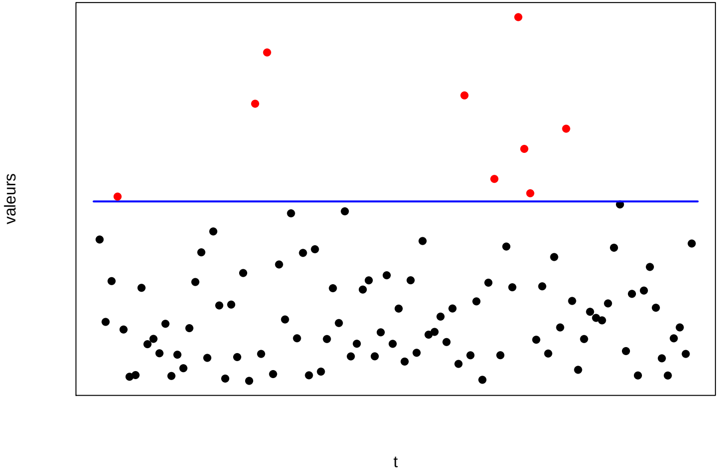
3. review some models suitable for **asymptotically independent data**

What observations are "extreme" ? – Maxima

We obtain a sample of maxima by considering block maxima over blocks of size n.



What observations are "extreme" ? – Threshold exceedances



Max-stable limit processes for pointwise maxima

Let be given i.i.d. copies $\mathbf{X_i}$ of a stochastic process $\mathbf{X} = \{X(s)\}$.

Maximum domain of attraction

If sequences $\sigma_n(s) > 0$, $\mu_n(s)$ exist such that

$$\left\{ \max_{i=1,\ldots,n} \sigma_n(s)^{-1} \left[X_i(s) - \mu_n(s) \right] \right\} \overset{fdd}{\to} \{Z(s)\}, \quad n \to \infty \tag{1}$$

with non-degenerate limit process $\boldsymbol{Z} = \{Z(s)\}$,
then $\boldsymbol{Z}$ is a max-stable process.
We say that $\boldsymbol{X}$ is in the max-domain of attraction of $\boldsymbol{Z}$.

Max-stability of $\boldsymbol{Z}$ means that (1) holds exactly for $\boldsymbol{Z}$.

Univariate limit distributions

The univariate limit distributions are **extreme-value** distributions $Z(s) \sim \mathrm{EV}_{\xi(s)}$.

We can choose $\sigma_n(s), \mu_n(s)$ such that

$$\mathrm{EV}_{\xi(s)}(z) = \exp\left(-T_{\xi(s)}(z)\right), \quad \mathrm{EV}_{\xi(s)}(z) = \exp\left(-T_{\xi(s)}(z)\right), \quad T_{\xi(s)}(z) = (1+\xi(s)z)_+^{-1/\xi(s)}$$

with shape parameter $\xi(s) \in \mathbb{R}$ that determines the tail decay rate

Tail dependence summaries

Multivariate extremal coefficients

Th extremal coefficient $\theta \in [1, D]$ of max-stable random vector $\boldsymbol{Z} = (Z_1, \ldots, Z_D)$ with $Z_j \overset{d}{=} Z_1$ is defined through

$$\mathrm{P}(Z_1 \leq x, \ldots, Z_D \leq x) = \mathrm{P}(Z_1 \leq x)^{\theta}. \quad (2)$$

Interpretation : D/θ is average "cluster size" of extreme events.

For stochastic processes, we can consider the **extremal coefficient function** $\theta(s_1, s_2) \in [1, 2]$ of bivariate extremal coefficients.

The **tail correlation function** $\lambda(s_1, s_2) = 2 - \theta(s_1, s_2) \in [0, 1]$ is a particular type of correlation function.

Asymptotic dependence and asymptotic independence

Asymptotic independence of $X(s_1)$ and $X(s_2)$ corresponds to **independence of** $Z(s_1)$ **and** $Z(s_2)$, *i.e.*, to $\theta(s_1, s_2) = 2$ and $\lambda(s_1, s_2) = 0$.

Otherwise ($\lambda > 0$), we observe **asymptotic dependence**.

More generally (even if a max-stable limit does not exist),

for standardized variables
$\boldsymbol{X}^\star = (X_1^\star, X_2^\star) = (1/(1 - F_{X(s_1)}(X(s_1))), 1/(1 - F_{X(s_2)}(X(s_2)))),$

we can define the tail correlation coefficient $\lambda(s_1, s_2)$ as

$$\lambda(s_1, s_2) = \lim_{x \to \infty} \mathrm{P}(X_1^\star > x \mid X_2^\star > x).$$

In spatial data, asymptotic independence means that the most extreme events become **more and more isolated in space**.

Note : Gaussian processes are **asymptotically independent** [Sibuya, 1960].

The spectral construction of max-stable processes

[de Haan, 1984]

A max-stable process $\boldsymbol{Z}^\star$ can be represented through a **spectral construction**

$$Z^\star(s) = \max_{i=1,2,\dots} \varepsilon_i(s)/U_i, \quad U_i \sim \mathrm{PPP}(\mathrm{d}u) \text{ on } [0,\infty),$$

- here : unit Fréchet marginal distribution, $\mathrm{pr}(Z^\star(s) \leq z) = \exp(-1/z)$, $z > 0$
- **profile processes** $\varepsilon_i(s)$, $i = 1, 2, \dots$ are i.i.d. with $\mathbb{E}\,\varepsilon_i(s)_+ = 1$

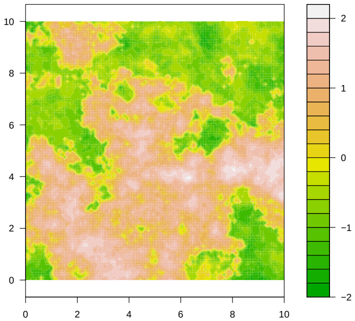
The spectral construction can be used to **construct spatial max-stable models** :

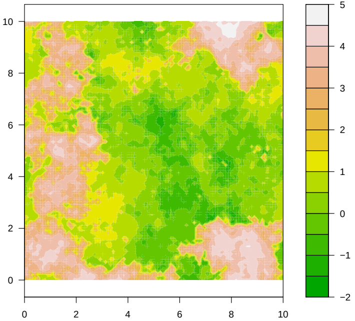
- centered Gaussian profile process : $\varepsilon_1(s) = W(s)_+^{\mathrm{df}}$, $\mathrm{df} > 0$
  $\rightsquigarrow$ **extremal-t process** $Z^\star(s)$ [Opitz, 2013, Thibaud and Opitz, 2016]
  $\mathrm{df} = 1$: **Schlather process** [Schlather, 2002]
- log-Gaussian profile process :
  $\varepsilon_1(s) = \exp(W(s) - \sigma^2(s)/2)$ where $\sigma^2(s) = \mathrm{Var}(W(s))$
  $\rightsquigarrow$ **Brown–Resnick type process** $Z^*(s)$ [Kabluchko et al., 2009]

Example 1 : Schlather process

[Schlather, 2002]

- extremal-t process with $\mathrm{df} = 1$
- simulation on $[0, 10] \times [0, 10]$
- exponential correlation function with range 3 in Gaussian profile process $W(s)_+^{\mathrm{df}}$
- plot of $\log(Z^\star(s))$

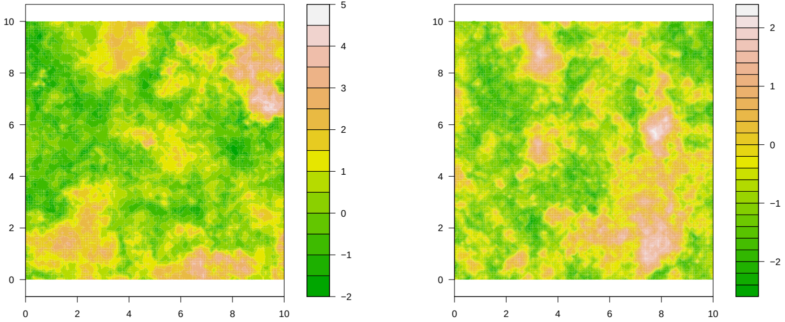




Example 2 : Extremal-t

[Opitz, 2013]

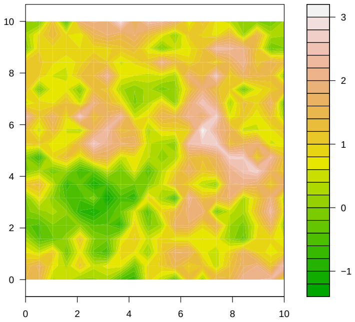
- extremal-t process with $\mathrm{df} = 4$
- simulation on $[0, 10] \times [0, 10]$
- exponential correlation function with range 3 in profile process $W(s)_+^{\mathrm{df}}$
- plot of $\log(Z^\star(s))$

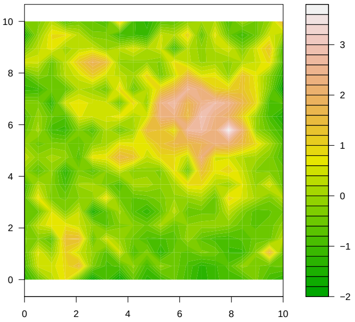


Example 3 : Brown–Resnick process

[Brown and Resnick, 1977, Kabluchko et al., 2009]

- simulation on $[0, 10] \times [0, 10]$
- profile process $\exp(W(s) - \sigma^2(s)/2)$ with fractional Brownian motion $W(s)$ (Hurst index 1/4)
- plot of $\log(Z^\star(s))$





Statistical inference

Idea : use max-stable limit distributions for observed maxima

- **cdf** of max-stable vector is

$$G(\boldsymbol{z}) = \exp\left(-V(T_{\xi_1}^{-1}(z_1), \ldots, T_{\xi_D}^{-1}(z_D)\right),$$

  where $V(\boldsymbol{z})$ with $V(t\boldsymbol{z}) = t^{-1}V(\boldsymbol{z})$ is the **exponent function**
- full likelihood inference impossible in high dimension since combinatorial explosion of terms when deriving cdf
- common likelihood-based inference through **pairwise likelihood approach**
- in **spatial modeling**, we need :
  - a marginal model for $\xi(s), \sigma(s), \mu(s)$
  - a dependence model (Schlather, Brown–Resnick, extremal-t, ...)
- **simulation** of max-stable processes is based on the spectral construction, conditional simulation is tricky but possible
- **extrapolation :**
  - if $\{Z(s)\} = \left\{\mu(s) + \sigma(s)T_{\xi(s)}^{-1}(Z^\star(s))\right\}$ models annual maxima, then $\left\{\mu(s) + \sigma(s)T_{\xi(s)}^{-1}(nZ^\star(s))\right\}$ models n-year maxima
  - often Monte–Carlo based calculation of complicated functionals $f(\boldsymbol{Z})$

From maxima to threshold exceedances

- **difficulties with maxima-based modeling**
  - pointwise maximum process $\boldsymbol{M}_n$ can be composed of components of different events
    $\rightsquigarrow$ not straightforward to derive results for individual extreme events
  - cannot capture non-i.i.d. behavior within blocks
- **equivalent limit relations exist for threshold exceedances**
  - in practice, can fix the **threshold value to balance bias and variance in estimation**
  - however, there is **no natural ordering relation for multivariate and spatial data**, making the definition of "extreme events" ambiguous

Genereralized Pareto limits for threshold exceedances

- the univariate max-domain of attraction condition holds **iff** threshold exceedances converge to a **generalized Pareto distribution** :

$$\sigma_u(s)^{-1}(X(s) - \mu_u(s)) \mid (X(s) > u) \to Y(s) \sim \mathrm{GP}_{\xi(s)}, \quad u \uparrow F_{X(s)}^{-1}(1^-)$$

  where $\mathrm{GP}_{\xi(s)}(y) = 1 - T_{\xi(s)}(y)/T_{\xi(s)}(u)$ with $T_{\xi(s)}(y) = (1+\xi(s)y)_+^{-1/\xi(s)}$

- for characterizing dependence and for estimating models, it is useful to **treat separately marginal and dependence behavior**
  - standardize margins : $X^\star(s) = 1/(1 - F_{X(s)}(X(s)))$
  - $X^\star(s)$ is standard Pareto distributed if $X(s)$ has continuous distribution
- define **extreme events** as **exceedances of a homogeneous risk functional** ℓ
  - $\boldsymbol{X}^\star$ is an extreme event if $\ell(\boldsymbol{X}^\star) > u$ with treshold $u > 0$
  - need **homogeneity** ($\ell(t\boldsymbol{x}) = t\ell(\boldsymbol{x})$) and **positivity** ($\ell(\boldsymbol{x}) > 0$ if $\boldsymbol{x} > \boldsymbol{0}$) for convergence
  - for instance, $\ell(\boldsymbol{x})$ given as $x(s)$, $\max_{s\in K} x(s)$, $\min_{s\in K} x(s)$ or $\mathrm{mean}_{s\in K} x(s_i)$

Limit processes for ℓ-exceedances

[Ferreira and De Haan, 2014, Dombry and Ribatet, 2015, Thibaud and Opitz, 2016]
If $\boldsymbol{X}$ is a the max-domain of attraction, we get

$$u^{-1}\boldsymbol{X}^{\star} \mid (\ell(\boldsymbol{X}^{\star}) > u) \to \boldsymbol{Y}^{\star} \sim \mathrm{GP}_{\ell}, \quad u \to \infty, \tag{3}$$

with an ℓ-Pareto process $\boldsymbol{Y}^{\star}$.

- if (3) holds for $\ell(\boldsymbol{x}) = \max_s x(s)$, the max-domain of attraction condition is satisfied.
- if $\ell(\boldsymbol{x}) = \max(x(s_1), \ldots, x(s_D))$, then $\mathrm{pr}(Y^{\star}(\boldsymbol{s}) \leq \boldsymbol{y}) = 1 - V(\boldsymbol{y})/V(\boldsymbol{u})$ for $\boldsymbol{y} > \boldsymbol{u}$
- we can retransform to the original scale of data,

  $$Y(s) = \mu(s) + \sigma(s)(Y^{\star}(s)^{\xi} - 1)/\xi,$$

  yielding a **generalized ℓ-Pareto process**

Censored likelihood inference for threshold exceedances

Assume data $\boldsymbol{x}_i = x_i(s_j)$, $i = 1, \ldots, n$ have been observed on sites $s_1, \ldots, s_D$.

Here we consider $\ell(\boldsymbol{x}) = \max(x^\star(s_1), \ldots, x^\star(s_D))$.

- using the ℓ-Pareto model amounts to $\mathrm{pr}(\boldsymbol{X}^* \not\leq \boldsymbol{x}^\star) = V(\boldsymbol{x}^\star)$ for $\boldsymbol{x}^\star > (u, \ldots, u)$
- non-extreme components $x_i^\star(s_j) < u$ are **censored**
- likelihood contribution of $\boldsymbol{x}_i^\star$:
  - when none of the components exceeds its threshold : $1 - V(u, \ldots, u)$
  - when w.l.o.g. components $x_i^\star(s_1)$, ..., $x_i^\star(s_{j_0})$ are exceedances :

$$-\frac{\partial^{j_0}}{\partial x_1 \times \ldots \times \partial x_{j_0}} V(x_i^\star(s_1), \ldots, x_i^\star(s_{j_0}), u, \ldots, u)$$

Modeling precipitation extremes around Zurich

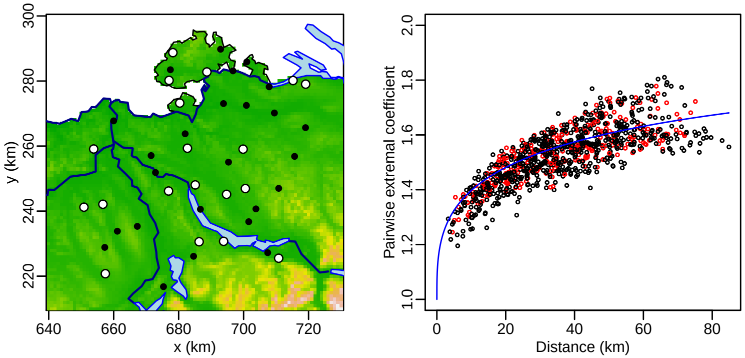
[Thibaud and Opitz, 2016]

- daily summer precipitation data (1962-2012) for 44 sites (25 for estimation, 19 for validation)
- ℓ-**Pareto model** with $\ell(\boldsymbol{x}) = \max(x(s_1), \ldots, x(s_D))$
  - marginal model :
    $\sigma(s) = \beta_{\sigma,0} + \beta_{\sigma,1}\mathtt{LAT} + \beta_{\sigma,2}\mathtt{LON} + \beta_{\sigma,3}\mathtt{ALT}$
    $\mu(s) = \beta_{\mu,0} + \beta_{\mu,1}\mathtt{LAT} + \beta_{\mu,2}\mathtt{LON} + \beta_{\mu,3}\mathtt{ALT}$
    $\xi(s) \equiv \xi_0$
  - extremal-t dependence with stable correlation function
- estimation :
  - threshold $u = 20$ (marginal 95%-quantile)
  - **two-step estimation :**
    independence likelihood for marginal parameters,
    full likelihood on standardized data for dependence parameters
  - $\hat{\mathrm{df}} = 6$
  - $\mathrm{Cor}(s_1, s_2) = \exp\left[-(\|h\|/\hat{\beta})^{\hat{\kappa}}\right]$ with $\hat{\beta} = 483(35)$ km, $\hat{\kappa} = 0.64(0.01)$

Goodness-of-fit of the extremal coefficient function

- good coverage of confidence intervals for conditional distributions on validation sites
- AIC-based model selection : extremal-t outperforms Brown–Resnick

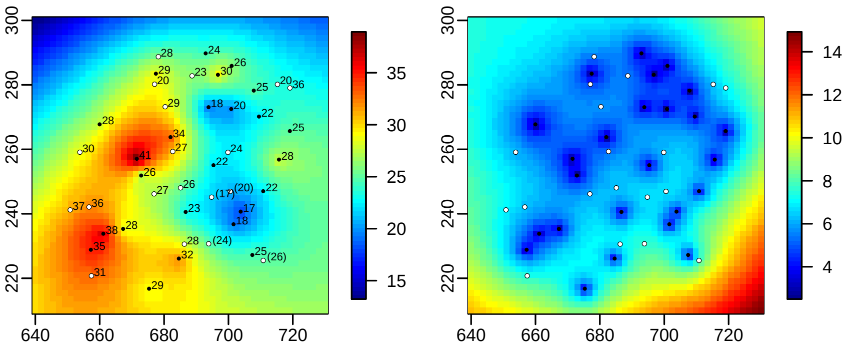
Extremal coefficient function : empirical vs. fitted



Spatial prediction with the fitted model (1 June 1962)

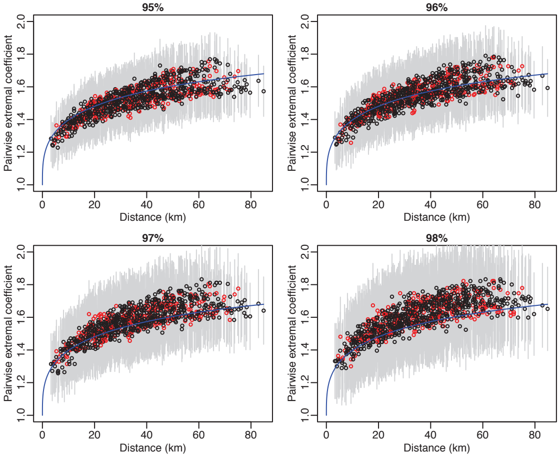
- condition on values at observed sites
- conditional process has (transformed) finite-dimensional *t*-distributions

Left : conditional mean ; right : standard deviation



Zurich data – have we reached asymptotics ?

Fitted extremal coefficient function **vs. Empirical** extremal coefficients (empirical coefficients calculated for different thresholds)

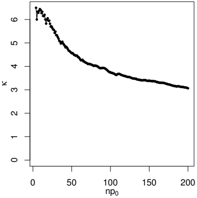


When going far into the tail of data, empirical coefficients tend to increase towards 2, the value for **asymptotic independence**.

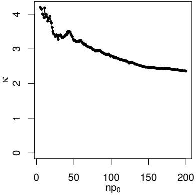
Other examples : asymptotic independence ?

Empirical estimates of extremal coefficients θ with respect to number of exceedances

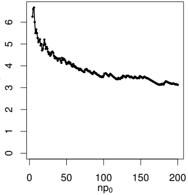
precipitation (Cévennes, $D = 21$)



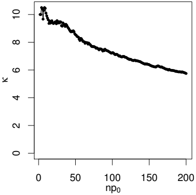
precipitation (Drenthe, $D = 15$)



wind gusts (Netherlands, $D = 24$)



temperature (France, $D = 33$)



Again, we observe a strong increase in estimates when going farther into the tail.

Can we handle asymptotic independence in spatial data ?

- **max-stable models** only allow **either asymptotic dependence or classical independence**, but nothing in between

- empirical evidence of many spatial data sets suggests that an asymptotic independent model would provide a better fit

- asymptotically dependent data : $\mathrm{pr}(X_1^\star > x, X_2^\star > x) \sim \lambda x^{-1}$ with $0 < \lambda \leq 1$

- **faster joint tail decay in asymptotically independent data**, *e.g.*

$$\mathrm{pr}(X_1^\star > x, X_2^\star > x) \sim h(x)x^{-\eta} \quad (4)$$

with $\eta > 1$ and slowly varying h, $h(tx)/h(t) \to 1$ $(t \to \infty)$

Can we have flexible spatial models for (4) **?**

Inverted max-stable processes

[Wadsworth and Tawn, 2012]

Given a max-stable process $Z^\star(s)$, the corresponding **inverted max-stable process**

$$X(s) = 1/Z^\star(s)$$

has exponential marginal distributions. Then $X^\star(s) = \exp(X(s))$.

Its **joint tail decay is**

$$\mathrm{pr}(X_1^\star > x, \ldots, X_D^\star > x) = x^{-D/\theta},$$

where $1 < \theta \leq D$ is the extremal coefficient of $(Z(s_1), \ldots, Z(s_D))$.

Since $D/\theta > 1$, the dependence of the original max-stable process determines the joint tail decay rate, leading to a **flexible framework for modeling asymptotic independence**.

We can estimate this tail dependence model by using threshold exceedances in data, *i.e.*, by censoring non-exceeding data values in the pairwise likelihood.

Gaussian scale mixture processes

[Opitz, 2016] + work in progress (joint with R. Huser, E. Thibaud)

Gaussian processes $W(s)$ are well-studied and well-tractable for inference, but they lack flexibility in the tail. We get more flexible tail behavior by **embedding a random variable R^2 with $R \geq 0$ for the variance** :

$$X(s) = RW(s)$$

- if R has power law tail, then $\boldsymbol{X}$ is asymptotically dependent with extremal-t limit
- if R has Weibull-type tail, $\mathrm{pr}(R > r) \sim r^{-c_1} \exp(-c_2 r^{-\alpha})$, then $\boldsymbol{X}$ is asymptotically independent with

  $$\mathrm{pr}(X_1^\star > x, X_2^\star > x) \sim h(x) x^{-[2/(1-\rho)]^{\alpha/(\alpha+2)}},$$

  yielding **flexible joint tail behavior** strongly determined by α
- can interpolate smoothly between Gaussian dependence and asymptotic dependence
- [Opitz, 2016] : **Laplace model** with $R^2 \sim \mathrm{Exp}$ has nice properties

Conclusion

- **Gaussian models are not flexible enough** to capture extreme value behavior
- spatial extreme value theory provides exploratory tools and asymptotic models
  - **max-stable models** are well studied and understood
  - **generalized Pareto models** are better suited for threshold exceedances
  - classical full likelihood inference is always tricky but possible in some cases, although computationally heavy
- **classical asymptotic models for spatial extremes** are well adapted to capture asymptotic dependence
- when convergence is not observed in data, we may need **subasymptotic models**, in particular for **asymptotic independence** to **avoid overestimation of extreme joint risks**

☺ – Merci – Danke – Thank you – ☺

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