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Divisor Arrangements and Algebraic Surfaces

by

Xavier ROULLEAU

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Abstract. We give a definition of arrangement of divisors on a surface that generalize the notion of arrangement of lines on the plane. To such an arrangement, we associate surfaces and we compute their Chern numbers. Using the arrangement of the 30 elliptic curves of the Fano surface of the Fermat cubic threefold, we obtain a surface with Chern ratio $2\frac{26}{27}$.

1. Introduction

In the spirit of the construction of surfaces by arrangement of lines on the plane done by of F. Hirzebruch, we give a definition of divisor arrangement on a smooth complex projective surface S. This is a given set Λ of curves $L_1, \dots, L_t$ on S and certain divisors $H_1, \dots, H_k$ linear combination of these curves. This definition leads to the construction of surfaces $S_n = S_n(\Lambda)$ for any integer $n > 1$ prime to a certain integer $m(\Lambda) > 0$ associated to the arrangement.

When n varies, the ratio of the Chern numbers of S_n are arbitrarily close to the ratio of the logarithmic Chern numbers of the divisor $L = \sum L_i$. This is the same fact for the surfaces recently constructed by cyclic covering (G. Urzua [13]).

This construction is done to obtain new examples of surfaces of general type with high Chern ratios. Recall that the upper bound for the ratios of the Chern numbers of a surface is 3 and we know only 3 arrangements of lines on the plane that reach this bound [3]. More generally, it is a difficult talk to construct surfaces with Chern ratio close to 3. We obtain:

THEOREM 1. *The Fano surface S of the Fermat cubic threefold possesses* 30 *elliptic curves that form an arrangement. The ratio of the Chern numbers of the associated surface S_2 is equal to $2\frac{26}{27}$.*

As a by product, we remark in paragraph 6 that our construction of surfaces enables us to recover the following two apparently very different constructions A) and B):

A) The surfaces $\mathcal{H}_n$ constructed by F. Hirzebruch by an arrangement of lines of the plane.

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B) Some surfaces $S_{\times C}C'$ (considered by Sommese [12]) where $f : S \to C$ is a fibration and $C' \to C$ is a particular covering.

2. Definition of divisor arrangement

Let S be a smooth complex projective surface. We denote by $\mathrm{Div}(S)$ the group of divisor on S.

Let $L_1, \dots, L_t$ be $t \geq 2$ smooth curves on S. If $H = \sum n_i L_i$ is a divisor, we note $v_i(H) = n_i$ the valuation at L_i of H.

DEFINITION 1. An arrangement $\Lambda = \Lambda(L_i, H_j)$ of the surface S is given by the smooth curves $L_1, \dots, L_t$ ($t \geq 3$) and effective divisors $H_0, \dots, H_k$ ($k \geq 1$) such that:

a) A singular point of $L = \sum_{i=1}^{i=t} L_i$ is the tranverse intersection point of exactly two curves L_i, L_j. We denote by I denote the set of singular points of L.

b) The divisors $H_0, \dots, H_k$ are linearly equivalent and

$$H_i = v_1(H_i)L_1 + \cdots + v_t(H_i)L_t\,.$$

c) The divisors $H_0, \dots, H_k$ are $\mathbb{Q}$ -linearly independents in $\mathrm{Div}(S) \otimes \mathbb{Q}$.

d) For each curve L_i, there exists a divisor H_a such that $v_i(H_a) = 1$.

e) If L_i and L_j cut each other in a point, then there exist 2 divisors H_a and H_b such that:

$$v_i(H_a) = 1\,, \quad v_j(H_a) = 0 \quad \text{and} \quad v_j(H_b) = 1$$

or such that:

$$v_i(H_b) = 0\,, \quad v_j(H_b) = 1 \quad \text{and} \quad v_i(H_a) = 1.$$

f) The linear system generated by $H_0, \dots, H_k$ is without base points.

We call $L_1, \dots, L_t$ (resp. $H_1, \dots, H_k$) the curves (resp. the divisors) of the arrangement $\Lambda = \Lambda(L_i, H_j)$.

REMARK 1. Let $L_1, \dots, L_t$ be curves and $H_1, \dots, H_k$ be divisors which verify hypothesis b), ..., f), such that the intersection of any two curves L_i, L_j is transverse but for which there exist points in which 3 or more curves L_i meet. Let $\pi : S' \to S$ be the blow-up of S at these points, L_i' the strict transform of L_i and $L_{t+1}', \dots, L_{t+r}'$ the exceptionnal divisors of π, then $\Lambda = \Lambda(L_i', \pi^* H_j)$ is an arrangement of S'.

3. Construction of surfaces by an arrangement of curves

Let us consider a surface S and an arrangement $\Lambda = \Lambda(L_i, H_j)$. Let $\mathcal{L}$ be the invertible sheaf $\mathcal{L} = \mathcal{O}_S(H_1)$. Let us denote by $s_0, \dots, s_k$ the sections of $\mathcal{L}$ such that $(s_i) = H_k$ (where (u) is the divisor of a section u). These sections define a morphism:

$$\begin{aligned} g : S &\to \mathbb{P}^k \\ s &\to (s_0 : \cdots : s_k)\,. \end{aligned}$$

Let $n > 1$ be an integer. Let S'_n be:

$$S'_n = S_{\times \mathbb{P}^k} \mathbb{P}^k$$

where the morphism $\mathbb{P}^k \to \mathbb{P}^k$ is the morphism

$$(x_0 : \cdots : x_k) \to (x_0^n : \cdots : x_k^n).$$

We define below (definition 2) an integer $m(\Lambda) \in \mathbb{N}^*$, named the number of the arrangement, such that if n is prime to $m(\Lambda)$, then S'_n is irreducible. In this case, we denote by S_n the desingularization of S'_n and

$$f : S_n \to S$$

the natural morphism. This is the surface that we will study.

REMARK 2. The idea of this study was found in [5] where M. N. Ishida consider $S = \mathbb{P}^2$ and $L_1 = H_1, \ldots, L_6 = H_6$ the lines of the complete quadrilateral. But our construction differs because it allows to use arrangements of curves L_i that are not linearly equivalent (see Theorem 3).

We denote by e the Euler characteristic. We have:

THEOREM 2. *Suppose that n is prime to $m(\Lambda)$. The morphism $f : S_n \to S$ has degree n^k. It is ramified with order n above $L = \sum_{i=1}^{i=t} L_i$ and with order n^2 above the set I of singular points of $L = \sum L_i$. The Euler number of S_n is:*

$$e(S_n) = n^{k-2}(n^2 e(S \setminus L) + ne(L \setminus I) + e(I)).$$

Let K_S be a canonical divisor of S and K_n a canonical divisor of S_n, then

$$(K_n)^2 = n^{k-2}(nK_S + (n-1)L)^2.$$

If $e(S) \neq e(L)$, then the limit of $\frac{K_n^2}{e(S_n)}$ (where n varies among the integers prime to $m(\Lambda) > 0$) is equal to the ratio

$$\frac{(K_S + L)^2}{e(S \setminus L)}$$

of the logarithmic Chern numbers of L.

Let us prove Theorem 2.

As a set, the surface S'_n is :

$$\{(s, (a_0 : \cdots : a_k))/a_i^n = s_i\}.$$

Let $F = \mathbb{C}(S)$ be the function field of S. For $1 \leq i \leq k$, denote by $f_i \in \mathbb{C}(S)$ the rational function such that $f_i = \frac{s_i}{s_0}$. The associated divisor of f_i is $H_i - H_0$. We want to compute the degree of the smallest field F' that contains the n^{th} roots of $f_1, \ldots, f_k$. To this aim, we use the following Proposition:

PROPOSITION 1 ([6], Appendix, Thm 10.3). *Let F be a field that contains the n^{th} roots of unity. Let $B \subset F^*$ be a finitely generated group and let F' be the smallest subfield* *(in an algebraic closure of F) that contains the n^{th} root of each element of B. The degree of F' over F is the order of the group $B/B \cap (F^*)^n$.*

We apply this Proposition to the group $B \subset F^*$ generated by $f_1, \dots, f_k$. We have a surjective morphism

$$\begin{aligned} \phi : \mathbb{Z}^k &\to B/B \cap (F^*)^n \\ (a_1, \dots, a_k) &\to f_1^{a_1} \cdots f_k^{a_k} . \end{aligned}$$

The kernel of ϕ contains $n\mathbb{Z}^k$ and by Proposition 1 the field F' has degree n^k if and only if $\operatorname{Ker}(\phi) = n\mathbb{Z}^k$.

For $g \in \mathbb{C}(S)$ let (g) be the associated divisor. Write $(f_j) = \sum_{i=1}^{i=t} m_{ij} L_i$ and let M be the $t \times k$ matrix $M = (m_{ij})_{1 \le i \le t; 1 \le j \le k}$. Let $m(\Lambda) \in \mathbb{N}$ be the integer such that the ideal generated by all the size k minors of M equals $m(\Lambda)\mathbb{Z}$.

DEFINITION 2. We call $m(\Lambda)$ the number of the arrangement Λ.

We denote by $\widetilde{M}$ the matrix with coefficients in $\mathbb{Z}/n\mathbb{Z}$ obtained by reduction modulo n of the coefficients of M. The Kernel of ϕ is equal to of $n\mathbb{Z}^k$ if and only if the equation:

$$\widetilde{M} A = 0$$

has no non trivial solution $A =^t (a_1, \dots, a_k)$ in $(\mathbb{Z}/n\mathbb{Z})^k$. This is so if and only if $m(\Lambda)$ and n are coprime. (Note that the hypothesis c) of definition 1 is necessary, otherwise, we have $m(\Lambda) = 0$).

Suppose now that $m(\Lambda)$ and n are coprime. Then S'_n is an irreducible surface and the function field of S_n and S'_n is:

$$\mathbb{C}(S)(f_1^{\frac{1}{n}}, \dots, f_k^{\frac{1}{n}}) .$$

As this Kummer extension of $\mathbb{C}(S)$ has degree n^k, the morphism f has degree n^k. Let s be a point of S and let t be a point of S_n above s. If s is not a point of L then the local ring of S_n at t is isomorphic to:

$$\mathcal{O}_{S,s}(f_1^{\frac{1}{n}}, \dots, f_k^{\frac{1}{n}}) .$$

The morphism f is not ramified above s. There exists an affine open U_s of S and an open V_s of S_n such that V_s is isomorphic to

$$\operatorname{Spec}(\mathcal{O}_S(U_s)(f_1^{\frac{1}{n}}, \dots, f_k^{\frac{1}{n}})) .$$

Let us suppose now that s is a point of I. Allowing that the indicies may be permuted, we can suppose that s is an intersection point of L_1 and L_2 and that it is not an element of H_0 (the system which contains $H_0, \dots, H_k$ is free from base points). By the condition e) of the definition, it can be also assumed that the divisors H_1 and H_2 verify:

$$v_1(H_1) = 1 , \quad v_2(H_1) = 0 \quad \text{and} \quad v_2(H_2) = 1 .$$

The regular element f_2 is written : $f_2 = f_1^{v_1(H_2)} g_2$ and (f_1, g_2) is a parameter system at s because L_1 and L_2 meet tranversally. Moreover f_1 (resp. g_2) is a local equation of L_1

(resp. L_2) in s. For $i \in \{3, \ldots, k\}$, there exists an invertible element $g_i \in \mathcal{O}_{S,s}$ such that:

$$f_i = f_1^{v_1(H_i)} g_2^{v_2(H_i)} g_i \, .$$

The local ring of S_p at t is isomorphic to the integral closure of

$$\mathcal{O}_{S,s}[f_1^{\frac{1}{n}}, f_2^{\frac{1}{n}}, \ldots, f_k^{\frac{1}{n}}]$$

and this ring is:

$$\mathcal{O}_{S,s}(g_3^{\frac{1}{n}}, \ldots, g_k^{\frac{1}{n}})[f_1^{\frac{1}{n}}, g_2^{\frac{1}{n}}] \, .$$

Hence f has ramification index n^2 at t. There exists an affine open U_s of S and an open V_s of S_n such that V_s is isomorphic to

$$\mathrm{Spec}(\mathcal{O}_S(U_s)(g_3^{\frac{1}{n}}, \ldots, g_k^{\frac{1}{n}})[f_1^{\frac{1}{n}}, g_2^{\frac{1}{n}}]) \, .$$

Let s be a point of L outside I. Allowing that the indicies may be permuted, we can suppose that s is a point of L_1 and that the divisors H_1 and H_0 verify:

$$v_1(H_1) = 1 \quad \text{and} \quad v_1(H_k) = 0 \, .$$

Let be $i \in \{2, \ldots, k\}$, there exist $h_i \in \mathcal{O}_{S,s}$ invertible such that:

$$f_i = f_1^{v_1(H_i)} h_i \, .$$

The local ring of S_n at t is isomorphic to:

$$\mathcal{O}_{S,s}(h_2^{\frac{1}{n}}, \ldots, h_k^{\frac{1}{n}})[f_1^{\frac{1}{n}}]$$

and f is ramified with order n in t. There exists an affine open U_s of S and an open V_s of S_n such that V_s is isomorphic to

$$\mathrm{Spec}(\mathcal{O}_S(U_s)(h_2^{\frac{1}{n}}, \ldots, h_k^{\frac{1}{n}})[f_1^{\frac{1}{n}}]) \, .$$

Now we calculate the Chern numbers of S_n. The morphism f is ramified with order n over L, hence K_n is numerically equivalent to:

$$f^*\left(K_S + \frac{n-1}{n} L\right) .$$

The morphism f is an étale covering of $f^{-1}(S \setminus L)$ of degree n^k. This is a covering of $f^{-1}(L \setminus I)$ of degree n^{k-1} and above each point of I, there are n^{k-2} points. The Euler number of S_n is equal to

$$n^k e(S \setminus L) + n^{k-1} e(L \setminus I) + n^{k-2} e(I) \, .$$

That ends the proof of Theorem 2.

4. Arrangement of the 30 elliptic curves on the Fano surface of the Fermat cubic threefold

Let S be the Fano surface that parametrizes the lines of the Fermat cubic threefold

$$F = \{x_1^3 + \cdots + x_5^3 = 0\} \hookrightarrow \mathbb{P}^4.$$

If s is a point of S, we denote by L_s the line on F that corresponds to the point s. Let μ_3 be the third roots of unity. For $1 \leq i < j \leq 5$, $\beta \in \mu_3$, the hyperplane $\{x_i + \beta x_j = 0\}$ cuts out a cone on F denoted by C_{ij}^{β}. The curve that parametrizes the lines on C_{ij}^{β} is an elliptic curve E_{ij}^{β} that is naturally embedded in the Fano surface S. These 30 cones are the only one contained in F, the 30 elliptic curves of S verify:

$$E_{ij}^{\beta} E_{st}^{\gamma} = \begin{cases} 1 & \text{if } \{i, j\} \cap \{s, t\} = \emptyset \\ -3 & \text{if } E_{ij}^{\beta} = E_{st}^{\gamma} \\ 0 & \text{otherwise} \end{cases}$$

and a canonical divisor K_S on S verifies $K_S^2 = 45$ (for these facts see [9]). For $1 \leq u < v \leq 5$, let B_{uv} be $B_{uv} = B_{vu} = \sum_{\mu_3} E_{uv}^{\beta}$.

THEOREM 3. *The* 10 *divisors*:

$$K_{ij} = 2B_{ij} + B_{rs} + B_{rt} + B_{st}$$

($\{i, j, r, s, t\} = \{1, \ldots, 5\}$) *are canonical divisors of S.*

The 6 *divisors* $K_{12}, K_{14}, K_{23}, K_{25}, K_{35}, K_{45}$ *and the* 30 *elliptic curves form an arrangement* Λ *of S. The number of this arrangement divides* 3. *Let n be an integer prime to* 3. *The Euler caracteristic of* S_n *is*:

$$e(S_n) = (162n^2 - 270n + 135)n^3$$

and the first Chern number is:

$$(K_n)^2 = 45(3n - 2)^2 n^3 .$$

We have $(K_2)^2/e(S_2) = 2\frac{26}{27}$ *and* $\lim_{n \notin 3\mathbb{Z}} \frac{(K_n)^2}{e(S_n)} = \frac{5}{2}$.

Proof. Let $\mathcal{O}_F(1)$ be the restriction of the sheaf $\mathcal{O}_{\mathbb{P}^4}(1)$ to $F \hookrightarrow \mathbb{P}^4$ and let Ω_S be the cotangent sheaf of S. By the tangent bundle Theorem 12.37 of [2], we can identify the forms $x_1, \ldots, x_5 \in H^o(F, \mathcal{O}_F(1))$ to a basis of $H^o(S, \Omega_S)$. This identification is as follows:

If ω_1, ω_2 are linearly independant elements of $H^o(S, \Omega_S)$, the closed set subjacent to the canonical divisor associated to $\omega_1 \wedge \omega_2 \in H^o(S, \wedge^2 \Omega_S)$ parametrizes the points s on S such that the line $L_s \hookrightarrow F$ cuts the space $\{\omega_1 = \omega_2 = 0\} \hookrightarrow \mathbb{P}^4$.

For $1 \leq i < j \leq 5$, the intersection of F and the plane $\{x_i = x_j = 0\} \hookrightarrow \mathbb{P}^4$ is a smooth elliptic curve E. Let $r < s < t$ be integers such that $\{i, j, r, s, t\} = \{1, 2, 3, 4, 5\}$. The curve E contains the 9 vertices of the cones

$$C_{rs}^{\beta}, \quad C_{rt}^{\beta}, \quad C_{st}^{\beta}, \quad \beta \in \mu_3$$

and E is the base curve of the cones $C_{ij}^{\beta} = F \cap \{x_i + \beta x_j = 0\}$, $\beta \in \mu_3$.

Let p be a point of F. The scheme S_p that parametrizes the lines on F going through p is an intersection of a cubic and a quadric in a plane (see [8]). If p is not the vertex of one of the 30 cones, S_p is finite of degree 6.

A line L on F is called double if there exists a plane X such that

$$XF = 2L + L'$$

where L' is the residual line. A double line going through a point p contribute for a degree at least 2 to the scheme S_p.

Suppose that p is a point of $E \hookrightarrow F$ hat is not the vertex of a cone. Then there is three lines through p that come from the 3 cones C_{ij}^{β}, $\beta \in \mu_3$. As each line of a cone is double, these 3 lines are the only one that goes through p.

That implies that each line L on F that goes through a point of $E = F \cap \{x_i = x_j = 0\}$ corresponds to a point s of one of the following 12 elliptic curves:

$$E_{ij}^{\beta}, \quad E_{rs}^{\beta}, \quad E_{rt}^{\beta}, \quad E_{st}^{\beta}, \quad \beta \in \mu_3 .$$

Hence the subjacent set to the canonical divisor K_{ij} associated to the 2-form $x_i \wedge x_j$ is the union of these 12 curves. The group of symmetries that preserves the plane $x_i = x_j = 0$ and the cubic F acts on the Fano surface and preserves the canonical divisor K_{ij}. That implies that there exist some integers a, b such that

$$K_{ij} = aB_{ij} + b(B_{rs} + B_{rt} + B_{st}) .$$

Let K_S be a canonical divisor on S. As $K_S^2 = K_S K_{ij} = 45$ and $K_S E_{uv}^{\beta} = 3$, we have:

$$9a + 27b = 45 .$$

Since a and b are positive integers, the unique solution is $a = 2$ and $b = 1$.

The 6 divisors $K_{12}, K_{14}, K_{23}, K_{25}, K_{35}, K_{45}$ and the 30 elliptic curves verify properties a), ... ,d) and f) of Definition 1.

For the property e), we consider the following tables:

12, 34	12, 35	12, 45	13, 24	13, 25	13, 45	14, 23	14, 25
35, 25	45, 14	35, 23	25, 35	45, 14	25, 23	25, 45	23, 14

and

14, 35	15, 23	15, 24	15, 34	23, 45	24, 35	25, 34
23, 12	23, 14	23, 35	23, 12	14, 12	35, 12	14, 12.

On the first line of these tables there are the indices ij, st such that the curve E_{ij}^{β} cuts the curve E_{st}^{γ} ($\gamma, \beta \in \mu_3$). The second line gives the indices uv, xy such that the divisors $H_a = K_{uv}$, $H_b = K_{xy}$ and the curves E_{ij}^{β} and E_{st}^{γ} verify the properties e) of the Definition 1. For example, we look at the 8$^{\text{th}}$ column of the first table. In that case the divisor K_{23} contains E_{14}^{β} with multiplicity 1 and does not contain E_{25}^{γ}, moreover K_{14} contains E_{25}^{γ} with multiplicity 1.

Thus the 6 divisors $K_{12}, K_{14}, K_{23}, K_{25}, K_{35}, K_{45}$ and the 30 elliptic curves form an arrangement. We easily check that the number of this arrangement divides 3.

The Euler characteristic of S is $e(S) = 27$ [2]. Each of the 30 elliptic curves on S cuts 9 elliptic curves and $L = \sum B_{ij}$ contains 135 singular points, hence

$$e(L \setminus I) = 30(0 - 9) = -270$$

and

$$e(L) = e(L \setminus I) + e(I) = -135$$

where I is the set of singular points of L. The calculation of $e(S_n)$ ensues. In order to simplify the computation of $(K_n)^2$, we can use the fact (proved in [9]) that L is a bicanonical divisor of S. □

5. Arrangements of hyperplane sections

Let S be a surface. For each integer $t \geq 3$, let $L_1, \dots, L_t$ be smooth hyperplane sections of an embedding of S such that for all $i \neq j$, the intersection of L_i and L_j is transverse and such that by a point of S goes at most 2 hyperplane sections. For $i \in \{1, \dots, t\}$, take $H_i = L_j$. The curves L_i and the divisors H_i form an arrangement Λ_t of S. The number of this arrangement is 1.

For $n \in \mathbb{N}^*$, let S_n^t be the surface associated to this arrangement. The numbers $K_S L$, L^2 and $e(S \setminus L)$, $e(L \setminus I)$ and $e(I)$ are easily calculated and we have $\lim_n \frac{c_1(S_n^n)^2}{e(S_n^n)} = 2$.

COROLLARY 1. *Let S be a smooth projective surface. There are smooth surfaces S' with a morphism $S' \to S$ such that the Chern ratios of S' are arbitrarilly close to 2.*

This fact was known for fibred surfaces [1]. We can ask what is the upper bound a of the ratio $c_1(S')/e(S')$ when S' varies among all surfaces of general type which have a morphism $S' \to S$. For the plane, the answer is $a = 3$.

6. Arrangements of lines on the plane and arrangements of smooth fibers of a fibration

Take $L_1, \dots, L_k$, k lines on the plane and let be $\pi : S \to \mathbb{P}^2$ the blow-up of points where 3 or more lines meet. The lines L_i and the divisors $H_i = L_i$ verifiy the properties b), ...,e) of the Definition 1. By Remark 1, we obtain an arrangement of S. The number of this arrangement is 1 and the surfaces S_n are isomorphic to the surfaces $\mathcal{H}_n$ constructed by F. Hirzebruch.

Now, let us consider S a surface with a fibration $f : S \to C$. Let $\delta_1, \dots, \delta_k$ be effective divisors of the same linear system on C such that the divisors $H_i = f^*\delta_i$ are smooth. Let $L_1, \dots, L_t$ be the irreducibles componants of the H_j. The divisors H_i and the fibers L_i form an arrangement $\Lambda(L_i, H_j)$ of S, with number $m(\Lambda) = 1$.

Let $C_n \to C$ be the ramified cover of C of degree n above the points subjacent to the divisor $\sum \delta_i$. The surface S_n is the fibred product

$$S \times_C C_n .$$

We obtain the same surfaces as Sommese did in [12].

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Max-Planck Institut Für Mathematik
Vivatgasse 7, 53111 Bonn, Germany
e-mail: roulleau@mpim-bonn.mpg.de