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CONIC CONFIGURATIONS VIA DUAL OF QUARTIC CURVES

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We construct special conic configurations from some point configurations which are the singularities of the dual of a quartic curve.

Introduction

We report some experiments on the construction of conic arrangements in the plane. Let us recall the very classical (and elementary) fact that through 5 points in general position there is a unique irreducible conic and there is no conic through 6 points in general position. In this note, we study and construct sets $\mathcal{C}$ of conics, each of which contains at least 6 points of a special set $\mathcal{P}$ of points. The idea and basic strategy for obtaining such sets of points grew from the following example: Let $C = \{f = 0\} \hookrightarrow \mathbb{P}^2$ be a smooth cubic curve and let $\check{C} \hookrightarrow \check{\mathbb{P}}^2$ be the dual curve (the image of C by the gradient map $(df/dx : df/dy : df/dz)$). The singularities of the degree 6 curve $\check{C}$ is a set $\mathcal{P}_9$ of 9 cusps (corresponding to the 9 flex lines of the cubic C). It turns out that a set of irreducible conics $\mathcal{C}_{12}$ that contains at least 6 points in $\mathcal{P}_9$ has cardinality 12, the sets $(\mathcal{P}_9, \mathcal{C}_{12})$ of points and conics form an interesting configuration of type

$$(9_8, 12_6),$$

described in [5]. In that paper, that conic arrangement appears in the context of generalized Kummer surfaces: the desingularization of the double cover branched over $\check{C}$ is a K3 surface X on which the pull-back of the conics are union of (-2)-curves which have as well an interesting configuration. The freeness of the arrangement of curves $\mathcal{C}_{12}$ is studied in [7], where we learned that this configuration has been also independently discovered in [3].

That example of 12 conics and 9 points provides an interesting example of points and conics in special position and motivated us to find other conic configurations with such a property. The dual curve $\check{C}$ of a smooth degree > 2 curve C is always a curve with singularities, and from the above example one may expect that the singular set is in special position with respect to the conics.

In this paper, we study the conics that go through at least 6 of the singularities of the dual of the Fermat quartic, the Klein quartic and a randomly chosen quartic curve which has 28 rational bitangent (corresponding to these 28 bitangent, there are 28 nodes on the dual curve). In each cases, we find a high number of smooth conics that contain at least 6 points in the singularity set of the curve. We compute the so-called Harbourne constant of the various conic arrangements obtained.

The literature on line arrangements is abundant, and their use occurs in many branches of mathematics. For example, special line arrangements are the main ingredient for the construction of ball quotient

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surfaces by Hirzebruch (see [4]). One of the most fascinating features of some line configurations and their set of singular points is their high degrees of symmetries. In this note, we obtain as well some very symmetric configurations of conics.

Let us describe a bit more the results obtained on a example. Let $\mathcal{P}_{28}$ be the set of 28 rational points corresponding dually to 28 rational bitangent of a particular smooth quartic curve $Q_{4/\mathbb{Q}}$. The set $\mathcal{C}_{1008}$ of conics in special position with respect to $\mathcal{P}_{28}$ has cardinality 1008. Each conic contains 6 points in $\mathcal{P}_{28}$, each point lies on 216 conics, that gives a $(28_{216}, 1008_6)$-configuration. This configuration is very symmetric, for example it contains 63 (16_6) and 63 $(12_{16}, 32_6)$ points and conic subconfigurations. We found all these configurations for various examples of quartics Q_4; we may ask if that is always the case for generic quartics and what could be the meaning of that phenomenon relative to these quartics.

One also may search for special configurations of higher degree curves, the next example being irreducible smooth cubic curves containing at least 10 points of a given set of points in $\mathbb{P}^2$. Searching such configurations is much more time consuming for the computer, but we give one example of a configuration of 63 smooth cubic curves defined over $\mathbb{Q}$ and 28 rational points in the plane (which is in fact the above set $\mathcal{P}_{28}$) forming a $(28_{27}, 63_{12})$-configuration. Apart from the 28 points of multiplicity 27, that arrangement of 63 cubics has 1008 triple points and 4725 double points.

This note is organized as follows: in Section 1 recall some preliminary materials, in Section 2, we describe special conic configurations obtained as the set of conics containing 6 or more points in the singular set of the dual of a quartic curve. In the Section 3, we study configurations of conics that contain 6 or more points in the singular set of some line arrangements, namely the mirror of complex reflection groups. In the Appendix, we give the algorithms (for magma software V2.24) used to compute the equations of the sets of conics, and the singularities of their union.

1. Preliminaries

1.1. Notations and conventions. The index $n \in \mathbb{N}$ of a set $\mathcal{S}_n$ indicates the cardinality of the set.

If $\mathcal{P}_n$ is a set of points in $\mathbb{P}^2$ and $\mathcal{C}_m$ an arrangement of curves (by which we mean a set of plane curves) such that each point $p \in \mathcal{P}$ is on r curves in $\mathcal{C}_m$ and each curve $C \in \mathcal{C}_m$ contains s points in $\mathcal{P}_n$, we say that the sets $(\mathcal{P}_n, \mathcal{C}_m)$ form a (n_r, m_s)-configuration. When $n = m$ (and thus $r = s$), one speaks of n_r-configurations. For more information and properties on abstract configurations in algebraic geometry, we refer to [2].

We remark that there is a natural identification between an arrangement and the associated reduced divisor which is the union of the curves in the arrangement; we will make no distinction between these two notions.

For an arrangement $\mathcal{C}$ of smooth irreducible curves and for $r > 1$ an integer,

$$t_r = t_r(\mathcal{C})$$

denotes the number of points of multiplicity r of the curve $\mathcal{C}$. That differs a bit from Hirzebruch's notations [4], where t_r stands for the number of ordinary singularities of multiplicity r.

Most of the time, we tried to use the following policy: when we study all the singularities of $\mathcal{C}$, a union of curves, we use the words "curve arrangement", when we study a pair $(\mathcal{P}, \mathcal{C})$ where $\mathcal{C}$ is a union of curves, and $\mathcal{P}$ is a fixed subset of the singular set, we use the words "curve configuration".

By “conic”, we mean a degree 2 smooth plane curve.

1.2. Harbourne constant of a curve. The H-constants have been introduced in [1]. Let C be a curve on a smooth surface X, let $f : \tilde{X} \to X$ be the blow-up of X at the singular points of C (supposed nonempty, of cardinality s) and let $\tilde{C} \hookrightarrow \tilde{X}$ be the strict transform of the curve C by f. The Harbourne constant of C at its singular set is

$$H(C) = \frac{(\tilde{C})^2}{s}.$$

For example, if $\mathcal{C} = C_1 + \cdots + C_k$ is a degree d plane curve which is the union of smooth curves C_j, having for $r > 1$, t_r points of multiplicity r, one has

$$H(C) = \frac{d^2 - \sum t_r r^2}{\sum t_r}.$$

If $\mathcal{C}_k$ is the union of k degree d smooth plane curves in general position then $H(\mathcal{C}_k) > -2$, and $\lim_k H(\mathcal{C}_k) = -2$. Obtaining arrangements of plane curves with H-constant lower than 2 is more difficult and it seems that no general construction is known (see [1] for results on line arrangements).

2. Dual of quartic curves

2.1. The Fermat quartic. Consider the curve $C = \{x^4 + y^4 + z^4 = 0\}$, its dual is the curve

$$\check{C} = \{x^{12} + 3x^8y^4 + 3x^4y^8 + y^{12} + 3x^8z^4 - 21x^4y^4z^4 + 3y^8z^4 + 3x^4z^8 + 3y^4z^8 + z^{12} = 0\}.$$

The set $\mathcal{P}_{28}$ of singular points of $\check{C}$ is the unions of the set $\mathcal{P}_{16}$ of nodes and the set $\mathcal{P}_{12}$ of E_6-singularities, where $\mathcal{P}_{16}$ is:

$$(-1:-1:1), (-1:1:1), (-1:-i:1), (-1:i:1), (1:-1:1),$$
$$(1:1:1), (1:-i:1), (1:i:1), (-i:-1:1), (-i:1:1), (-i:-i:1),$$
$$(-i:i:1), (i:-1:1), (i:1:1), (i:-i:1), (i:i:1).$$

and $\mathcal{P}_{12}$ is:

$$(0:-w:1), (0:w:1), (0:-w^3:1), (0:w^3:1), (-w:0:1), (w:0:1),$$
$$(-w^3:0:1), (w^3:0:1), (-w:1:0), (w:1:0), (-w^3:1:0), (w^3:1:0),$$

here $i^2 = -1$, $w = \frac{\sqrt{2}}{2}(1+i)$ (thus $w^2 = i$).

We compute that the set $\mathcal{C}$ of irreducible conics that contains at least 6 points in $\mathcal{P}_{28}$ has cardinality 2736, these conics are as follow:

- 2616 conics contain exactly 6 points in $\mathcal{P}_{28}$.
- 96 conics contain exactly 7 points in $\mathcal{P}_{28}$.
- 24 conics contain exactly 8 points in $\mathcal{P}_{28}$.

Through a point in $\mathcal{P}_{16}$ there are 546 conics in $\mathcal{C}$ and through a point in $\mathcal{P}_{12}$ there are 652 conics in $\mathcal{C}$.

2.1.1. *A* $(16_{24}, 64_6)$ *point-conic configuration.* Since the behavior of the number of conics through nodes and E_6-singularities is different, let us study the set of irreducible conics that contain at least 6 of the 16 nodes of $\check{C}$. We obtain that there is a set $\mathcal{C}_{64}$ of 64 conics that contain at least 6 points in $\mathcal{P}_{16}$. Through each of the 16 points there are 24 conics in $\mathcal{C}_{64}$, and each conic contains exactly 6 points among the 16. The sets $\mathcal{P}_{16}$ and $\mathcal{C}_{64}$ of points and conics form a

$$(16_{24}, 64_6)$$

configuration. The configuration has singularities

$$t_2 = 2832, \quad t_3 = 96, \quad t_4 = 72, \quad t_{24} = 16.$$

The H-constant is

$$H = -\tfrac{772}{377} \simeq -2.04.$$

2.1.2. *A* $(12_{28}, 56_6)$ *point-conic configuration.* If instead we consider the set $\mathcal{P}_{12}$ of 12 points which are E_6 singularities of $\check{C}$, we obtain that there is a set $\mathcal{C}_{56}$ of 56 conics such that each conic contains at least 6 points in $\mathcal{P}_{12}$. In fact, each conic contains exactly 6 points and through each points there are 28 conics, thus, we obtain a

$$(12_{28}, 56_6)$$

configuration of points and conics. We can check that when 6 points in $\mathcal{P}_{12}$ are on a conic in $\mathcal{C}_{56}$, then there is a unique conic in $\mathcal{C}_{56}$ that contain the 6 complementary points. On the union of the conics there are 44 nonordinary singularities, which are tacnodes (i.e., the local intersection multiplicity is 2). One has

$$t_2 = 1520, \quad t_{28} = 12,$$

therefore the H-constant is

$$H = \tfrac{112^2 - 1520\cdot 2^2 - 12\cdot 28^2}{1532} = -\tfrac{736}{383} \simeq -1.92.$$

Remark 1. For an arrangement of k conics with ordinary singularities one has

$$4\binom{k}{2} = \sum_{r\geq 2}\binom{r}{2} t_r,$$

but in that example, the left-hand side equals 6160, whereas the right-hand side equals to 6056. This is due to the fact that some singularities are not ordinary.

2.1.3. *A* $(32_3, 24_4)$ *point-line configuration.* In the above example there are 44 nonordinary singularities; 12 of them are in the set $\mathcal{P}_{12}$. Let us denote by $\mathcal{P}_{32}$ the complementary set. The set of lines $\mathcal{L}_{24}$ lines that contain strictly more that 2 points in $\mathcal{P}_{32}$ has cardinality 24, the union of these 24 lines has equation

$$x^{16}y^8 + x^{12}y^{12} + x^8y^{16} + x^{16}y^4z^4 + 2x^{12}y^8z^4 + 2x^8y^{12}z^4 + x^4y^{16}z^4 + x^{16}z^8 + 2x^{12}y^4z^8 + 3x^8y^8z^8 \\ + 2x^4y^{12}z^8 + y^{16}z^8 + x^{12}z^{12} + 2x^8y^4z^{12} + 2x^4y^8z^{12} + y^{12}z^{12} + x^8z^{16} + x^4y^4z^{16} + y^8z^{16} = 0.$$

The multiplicities of the singularities of the union are

$$t_2 = 96, \quad t_3 = 32, \quad t_8 = 3.$$

Each of the 24 lines contains 13 singularities. The sets $\mathcal{P}_{32}$ and $\mathcal{L}_{24}$ form a

$$(32_3, 24_4)$$

configuration of lines and points. Each line contains 8 double points, 4 triple points and one octuple point. The H-constant is

$$H = -\tfrac{288}{131} \simeq -2.19.$$

2.1.4. $(12_8, 24_4)$ *and* $(16_6, 24_4)$ *point-conic configurations.* The set $\mathcal{C}_{24}$ of 24 conics that contain 8 points in $\mathcal{P}_{28}$ and the set $\mathcal{P}_{12} \subset \mathcal{P}_{28}$ form a

$$(12_8, 24_4)$$

configuration. The sets $\mathcal{C}_{24}$ and $\mathcal{P}_{16} \subset \mathcal{P}_{28}$ form a

$$(16_6, 24_4)$$

configuration. The union of the 24 conics has 292 singular points, one has

$$t_2 = 216, \quad t_4 = 48, \quad t_6 = 16, \quad t_8 = 12.$$

There are 24 points where the intersection is not transverse, at these points only two conics go through (thus $t_2 = 196 + 24$). The H-constant is

$$H = \tfrac{48^2 - 216\cdot 2^2 - 48\cdot 4^2 - 16\cdot 6^2 - 12\cdot 8^2}{292} = -\tfrac{168}{73} \simeq -2.30.$$

2.2. The Klein quartic. Let $C = \{x^3y + y^3z + z^3x = 0\}$, its dual $\check{C}$ is a degree 12 curve with a set of 52 singularities, which is the union of:

- A set $\mathcal{P}_{28}$ of 28 nodes, corresponding to the 28 bitangent.
- A set $\mathcal{P}_{24}$ of 24 cusps, which correspond to flex points on the quartic.

These 52 points are defined over $\mathbb{Q}(\zeta_7)$.

2.2.1. *A* $(28_6, 21_8)$ *point-conic configuration.* There are 8 lines that contain at least 3 points of $\mathcal{P}_{28}$. There are 3129 irreducible conics that contain at least 6 points. These conics contain 6, 7 or 8 points. The set $\mathcal{C}_{21}$ of conics that contain 8 points in $\mathcal{P}_{28}$ has cardinality 21. Through each points in $\mathcal{P}_{28}$ there are 6 conics, thus the sets $\mathcal{P}_{28}, \mathcal{C}_{21}$ form a

$$(28_6, 21_8)$$

configuration. The singularities are

$$t_2 = 420, \quad t_6 = 28,$$

which gives

$$H = -\tfrac{33}{16} \simeq -2.06.$$

The intersections are transverse. The log-Chern ratio of the arrangement is

$$\tfrac{1161}{543} \simeq 2.13.$$

2.2.2. *A* $(24_{49}, 147_8)$ *point-conic configuration.* The set $\mathcal{C}_{147}$ of conics that contain 8 points in $\mathcal{P}_{24}$ has cardinality 147; the sets $\mathcal{P}_{24}$ and $\mathcal{C}_{147}$ form a

$$(24_{49}, 147_8)$$

configuration. We have

$$t_2 = 14280, \quad t_3 = 42, \quad t_{49} = 24,$$

thus the H-constant is

$$H = \tfrac{(147\cdot 2)^2 - 14280\cdot 2^2 - 42\cdot 3^2 - 24\cdot 49^2}{14346} = -\tfrac{4781}{2391} \simeq -1.99.$$

There are some nonordinary singularities.

The subset $\mathcal{C}_{49}$ of conics in $\mathcal{C}_{147}$ that go through the same point in $\mathcal{P}_{49}$ has cardinality 49 and the singularities of their union are as follow:

$$t_2 = 1127, \quad t_{14} = 2, \quad t_{15} = 21, \quad t_{49} = 1,$$

thus the H-constant is

$$H = \tfrac{98^2 - 1127\cdot 2^2 - 2\cdot 14^2 - 21\cdot 15^2 - 49^2}{1151} = -\tfrac{2422}{1151} \simeq -2.10.$$

There are 2 nonordinary singularities (through which pass 14 conics).

2.3. A quartic with 28 rational bitangents.

2.3.1. *Shioda's construction of quartics with* 28 *rational bitangents.* To a set $(u_1, \dots, u_6)$ of rational numbers in a specified open set of $\mathbb{A}^6$, Shioda (see [8]) associates a quartic with 28 rational bitangents and gives the coefficients of the equations of the bitangents as degree 4 polynomial functions of the u_i.

The affirmations of the following section have been obtained for the quartic Q_4 associated to the randomly chosen parameters

$$u_1 = -\tfrac{10}{9}, \quad u_2 = \tfrac{1}{9}, \quad u_3 = -\tfrac{6}{23}, \quad u_4 = -\tfrac{7}{9}, \quad u_5 = \tfrac{10}{9}, \quad u_6 = \tfrac{9}{19}.$$

We tested several other examples of the parameters u_i and obtained the same abstract point-conic configurations $(28_{216}, 1008_6)$ and point-cubic configurations $(28_{27}, 63_{12})$ we describe below. It would be interesting to understand if that is always the case for the generic parameters u_i, and why that phenomenon happens.

2.3.2. *A* $(28_{216}, 1008_6)$ *point-conic configuration.* Let $\mathcal{P}_{28}$ be the set of the 28 rational nodes of the dual curve of Q_4, nodes which corresponds to the 28 rational bitangents. Let $\mathcal{C}_{1008}$ be the set of conics that contain at least 6 points in $\mathcal{P}_{28}$. Each conic in $\mathcal{C}_{1008}$ contains 6 points in $\mathcal{P}_{28}$ and through each point there are 216 conics, thus, we obtain a

$$(28_{216}, 1008_6)$$

configuration of points and conics.

Remark 2. The 28 points in $\mathcal{P}_{28}$ are in general position with respect to the lines: no line contains 3 points from $\mathcal{P}_{28}$.

2.3.3. *Sixty three* $(12_{16}, 32_6)$ *point-conic subconfigurations.* Let us fix one of the conic $C_o \in \mathcal{C}_{1008}$ (we checked that the following is true for any such conic). Consider the 22 points that are the complementary set of points to the 6 points in $\mathcal{P}_{28}$ on C_o. Then there are 152 conics that contain at least 6 points in that set of 22 points, moreover there are 12 points through which there are 51 conics and 10 points through which there are 30 conics. Consider that set $\mathcal{P}_{12} = \mathcal{P}_{12}(C_o)$ of 12 points. Then there is a set of 32 conics in $\mathcal{C}_{1008}$ such that each conic contains 6 points in $\mathcal{P}_{12}$, and through each point in $\mathcal{P}_{12}$ there are 16 conics, thus, we get a configuration

$$(12_{16}, 32_6)$$

of points and conics.

We checked that on the union of the 32 conics, the 12 points have multiplicity 16, there are 544 double points and no other intersection points. The singularities are ordinary and the H-constant is

$$H = \tfrac{64^2-544\cdot 2^2-12\cdot 16^2}{544+12} = -\tfrac{288}{139} \simeq -2.07.$$

The configuration $(12_{16}, 32_6)$ has the following symmetries: the set of 16 conics that go through one point p_0 contains only 11 points among the 12. Let q_0 be the complementary point; then the set of 16 conics that go through q_0 is complementary to the set of 16 conics through p_0. In that way, we get two configurations of type $(10_8, 16_5)$ for the points different than p_0, q_0. That new configuration has

$$t_2 = 80, \quad t_8 = 10, \quad t_{16} = 1,$$

thus

$$H = \tfrac{32^2-80\cdot 2^2-10\cdot 8^2-16^2}{80+10+1} = -\tfrac{192}{91} \simeq -2.109.$$

2.3.4. *A* $(28_{27}, 63_{12})$ *point-cubic configuration.* It turns out that there are exactly 63 sets of 12 points of the form $\mathcal{P}_{12} = \mathcal{P}_{12}(C_o)$ (when C_o varies in $\mathcal{C}_{1008}$). Moreover, we obtain that through each set $\mathcal{P}_{12}$ of 12 points there is a unique cubic curve, which is smooth. Let us denote by $\mathcal{E}_{63}$ the set of such cubic curves. The sets $(\mathcal{P}_{28}, \mathcal{E}_{63})$ form a

$$(28_{27}, 63_{12})$$

configuration. The 63 elliptic curves in $\mathcal{E}_{63}$ are not isomorphic, and by taking their reduction modulo a prime, one can even show that they are not isogenous. The singularities of the cubic curve configuration $\mathcal{E}_{63}$ are ordinary with $t_2 = 4725$, $t_3 = 1008$, $t_{27} = 28$ and H-constant

$$H = -\tfrac{1809}{823} \simeq -2.19.$$

Remark 3. The elliptic curves in $\mathcal{E}_{63}$ contain naturally 12 rational points. These 12 points are the origin, a 2-torsion point t_2, 5 points $p_1, \dots, p_5$ and the points $t_2 + p_j$, $1 \leq j \leq 5$. One may ask how large is its Mordell–Weil group; computations with a computer never finished, but we made some experiments which seems to show that the Mordell–Weil group has rank at least 5.

2.3.5. *Sixty three* (16_6) *point-conic configurations.* We defined above 63 sets $\mathcal{P}_{12}$ of cardinality 12. For such fixed set $\mathcal{P}_{12}$, let us consider its complementary $\mathcal{P}_{16} \subset \mathcal{P}_{28}$. We obtain that the set $\mathcal{C}_{16}$ of conics in $\mathbb{P}^2$ that contain (at least) 6 points in $\mathcal{P}_{16}$ has cardinality 16 and the sets $\mathcal{P}_{16}, \mathcal{C}_{16}$ form a 16_6 configuration: each point is contained in 6 conics, each conic contains 6 points. The singularities are ordinary, with

$$t_2 = 240, \quad t_6 = 16,$$

its H-constant is $H = -2$.

3. Complex reflection groups

In the above conic configurations linked to the dual of a quartic, the conics contain at most 8 of the points in the special set $\mathcal{P}$. This is why we also searched for conic configurations related to complex reflection group: below we find a set of conics, each one containing 10 points of a set $\mathcal{P}$. We study exceptional three dimensional irreducible complex reflection groups; there are 5 such groups, which are classically denoted by $G_{23}, \dots, G_{27}$.

3.1. The Group G_{23}. The complex reflection group G_{23}, of order 120, has 15 mirrors (of order 2). The union of the 15 mirrors has a set $\mathcal{P}_{15}$ of 15 double points, a set $\mathcal{P}_{10}$ of 10 triple points and a set $\mathcal{P}_6$ of 6 quintuple points. There are 2345 conics containing at least 6 points in the union $\mathcal{P}_{15} \cup \mathcal{P}_{10} \cup \mathcal{P}_6$.

There are 25 conics that contain at least 6 points in $\mathcal{P}_{10}$. That set of conics $\mathcal{C}_{25}$ and the set $\mathcal{P}_{10}$ form a $(10_{15}, 25_6)$-configuration. The number and multiplicities of the singularities are

$$t_2 = 150, \quad t_{15} = 10$$

and the H-constant is $-\frac{35}{16} \simeq -2.18$.

3.2. The reflection group G_{24} of cardinality 168, the automorphism group of the Klein quartic. The complex reflection group G_{24} is the order 168 automorphism group of the Klein quartic curve. The union of the 21 mirrors (of order 2) has a set $\mathcal{P}_{21}$ of 21 quadruple points and a set $\mathcal{P}_{28}$ of 28 triple points.

There are 133 irreducible conics containing at least 6 points in $\mathcal{P}_{21}$. That set of conics is the union of the set $\mathcal{C}_{21}$ of the 21 conics that contain 8 points in $\mathcal{P}_{21}$ and the set $\mathcal{C}_{112}$ of conics that contain 6 points in $\mathcal{P}_{21}$. The sets $\mathcal{P}_{21}$ and $\mathcal{C}_{21}$ form a

$$21_8$$

configuration, the number and multiplicities of the singularities are

$$t_2 = 168, \quad t_8 = 21$$

and the H-constant is $-\frac{4}{3}$. Some singularities are nonordinary.

Remark 4. The Klein configuration of 21 lines and 21 points is a 21_4-configuration.

The union of the 21 lines and 21 conics in $\mathcal{C}_{21}$ is a curve with

$$t_2 = 378, \quad t_3 = 28, \quad t_{12} = 21$$

and H-constant equal to $-\frac{117}{61} \simeq -1.91$.

In [6, Section 3] is given another arrangement of 21 conics related to the group G_{24}, however these 21 conics are different from ours; their union have singularities: $t_2 = 168$, $t_3 = 224$.

3.3. The reflection groups G_{25} and G_{26}, the automorphism group of the Hesse pencil. The configuration of 21 lines obtained as the mirrors (of order 2 and 3) of the complex reflection group G_{26} have singularity set with

$$t_2 = 36, \quad t_4 = 9, \quad t_5 = 12.$$

There are no irreducible conics containing at least 6 points in the set $\mathcal{P}_9$ of points with multiplicity 4, neither there are for the set $\mathcal{P}_{12}$ of 12 points with multiplicity 5. However, through the set $\mathcal{P}_{21} = \mathcal{P}_9 \cup \mathcal{P}_{12}$

there pass 108 irreducible conics that contain 6 points. We denote by $\mathcal{C}_{108}$ that set of conics. Through each point in $\mathcal{P}_9$ there are 24 conics, and through each points in $\mathcal{P}_{12}$ there are 36 conics. Each of the conics contains 2 points in $\mathcal{P}_9$ and 4 points in $\mathcal{P}_{12}$, thus, we get

$$(9_{24}, 108_2) \quad \text{and} \quad (12_{36}, 108_4)$$

configurations.

The group G_{25} is implicitly studied since its projectivization is the same as G_{26}, and the 12 mirrors of G_{25} are the 12 order 3 mirrors of G_{26}: the 21 singularities of the line arrangement of G_{25} form the set $\mathcal{P}_{21}$.

3.4. The Group G_{27}, the Valentiner–Wiman group. The group G_{27} has 45 mirrors, the singularities of the union of these 45 lines are

$$t_3 = 120, \quad t_4 = 45, \quad t_5 = 36.$$

Let $\mathcal{P}_{36}$ be the set of multiplicity 5 singularities. There are 13062 irreducible conics that contain at least 6 points in $\mathcal{P}_{36}$. Through each point there are 2200 such conics. The conics contain 6, 8 or 10 points in $\mathcal{P}_{36}$. The set $\mathcal{C}_{72}$ of conics that contain 10 points has cardinality 72. Through each point there are 20 conics, thus the sets $\mathcal{P}_{36}, \mathcal{C}_{72}$ form a

$$(36_{20}, 72_{10})$$

configuration of points and conics. The singularities of that configuration are

$$t_2 = 3312, \quad t_{20} = 36,$$

there are 72 tacnodes and the H-constant equals to $-\frac{64}{31} \simeq -2.06$.

The set of conics containing 8 points has cardinality 270, through each points in $\mathcal{P}_{36}$ there are 60 conics, thus, we get a $(36_{60}, 270_8)$ configuration. The remaining conics and the points in $\mathcal{P}_{36}$ form a $(36_{2120}, 12720_6)$-configuration.

Appendix: Algorithm used to compute the conics and their intersection points

Let us recall the following elementary facts:

Lemma 5. *Let us consider* 5 *points in the plane. Assume that no three of these points are on a line. Then there is a unique conic that contains these points and that conic is irreducible. If three of these points lie on a line, then the conic is reducible, and may or may not be unique. If no four points are collinear, then the five points define a unique conic* (*degenerate if three points are collinear, but the other two points determine the unique other line*). *If four points are collinear, then there is not a unique conic passing through them — one line passing through the four points, and the remaining line passes through the other point, leaving* 1 *parameter free. If all five points are collinear, then the remaining line is free, which leaves* 2 *parameters free.*

In the following, we describe two algorithms used to compute the set of conics containing at least 6 points of a set of points in $\mathbb{P}^2$ and the singularity set of their union (with the multiplicities of these points); we use Magma software V. 2.24.

The entry of the first algorithm is a set P_0 of points in the plane, defined over a field K. The output is the set of irreducible reduced conics that contain at least 6 points in P_0. This algorithm is as follows:

```
function ConicsThrough6Points(Po)
  P2<x,y,z>:=Scheme(Po[1]);
  L2:=LinearSystem(P2,2);
```

Here L_2 is the linear system of conics in the plane. Rather than computing the set of conics containing 6 points ($\#P_0$ choose 6 possibilities), we start by studying the set of conics that contains at least 5 points in P_0 ($\#P_0$ choose 5 possibilities) and then we sort the conics that contain 6 points. Doing so we gain some speed:

```
  E:=SetToSequence(Subsets({1..#Po}, 5));
  Conics5:=[LinearSystem(L2,[P2!Po[k] : k in q]) : q in E];
```

The set Conics5 contains the linear systems of conics that goes through the points $P_0[k_1], \dots, P_0[k_5]$, where $\{k_1, \dots, k_5\}$ varies into the set E of 5-tuples of elements of $\{1, \dots, \#P_o\}$. We then select among them the 1-dimensional linear systems (a necessarily condition in order that the conic is irreducible) and define the list `Conics5P` of conics that contains at least 5 points:

```
  Conics5P:=[];
  for ru in [1..#E] do
    q:=Sections(Conics5[ru]);
    if #q eq 1 then
      Append(~Conics5P,Conics5[ru]);
    end if;
  end for;
  Conics5P:=[Scheme(P2,Sections(q)[1]): q in Conics5P}];
```

Here we construct the set of irreducible reduced conics that contains at least 6 points in P_0:

```
  Conics6:={q : q in Conics5P | #{p : p in Po | p in q} gt 5};
  Conics6:=SetToSequence({q : q in Conics6 | IsIrreducible(q)
and  IsReduced(q)});
```

The following lines compute the number of points in P_0 per conics, the number of conics through points in P_0, the set S_e of number of points in the conics, and then output the results:

```
  NumbPoInConics:=[#{p : p in Po| p in q}: q in Conics6];
  NumbConicsTroughPo:=[#{q : q in Conics6| p in q}: p in Po];
  Se:= Sort(SetToSequence({p : p in NumbPoInConics}));
  return Conics6,NumbConicsTroughPo,Se,NumbPoInConics;
end function;
```

The second algorithm takes as entry a set $\mathcal{C}$ of irreducible curves and computes the singularities of the curve $\sum_{q\in\mathcal{C}} q$ with their number and multiplicities. For $r \geq 2$, we denote by t_r the number of points of multiplicity r in the curve $\sum_{q\in\mathcal{C}} q$:

```
function ConfigurationOfCurves(ConfOfCur);
  P2<x,y,z>:=Ambient(ConfOfCur[1]);
  K:=BaseField(P2);
  R1<X>:=PolynomialRing(K);
  AllDeg:=[Degree(q) : q in ConfOfCur];  ma:=Max(AllDeg);
```

We start by computing the set SetSchP of irreducible components over the base field K of the intersections of two different curves; these are 0-dimensional scheme, maybe not reduced if the intersection points of the two curves are not transverse:

```
  SetSchP:={};
  for k1 in [1..#ConfOfCur-1] do
    for k2 in [k1+1..#ConfOfCur] do
      U:=IrreducibleComponents(ConfOfCur[k1] meet ConfOfCur[k2]);
      SetSchP:=SetSchP join {q:q in U};
    end for;
  end for;
```

We put away the reduced subschemes of the 0-dimensional schemes in the list SetSchP according to their degree and keep in a list the nonreduced intersection points:

```
  IntPoSch:=[[]: a in [1..ma^2]];
  NotRedIntPo:={};
  for q in SetSchP do
    dq1:=Degree(q);
    if dq1 eq 1 then Append(~IntPoSch[1],q);
      else
      qr:=ReducedSubscheme(q);
      dq2:=Degree(qr);
        if dq1 ne dq2 then
          NotRedIntPo:= {qr} join NotRedIntPo;
        end if;
      Append(~IntPoSch[dq2],qr);
    end if;
  end for;
  for k in [1..ma^2] do
    W:=IntPoSch[k];
    IntPoSch[k]:=SetToSequence({q : q in W});
  end for;
```

In the last above lines we removed the eventual repetitions. The k-th element in the list IntPoSch is the list of degree k reduced singular points of the curve $\sum_{q \in C} q$ over K. For each of these 0-dimensional schemes p, we compute the number of curves q such that p is a subscheme of q. In the following lines, the polynomial Poly is

$$\texttt{Poly} = \sum t_k X^k.$$

We put a multiplicity μ equal to the degree of the 0-dimensional scheme p: if there are k conics containing p and $p_1, \dots, p_\mu$ are the (degree 1) points in $\overline{K}$ over p, each p_j has multiplicity k.

```
  Poly:=R1!0;
  NumCurByPo:=[];
  for mu in [1..ma^2] do
    W:=IntPoSch[mu];
    if #W ne 0 then
      NumCurByPo[mu]:=[#[q : q in ConfOfCur | IsSubscheme(p,q)] :  p in W];
      Poly:=Poly + &+ [mu*X^u : u in   NumCurByPo[mu]];
    end if;
  end for;
```

Then, we extract the data for the list T_r of the integers $t_2, t_3, \dots$:

```
  CoPoly:=Coefficients(Poly);
  Tr:=[[t-1,CoPoly[t]]: t in [1..#CoPoly]| CoPoly[t] ne 0];
```

Finally, we compute the H-constant of the configuration and we return the data:

```
  Hcst:=((&+AllDeg)^2-&+[q[1]^2*q[2] : q in Tr])/(&+[q[2] : q in Tr]);
  return Tr,Hcst,IntPoSch,NumCurByPo,NotRedIntPo;
end function;
```


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