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Construction of Nikulin configurations on some Kummer surfaces and applications

Xavier Roulleau[1] · **Alessandra Sarti**[2]

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Abstract

A Nikulin configuration is the data of 16 disjoint smooth rational curves on a K3 surface. According to a well known result of Nikulin, if a K3 surface contains a Nikulin configuration $\mathcal{C}$, then X is a Kummer surface $X = \mathrm{Km}(B)$ where B is an Abelian surface determined by $\mathcal{C}$. Let B be a generic Abelian surface having a polarization M with $M^2 = k(k+1)$ (for $k > 0$ an integer) and let $X = \mathrm{Km}(B)$ be the associated Kummer surface. To the natural Nikulin configuration $\mathcal{C}$ on $X = \mathrm{Km}(B)$, we associate another Nikulin configuration $\mathcal{C}'$; we denote by B' the Abelian surface associated to $\mathcal{C}'$, so that we have also $X = \mathrm{Km}(B')$. For $k \geq 2$ we prove that B and B' are not isomorphic. We then construct an infinite order automorphism of the Kummer surface X that occurs naturally from our situation. Associated to the two Nikulin configurations $\mathcal{C}, \mathcal{C}'$, there exists a natural bi-double cover $S \to X$, which is a surface of general type. We study this surface which is a Lagrangian surface in the sense of Bogomolov-Tschinkel, and for $k = 2$ is a Schoen surface.

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✉ Xavier Roulleau
Xavier.Roulleau@univ-amu.fr
http://www.i2m.univ-amu.fr/perso/xavier.roulleau

Alessandra Sarti
sarti@math.univ-poitiers.fr
http://www-math.sp2mi.univ-poitiers.fr/ sarti/

1 Aix-Marseille Université, CNRS, Centrale Marseille, I2M UMR 7373, 13453 Marseille, France

2 Laboratoire de Mathématiques et Applications, UMR CNRS 7348, Université de Poitiers, Boulevard Marie et Pierre Curie, Téléport 2, Futuroscope, 86962 Chasseneuil, France

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1 Introduction

To a set $\mathcal{C}$ of 16 disjoint smooth rational curves $A_1, \ldots, A_{16}$ on a K3 surface X, Nikulin proved that one can associate a double cover $\tilde{B} \to X$ branched over the curve $\sum A_i$, such that the minimal model B of $\tilde{B}$ is an Abelian surface and the 16 exceptional divisors of $\tilde{B} \to B$ are the curves above $A_1, \ldots, A_{16}$. The K3 surface X is thus a Kummer surface.

We call a set of 16 disjoint (-2)-curves on a K3 surface a *Nikulin configuration*. Let us recall a classical construction of Nikulin configurations. The Kummer surface $X = \mathrm{Km}(B)$ of a Jacobian surface B can be embedded birationally onto a quartic Y of $\mathbb{P}^3$ with 16 nodes. Projecting from one node one gets another projective model for X, this is a double cover $Y' \to \mathbb{P}^2$ of the plane branched over 6 lines tangent to a conic. The strict transform (in X) of that conic is the union of two (-2)-curves A_1, A_1', with $A_1 A_1' = 6$. One of these two curves, A_1 say, corresponds to the node from which we project. Above the 15 intersection points of the 6 lines there are 15 disjoint (-2)-curves $A_2, \ldots, A_{16}$ on X, which corresponds to the 15 other nodes of the quartic Y.

The divisors $\mathcal{C} = \sum_{i=1}^{16} A_i$, $\mathcal{C}' = A_1' + \sum_{i=2}^{16} A_i$ are two Nikulin configurations. The Abelian surface B is then the Jacobian of the double cover of A_1 branched over $A_1 \cap A_1'$.

Let now $k > 0$ be an integer and let (B, M) be a polarized Abelian surface with $M^2 = k(k+1)$, such that B is generic, i.e. $\mathrm{NS}(B) = \mathbb{Z}M$. Let $X = \mathrm{Km}(B)$ be the associated Kummer surface, let $L \in \mathrm{NS}(X)$ be the class corresponding to M (so that $L^2 = 2M^2$), and let $\mathcal{C} = A_1 + \cdots + A_{16}$ be the natural Nikulin configuration on $\mathrm{Km}(B)$ (the class L is orthogonal to the A_i's). We obtain the following results, which for $k = 1$ are the results we recalled for Jacobian Kummer surfaces:

Theorem 1 *Let be $t \in \{1, \ldots, 16\}$. There exists a (-2)-curve A_t' on $\mathrm{Km}(B)$ such that $A_t A_t' = 4k + 2$ and $\mathcal{C}_t = A_t' + \sum_{j \neq t} A_j$ is another Nikulin configuration.*

The numerical class of A_t' is $2L - (2k+1)A_t$; the class

$$L_t' = (2k+1)L - 2k(k+1)A_t$$

generates the orthogonal complement of the 16 curves A_t' and $\{A_j \mid j \neq t\}$; moreover $L_t'^2 = L^2$.

A *Kummer structure* on a Kummer surface X is an isomorphism class of Abelian surfaces B such that $X \simeq \mathrm{Km}(B)$. It is known that Kummer structures on X are in one-to-one correspondence with the orbits of Nikulin configurations by the action of the automorphism group of X (see Proposition 21). In [29, Question 5], Shioda raised the question whether if there could be more than one Kummer structure on a Kummer surface. In [10], Gritsenko and Hulek noticed that $\mathrm{Km}(B) \simeq \mathrm{Km}(B^*)$, where B^* is the dual of B, a $(1, t)$-polarized Abelian surface (thus $B \not\simeq B^*$ if $t > 1$). In [12] Hosono, Lian, Oguiso and Yau proved that the number of Kummer structures is always finite and they construct for any $N \in \mathbb{N}^*$ a Kummer surface of Picard number 18 with at least N Kummer structures. When the Picard number is 17 (which is the case of our paper), by results of Orlov [20] on derived categories, the number of Kummer structures on

X equals 2^s where s is the number of prime divisors of $\frac{1}{2}M^2$. In Sect. 3.3, we obtain the following result

Theorem 2 *Suppose $k \geq 2$. There is no automorphism of X sending the Nikulin configuration $\mathcal{C} = \sum_{j=1}^{16} A_j$ to the configuration $\mathcal{C}_t = A'_t + \sum_{j \neq t} A_j$.*

Therefore the two configurations $\mathcal{C}$, $\mathcal{C}_t$ belong in two distinct orbits of Nikulin configurations under the action of $\mathrm{Aut}(X)$. As far as we know, Theorem 2 gives the first explicit construction of two distinct Kummer structures on a Kummer surface: the constructions in [10,12] use lattice theory and do not give a geometric description of the Nikulin configurations.

We already recalled that when X is a Jacobian Kummer surface, there exists a non-symplectic involution ι on X such that the double cover $\pi : X \to \mathbb{P}^2$ is the quotient of X by ι (after contraction of the 16 (-2)-curves). That involution exchanges the (-2)-curves A_1 and A'_1 and fixes the 15 other curves $\{A_j \mid j \neq 1\}$. For X a K3 surface with a polarization L such that $L^2 = 2k(k+1)$ and $t \in \{1, \dots, 16\}$, let θ_t be the involution of $\mathrm{NS}(X) \otimes \mathbb{Q}$ defined by $L \to L'_t$, $A_t \to A'_t$ (as defined in Theorem 1), and $\theta_t(A_j) = A_j$ for $j \neq t$. When $k = 1$, θ_1 is in fact the action of the involution ι on $\mathrm{NS}(X) : \iota^* = \theta_1$. We do not have such an interpretation when $k > 1$ (this is in fact the content of Theorem 2), but we obtain the following result on the product $\theta_i \theta_j$:

Theorem 3 *For $1 \leq i \neq j \leq 16$ there exists an infinite order automorphism μ_{ij} of X such that the action of μ_{ij} on $\mathrm{NS}(X)$ is $\mu_{ij}^* = \theta_i \theta_j$.*

The classification of the automorphism group of a generic Jacobian Kummer surface has been completed by Keum [13] (who constructed the last unknown automorphisms) and by Kondo [14] (who proved that there was indeed no more automorphisms). We are far from such a knowledge for non Jacobian Kummer surfaces, thus it is interesting to have a construction of such automorphisms μ_{ij}. Let A be an Abelian variety. In [18], Narasimhan and Nori prove that the orbits by $\mathrm{Aut}(A)$ of the principal polarisations in the Néron-Severi group $\mathrm{NS}(A)$ are finite. Similarly, one could think to prove that the number of Kummer structures on a K3 is finite by associating to each Nikulin configuration $\mathcal{C}$ the pseudo-ample divisor $L_{\mathcal{C}}$ orthogonal to $\mathcal{C}$ and by proving that the number of orbits of such $L_{\mathcal{C}}$ under the action of $\mathrm{Aut}(X)$ in $\mathrm{NS}(X)$ is finite. Our approach is closer to that idea than to the solutions previous proposed e.g. in [12] or [10], and it gives us more informations on $\mathrm{Aut}(X)$.

Observe that one can repeat the construction in Theorem 1, starting with configuration $\mathcal{C}_i$ instead of $\mathcal{C}$, but Theorem 3 tells us that the Nikulin configurations so obtained will be in the orbit of the Nikulin configuration $\mathcal{C}$ under the automorphism group X, thus we do not obtain new Kummer structures (tu l'as déjà signalé!) in that way (observe also that $\mathcal{C}_t$ and $\mathcal{C}_{t'}$ $(t \neq t')$ are in the same orbit).

The paper is organized as follows: In Sect. 2 we construct the curve A'_i such that $A_i A'_i = 4k + 2$ and we prove Theorem 1. This is done by geometric considerations on the properties of the divisor L'_i, which we prove is big and nef.

In Sect. 3, we construct the automorphisms mentioned in Theorem 3. This is done by using the Torelli Theorem for K3 surfaces. We then prove Theorem 2, which is obtained by considerations on the lattice $H^2(X, \mathbb{Z})$.

In Sect. 4, we study the bi-double cover $Z \to X$ associated to the two Nikulin configurations $\mathcal{C} = \sum_{i=1}^{16} A_i$, $\mathcal{C}' = A'_1 + \sum_{i=2}^{16} A_i$. When $k = 2$, Y is a so-called

Schoen surface, a fact that has been already observed in [24]. Schoen surfaces carry many remarkable properties (see e.g. [7,24]). For example the kernel of the natural map

$$\wedge^2 H^0(Z, \Omega_Z) \to H^0(Z, K_Z)$$

is one dimensional, and is not of the form $w_1 \wedge w_2$, i.e. by the Castelnuovo De Franchis Theorem, it does not come from a fibration of Z onto a curve of genus ≥ 2. Surfaces with this property are called Lagrangian. We will see that for the other $k > 1$, the surfaces are also Lagrangian.

In Sect. 4.1, we discuss the singularities of the curve $A_i + A_i'$. The transversality of the intersection of two rational curves on a K3 surface is an interesting but open problem in general (see e.g. [11]). We also study the curve Γ_i on the Abelian surface B coming from the pull-back of the curve A_i'. That curve Γ_i is hyperelliptic and has a unique singularity, which is a point of multiplicity $4k + 2$, and therefore Γ_i has geometric genus $\leq 2g$. In the case of a Jacobian surface, Γ_i has been used as the branch locus of covers of B by Penegini [22] and Polizzi [21], for creating new surfaces of general type. We end this paper by remarking that Γ_i is a curve with the lowest known H-constant (see [25] for definitions and motivations) on an Abelian surface.

2 Two Nikulin configurations on Kummer surfaces

2.1 Two rational curves A_1, A_1' such that $A_1 A_1' = 2(2k + 1)$

Let $k > 0$ be an integer and let B be an abelian surface with a polarization M such that $M^2 = k(k + 1)$. We suppose that B is generic so that M generates the Néron-Severi group of B. Let $X = \mathrm{Km}(B)$ be the associated Kummer surface and $A_1, \dots, A_{16}$ be its 16 disjoint (-2)-curves coming from the desingularization of $B/[-1]$.

By [17, Proposition 3.2], [9, Proposition 2.6], corresponding to the polarization M on B, there is a polarization L on $\mathrm{Km}(B)$ such that

$$L^2 = 2k(k + 1)$$

and $LA_i = 0,\ i \in \{1, \dots, 16\}$. The Néron-Severi group of $X = \mathrm{Km}(B)$ satisfies:

$$\mathbb{Z}L \oplus K \subset \mathrm{NS}(X),$$

where K denotes the Kummer lattice (the saturated sub-lattice of $NS(X)$ containing the 16 classes A_i). For B generic among polarized Abelian surfaces $\mathrm{rk}(\mathrm{NS}(X)) = 17$ and $\mathrm{NS}(X)$ is an overlattice of finite index of $\mathbb{Z}L \oplus K$ which is described precisely in [9], in particular we will use the following result:

Lemma 4 *([9, Remarks 2.3 & 2.10]) An element $\Gamma \in \mathrm{NS}(X)$ has the form $\Gamma = \alpha L - \sum \beta_i A_i$ with $\alpha, \beta_i \in \frac{1}{2}\mathbb{Z}$. If α or β_i for some i is in $\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, then at least 4 of the β_j's are in $\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, if moreover $\alpha \in \mathbb{Z}$, at least 8 of the β_j's are in $\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$.*

The divisor

$$A_1' = 2L - (2k+1)A_1$$

is a (-2)-class, indeed:

$$(2L - (2k+1)A_1)^2 = 8k(k+1) - 2(2k+1)^2 = -2,$$

and one has $A_1'A_i = 0$ for $i = 2, \dots, 16$. By the Riemann-Roch Theorem and since $LA_1' > 0$, the class A_1' is represented by an effective divisor. Let us prove the following result

Theorem 5 *The class A_1' can be represented by a (-2)-curve and $A_1A_1' = 2(2k+1)$. The set of (-2)-curves*

$$A_1', A_2, \dots, A_{16}$$

is another Nikulin configuration on X.

In order to prove Theorem 5, let us define

$$L' = (2k+1)L - 2k(k+1)A_1.$$

One has $L'A_1' = 0$ and

$$L'^2 = (2k+1)^2 2k(k+1) - 8k^2(k+1)^2 = 2k(k+1) = L^2.$$

First let us prove:

Proposition 6 *One has:*

a) The divisor L' is nef and big. Moreover a (-2)-class Γ satisfies $\Gamma L' = 0$ if and only if $\Gamma = A_1'$ or $\Gamma = A_j$ for j in $\{2, \dots, 16\}$.
b) The linear system $|L'|$ has no base components.
c) The linear system $|L'|$ defines a morphism from $X = \mathrm{Km}(B)$ to $\mathbb{P}^{k^2+k+1}$ which is birational onto its image and contracts the divisor A_1' and the 15 (-2)-curves A_i, $i \geq 2$.

Proof **Proof of a).** We already know that $L'^2 = 2k(k+1) > 0$. By the Riemann-Roch Theorem either L' or $-L'$ is effective. Since $LL' > 0$, we see that L' is effective. On a K3 surface, the (-2)-curves are the only irreducible curves with negative self-intersection, thus L' is nef if and only if $L'\Gamma \geq 0$ for each irreducible (-2)-curve Γ. Let

$$\Gamma = \alpha L - \sum_{i=1}^{16} \beta_i A_i, \qquad \alpha, \beta_i \in \frac{1}{2}\mathbb{Z}$$

be the class of Γ in $\mathrm{NS}(X)$. Since Γ represents an irreducible curve we have $\alpha \geq 0$. Moreover if $\Gamma = A_i$ then the condition $L'\Gamma \geq 0$ is trivially verified so that we can assume $\Gamma A_i \geq 0$, which gives $\beta_i \geq 0$. From the condition $\Gamma^2 = -2$, we get

$$k(k+1)\alpha^2 - \sum_i \beta_i^2 = -1 \tag{2.1}$$

Assume that the (-2)-curve Γ satisfies $L'\Gamma < 0$. We have

$$0 > L'\Gamma = ((2k+1)L - 2k(k+1)A_1)\,\Gamma = 2\alpha k(k+1)(2k+1) - 4k(k+1)\beta_1,$$

thus

$$\beta_1 > \frac{(2k+1)}{2}\alpha.$$

Combining with Eq. (2.1) we get

$$-1 = k(k+1)\alpha^2 - \sum_i \beta_i^2 < -\frac{1}{4}\alpha^2 - \sum_{i=2}^{16} \beta_i^2.$$

which is

$$\frac{1}{4}\alpha^2 + \sum_{i=2}^{16} \beta_i^2 < 1 \tag{2.2}$$

thus $\alpha \in \{0, 1/2, 1, 3/2\}$.

If $\alpha = 0$, by (2.1) either exactly one of the $\beta_i = 1$ (but this is not possible since it would give $\Gamma = -A_i$) or exactly 4 of the $\beta_i's$ are equal to $\frac{1}{2}$ and the others are 0 but such a class is not contained in $\mathrm{NS}(X)$ by Lemma 4.

If $\alpha = \frac{1}{2}$, then from inequality (2.2), $\beta_i \in \{0, \frac{1}{2}\}$ for $i \geq 2$ and at most 3 of these β_i's equal $\frac{1}{2}$. By Lemma 4 at least 4 of the β_i are in $\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, thus 3 of the β_i, $i \geq 2$ equals $\frac{1}{2}$ and the others are 0. Then from Eq. (2.1), we get:

$$\beta_1^2 = \frac{k^2+k+1}{4}.$$

Suppose that there exists $n \in \mathbb{N}$ such that $k^2 + k + 1 = n^2$. Then $n > k$, but since $n^2 \geq (k+1)^2 > k^2 + k + 1$, we get a contradiction. Hence $\forall k \in \mathbb{N}^*$, the integer $k^2 + k + 1$ is never a square and therefore the case $\alpha = \frac{1}{2}$ is impossible.

If $\alpha = 1$, at most 2 of the β_i's with $i > 1$ are equal $\frac{1}{2}$ and the others are 0, by applying Lemma 4 we get $\beta_i = 0$ for $i > 1$ and $\beta_1 \in \mathbb{N}$. Then Eq. (2.1) implies

$$\beta_1^2 = k^2 + k + 1,$$

which we know has no integral solutions for $k > 0$.

If $\alpha = \frac{3}{2}$, at most 1 of the β_i's with $i > 1$ is $\frac{1}{2}$, this is also impossible by Lemma 4, therefore such Γ does not exist and this concludes the proof that L' is big and nef for all $k \geq 1$.

Assume that the (-2)-curve Γ satisfies $L'\Gamma = 0$ and is not A_j for $j \geq 2$. Then one has $\beta_1 = \frac{(2k+1)}{2}\alpha$, and one computes that either $\alpha = 2$, $\beta_1 = 2k+1$ and $\Gamma = A'_1$, or $\alpha = 1$, $\beta_1 = \frac{(2k+1)}{2}\alpha$ and (up to re-ordering) $\beta_2 = b_3 = b_4 = 1/2$. Since α is an integer the second case is impossible by Lemma 4.

Proof of b). By [23, Section 3.8] either $|L'|$ has no fixed part or $L' = aE + \Gamma$, where $|E|$ is a free pencil, and Γ a (-2)-curve with $E\Gamma = 1$. In that case, write $\Gamma = \alpha L - \sum \beta_i A_i$. Then

$$2k(k+1) = L'^2 = 2a - 2$$

gives $a = k^2 + k + 1$. In particular, a is odd. But

$$a - 2 = L'\Gamma = 2k(k+1)(2k+1)\alpha - 4k(k+1)\beta_1$$

and since $\alpha, \beta_1 \in \frac{1}{2}\mathbb{Z}$, one gets that a is even, which yields a contradiction. Therefore $|L'|$ has no base components. By [27, Corollary 3.2], it then has no base points.

Proof of c). The linear system $|L'|$ is big and nef without base points. We have to show that the resulting morphism has degree one, i.e. that $|L'|$ is not hyperelliptic (see [27, Section 4]). By loc. cit., $|L'|$ is hyperelliptic if there exists a genus 2 curve C such that $L' = 2C$ or there exists an elliptic curve E such that $L'E = 2$.

In the first case $L'^2 = 8$, but since $L'^2 = 2k(k+1)$, that cannot happen. Assume now

$$E = \alpha L - \sum \beta_i A_i,$$

for E with $EL' = 2$, we get

$$2 = \left(\alpha L - \sum \beta_i A_i\right)((2k+1)L - 2k(k+1)A_1) = k(k+1)\left(2(2k+1)\alpha - 2\beta_1\right).$$

Since $\alpha, \beta_1 \in \frac{1}{2}\mathbb{Z}$, $2(2k+1)\alpha - 2\beta_1$ is an integer, thus we get $k = 1$ and $6\alpha - 2\beta_1 = 1$. Since $E^2 = 0$, one obtain

$$2\alpha^2 = \sum \beta_i^2,$$

using $\beta_1 = 3\alpha - \frac{1}{2}$, one reaches a contradiction.

Therefore $|L'|$ defines a birational map $X \to \mathbb{P}^N$ onto its image, contracting the (-2)-curves Γ such that $L'\Gamma = 0$, moreover $N = h^0(L') - 1 = \frac{L'^2}{2} + 1 = k^2 + k + 1$. □

We can now prove Theorem 5:

Proof We proved that the only (-2)-classes that are contracted by L' are A_1', A_2, $\dots, A_{16}$. We know moreover that $A_1'A_j = A_iA_j = 0$ for $2 \le i \neq j \le 16$. Since one has $L'A_1' = 0$ the base point free linear system $|L'|$ contracts the connected components of A_1' to some points. Therefore by the Grauert contraction Theorem (see [4, Chapter III, Theorem 2.1]), the support of A_1' is the union of irreducible curves $(C_i)_{i\in\{1,\dots,m\}}$ (for $m \in \mathbb{N}$, $m \neq 0$) such that the intersection matrix (C_iC_j) is negative definite.

Since X is a K3 surface, the curves C_i are (-2)-curves. Since L' only contracts the (-2)-classes $A_1', A_2, \dots, A_{16}$ that are disjoint, we get that $m = 1$ and we conclude that A_1' is the class of a (-2)-curve C_1. □

2.2 A projective model of the surface Km(*B*)

Let us describe a natural map from $\mathrm{Km}(B)$ to $\mathbb{P}^{k+1}$, which is birational for $k > 1$:

Theorem 7 *The class $D = L - kA_1$ is big and nef with*

$$(L - kA_1)^2 = 2k$$

and for $k \ge 2$ it defines a birational map

$$\phi : \mathrm{Km}(B) \to \mathbb{P}^{k+1}$$

onto its image X such that X (of degree $2k$) has 15 ordinary double points and moreover the curves A_1' and A_1 are sent to two rational curves of degree $2k$ such that $A_1A_1' = 2(2k+1)$.

Remark 8 We have

$$A_1' + A_1 = 2(L - kA_1)$$

so that $A_1' + A_1$ is cut out by a quadric of $\mathbb{P}^{k+1}$ and is 2-divisible.

Proof We proceed as in Proposition 6.

Let us show that D is nef and big. We have to prove that $D\Gamma \ge 0$ for each irreducible (-2)-curve Γ. As above, let

$$\Gamma = \alpha L - \sum \beta_i A_i, \qquad \alpha, \beta_i \in \frac{1}{2}\mathbb{Z},$$

be such that $\Gamma D < 0$. Then

$$\Gamma D = 2\alpha k(k+1) - 2k\beta_1 < 0,$$

implies $\beta_1 > (k+1)\alpha$.

Combining with the Eq. (2.1), we get

$$1 > (k+1)\alpha^2 + \sum_{i\geq 2} \beta_i^2,$$

thus $\alpha < 1$. As in Proposition 6, the case $\alpha = 0$ is impossible. If $\alpha = \frac{1}{2}$, then $k \in \{1, 2\}$, but as above, Lemma 4 implies that this is not possible. Thus D is nef and big.

Let us now suppose $k > 1$. **Let us show that $|D|$ has no base components**. Suppose that there is a base component. Then $D = aE + \Gamma$, where $a \in \mathbb{N}$, $|E|$ is a free pencil, Γ is a (-2)-curve and $E\Gamma = 1$. One has

$$2k = D^2 = 2a - 2,$$

thus $a = k + 1$, so that

$$L - kA_1 = (k+1)E + \Gamma.$$

Suppose that $\Gamma = A_1$, then $2k = A_1 D = k - 1$ and $k = -1$, which is impossible. If $\Gamma = A_i$, $i \geq 2$, then $0 = DA_i = k - 1$, thus $k = 1$, but we assumed that $k > 1$.

Thus we can assume that Γ is not one of the A_i and write $\Gamma = \alpha L - \sum \beta_i A_i$ with $\alpha, \beta_i \geq 0$. One has

$$2k = DA_1 = (k+1)EA_1 + 2\beta_1, \tag{2.3}$$

moreover

$$2k(k+1) = (L - kA_1)L = (k+1)EL + 2k(k+1)\alpha. \tag{2.4}$$

Since $EA_1 \geq 0$ we obtain from Eq. (2.3) that either $\beta_1 = k$ (and $EA_1 = 0$) or $\beta_1 = \frac{k-1}{2}$ and $EA_1 = 1$, in that second case since

$$E(L - kA_1) = E((k+1)E + \Gamma) = 1$$

one obtains $EL = k + 1$.

Since $EL \geq 0$, we obtain from Eq. (2.4) that $\alpha \in \{0, \frac{1}{2}, 1\}$, but as in Proposition 6, $\alpha = 0$ is not possible. Moreover if $\alpha = 1$, $EL = 0$, but this contradicts the Hodge Index Theorem since $E^2 = 0$ and $L^2 > 0$, therefore $\alpha = \frac{1}{2}$. If $\beta_1 = k$, from $\Gamma^2 = -2$, one gets

$$\frac{k(k+1)}{4} - k^2 - \sum_{i\geq 2} \beta_i^2 = -1$$

which is

$$\sum_{i\geq 2} \beta_i^2 = \frac{1}{4}(-3k^2 + k + 4).$$

But for $k > 1$, $-3k^2 + k + 4 < 0$ and we obtain a contradiction. If now $\beta_1 = \frac{k-1}{2}$, then $EL = k + 1$, but Eq. (2.4) gives $EL = k$, contradiction. Therefore $|D|$ has no base component.

Let us show that $|D|$ defines a birational map. We have to show that $|D|$ is not hyperelliptic. Suppose that $D = 2C$ where C is a genus 2 curve. Then $D^2 = 8$; since $D^2 = 2k$, we get $k = 4$. One has $D = L - 4A_1$ and the class of C is $\frac{1}{2}L - 2A_1$. Then $\frac{1}{2}L \in \mathrm{NS}(X)$, which contradicts the fact that L generates the orthogonal complement of the Kummer lattice K in NS(KM(B)), and so L is primitive. Suppose now that there exists an elliptic curve E such that $DE = 2$. Let

$$E = \alpha L - \sum \beta_i A_i,$$

with $\alpha \in \frac{1}{2}\mathbb{Z}$. Since $D = L - kA_1$, one has

$$DE = 2k(k+1)\alpha - 2k\beta_1,$$

therefore $k(k+1)\alpha - k\beta_1 = 1$. If $\alpha \in \mathbb{Z}$, then if $\beta_1 \in \mathbb{Z}$, one gets $k = 1$, if $\beta_1 = \frac{b}{2}$ with b odd, then

$$k(2(k+1)\alpha - b) = 2$$

and $k = 2$ (we supposed $k > 1$), $6\alpha - b = 2$, which is impossible since b is odd. If $\alpha = \frac{a}{2}$ with $a \in \mathbb{Z}$ odd , then $k((k+1)a - 2\beta_1) = 2$. Then since $2\beta_1 \in \mathbb{Z}$ and $k > 1$, one has $k = 2$ and $3a - 2\beta_1 = 1$, thus $\beta_1 = \frac{3a-1}{2} = 3\alpha - \frac{1}{2} \in \mathbb{Z}$. We have moreover (since $k = 2$):

$$0 = E^2 = 6\alpha^2 - \sum \beta_i^2$$

thus

$$9\alpha^2 - 3\alpha + \frac{1}{4} + \sum_{i\geq 2} \beta_i^2 = 6\alpha^2,$$

and $3\alpha^2 - 3\alpha + \frac{1}{4} \leq 0$, the only possibility is $\alpha = \frac{1}{2}$, but then $\sum_{i\geq 2} \beta_i^2 = \frac{1}{2}$, which is impossible since, by Lemma 4, there is no class with $\beta_i = \frac{1}{2}$ for only 2 indices i. Therefore when $k > 1$, $|D|$ defines a birational map to $\mathbb{P}^N$, with $N = \frac{D^2}{2} + 1 = k+1$. That map contracts the curves Γ with $\Gamma D = 0$, ie $A_2, \ldots, A_{16}$.

One has

$$A_1(L - kA_1) = 2k = A_1'(L - kA_1),$$

thus the curves A_1, A_1' in $\mathbb{P}^{k+1}$ have degree $2k$. Moreover $A_1 A_1' = 2(2k+1)$.

Let us prove that the 15 (-2)-curves A_i, $i > 1$ are the only ones contracted i.e. they are the only solutions of the equation $\Gamma D = 0$, $(D = L - kA_1)$. Suppose $\Gamma \neq A_i$, $\Gamma = \alpha L - \sum \beta_i A_i$. One has $\Gamma D = 0$ if and only if

$$\alpha(k+1)=\beta_1,$$

and $\alpha^2 k(k+1)-\sum\beta_i^2=-1$, which gives

$$(k+1)\alpha^2+\sum_{i>1}\beta_i^2=1,$$

which has no solutions by Lemma 4. □

Remark 9 To the pair (L, A_1) one can associate the pair (L', A_1'), with

$$L'=(2k+1)L-2k(k+1)A_1,\ \ A_1'=2L-(2k+1)A_1$$

with the same numerical properties

$$L^2=L'^2=2k,\ LA_1=0=L'A_1',\ LA_1'=4k(k+1)=L'A_1.$$

The polarization L' comes from a polarization M' on the Abelian surface B' associated to the Nikulin configuration $A_1', A_2,\dots,A_{16}$. We will see that for $k=1$ the mapping $\Psi:(L,A_1)\to(L',A_1')$ is an involution of $\mathrm{NS}(X)$ which comes from an involution of X, and the Abelian surfaces B, B' are isomorphic.

One can repeat the construction with (L', A_2) instead of L, A_1 etc... Let us define the maps $\Psi_i,\ \Psi_j,\ \{i,j\}=\{1,2\}$ by $\Psi_i(L)=(2k+1)L-2k(k+1)A_i$, $\Psi_i(A_i)=2L-(2k+1)A_i$, $\Psi_i(A_j)=A_j$. It is easy to check that $\Psi_1\circ\Psi_2$ has infinite order, and we therefore obtain in that way an infinite number of Nikulin configurations. For any $k\in\mathbb{N},\ k\neq 0$, we will see that the map $\Psi_i\circ\Psi_j$ for $i\neq j$ is in fact the restriction of the action of an automorphism of X on $\mathrm{NS}(X)$.

2.3 The first cases $k=1,2,3,4$

In this subsection, we give a more detailed description of our construction when k is small. One has

k	1	2	3	4
A_1A_1'	6	10	14	18
L^2	4	12	24	40

and the morphism ϕ associated to the linear system $|L-kA_1|$ is from $\mathrm{Km}(B)$ to $\mathbb{P}^{k+1}$, with $k+1=2,3,4,5$ (which produce the most famous geometric examples of K3 surfaces).

The case $k=1$ has been discussed in the Introduction.

For $k=2$, the result was already observed in [24]. The image of ϕ is a 15-nodal quartic $Q=Q_4$ in $\mathbb{P}^3$, the curves A_1, A_1' are sent to two degree 4 rational curves (denoted by the same letters) meeting in 10 points. As we already observed, the divisor $A_1 + A_1'$ is a *2-divisible class*. The double cover $Y \to Q$ branched over $A_1 + A_1'$ has 40 ordinary double points coming from the 15 singular points on Q and from the 10 intersection points of A_1 and A_1'. This surface Y is described in [24]. It is a general type surface, a complete intersection in $\mathbb{P}^4$ of a quadric and the Igusa quartic. It is the canonical image of its minimal resolution. The double cover S of Y branched over the 40 nodes is a so-called Schoen surface. It is a surface with $p_g(S) = p_g(Y) = 5$, thus the canonical image of S is Y and the degree of the canonical map of the Schoen surface is 2.

For $k = 3$, one get a model Q_6 of X in $\mathbb{P}^4$ which is the complete intersection of a quadric and a cubic. In a similar way as before, Q_6 has 15 ordinary double points and A_1 and A_1' are sent by $|L - 3A_1|$ to two rational curves of degree 6 with intersection number 14.

For $k = 4$, one get a degree 8 model Q_8 of X in $\mathbb{P}^5$ which is the complete intersection of 3 quadrics. That model has 15 ordinary double points and the curves A_1 and A_1' are sent by $|L - 4A_1|$ to two rational curves of degree 8 with intersection number 18.

3 Nikulin configurations and automorphisms

3.1 Construction of an infinite order automorphism

Let us denote by K_{abcd} with $a, b, c, d \in \{0, 1\}$ the 16 (-2)-curves on the K3 surface $X = \mathrm{Km}(A)$, and as before let L be the polarization coming from the polarization of A.

Let K be the lattice generated by the following 16 vectors $v_1, \dots, v_{16}$:

$$\frac{1}{2}\sum_{p\in A[2]} K_p,\ \frac{1}{2}\sum_{W_1} K_p,\ \frac{1}{2}\sum_{W_2} K_p,\ \frac{1}{2}\sum_{W_3} K_p,\ \frac{1}{2}\sum_{W_4} K_p,\ K_{0000},$$
$$K_{1000},\ K_{0100},\ K_{0010},\ K_{0001},\ K_{0011},\ K_{0101},\ K_{1001},\ K_{0110},\ K_{1010},\ K_{1100}$$

where $W_i = \{(a_1, a_2, a_3, a_4) \in (\mathbb{Z}/2\mathbb{Z})^4 \mid a_i = 0\}$. By results of Nikulin, [19], the lattice K is the minimal primitive sub-lattice of $H^2(X, \mathbb{Z})$ containing the (-2)-curves K_{abcd}. The discriminant group $K^\vee/K$ is isomorphic to $(\mathbb{Z}_2)^6$ and the discriminant form of K is isometric to the discriminant form of $U(2)^{\oplus 3}$.

Lemma 10 (See [9, Remark 2.3]) *The Néron-Severi group* $\mathrm{NS}(X)$ *is generated by* K *and* $v_{17} := \frac{1}{2}(L + \omega_{4d})$, *where* L *is the positive generator of* $K^\perp$ *with* $L^2 = 4d$ *(here* $d = \frac{k(k+1)}{2}$*), and if* $L^2 = 0 \bmod 8$,

$$\omega_{4d} = K_{0000} + K_{1000} + K_{0100} + K_{1100},$$

if $L^2 = 4 \bmod 8$,

$$\omega_{4d} = K_{0001} + K_{0010} + K_{0011} + K_{1000} + K_{0100} + K_{1100}.$$

One has moreover

Lemma 11 ([9, Remark 2.11]) *The discriminant group of* NS(X) *is isomorphic to* $(\mathbb{Z}/2\mathbb{Z})^4 \times \mathbb{Z}/4d\mathbb{Z}$. *Suppose that* $d = 4$ *mod* 8. *Then* $\mathrm{NS}(X)^\vee/\mathrm{NS}(X)$ *is generated by*

$$w_1 = \frac{1}{2}(v_6 + v_8 + v_{10} + v_{12}), \quad w_2 = \frac{1}{2}(v_{12} + v_{13} + v_{14} + v_{15}),$$
$$w_3 = \frac{1}{2}(v_{11} + v_{13} + v_{14} + v_{16}), \quad w_4 = \frac{1}{2}(v_9 + v_{10} + v_{12} + v_{13}),$$
$$w_5 = \frac{1}{2}(v_6 + v_{12} + v_{13}) + \frac{1}{4d}(v_7 + v_8 + v_9 + v_{10} + (1 + 2d)v_{11} + v_{16} - 2v_{17})$$

Suppose that $d = 0$ *mod* 8. *Then* $\mathrm{NS}(X)^\vee/\mathrm{NS}(X)$ *is generated by*

$$w_1 = \frac{1}{2}(v_6 + v_{12} + v_{14} + v_{16}), \quad w_2 = \frac{1}{2}(v_6 + v_{13} + v_{15} + v_{16}),$$
$$w_3 = \frac{1}{2}(v_6 + v_8 + v_{10} + v_{12}), \quad w_4 = \frac{1}{2}(v_6 + v_8 + v_9 + v_{13}),$$
$$w_5 = \frac{1}{2}(v_{11} + v_{12} + v_{13}) + \frac{1}{4d}((1 + 2d)v_6 + v_7 + v_8 + v_{16} - 2v_{17})$$

In both cases, the discriminant form of NS(X) *is isometric to the discriminant form of* $U(2)^{\oplus 3} \oplus \langle 4d\rangle$ *and the transcendental lattice* $T_X = \mathrm{NS}(X)^\perp$ *is isomorphic to* $U(2)^{\oplus 3} \oplus \langle -4d\rangle$.

Proof The columns of the inverse of the intersection matrix $(v_i v_j)_{1\le i,j\le 17}$ is a base of $\mathrm{NS}(X)^\vee$ in the base $v_1, \dots, v_{17}$. From that data we obtain the generators $w_1, \dots, w_5$ of $\mathrm{NS}(X)^\vee/\mathrm{NS}(X)$. The matrix $(w_i w_j)_{1\le i,j\le 5}$ is

$$\begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4d} \end{pmatrix} \in M_5(\mathbb{Q}/\mathbb{Z}),$$

one has moreover $w_i^2 = 0 \bmod 2\mathbb{Z}$ for $1 \le i \le 4$ and $w_5^2 = \frac{1}{4d} \bmod 2\mathbb{Z}$. Thus the discriminant form

$$q : \mathrm{NS}(X)^\vee/\mathrm{NS}(X) \to \mathbb{Q}/2\mathbb{Z}$$

is isometric to the discriminant form of $U(2)^{\oplus 3} \oplus \langle 4d\rangle$. Since $H^2(X, \mathbb{Z})$ is unimodular, and $U(-2) \simeq U(2)$, we obtain T_X (for more details see e.g. [11, Chap. 14, Proposition 0.2]). □

In Sect. 2, we associated to L and to A_j the divisors

$$L_j = (2k + 1)L - 2k(k + 1)A_j, \;\; A'_j = 2L - (2k + 1)A_1.$$

The vector space endomorphism

$$\theta_j : \mathrm{NS}(X) \otimes \mathbb{Q} \to \mathrm{NS}(X) \otimes \mathbb{Q}$$

defined by $\theta_j(A_i) = A_i$ for $i \neq j$ and

$$\theta_j(A_j) = A'_j,\ \theta_j(L) = L_j$$

is an involution, and we will see that it is an isometry (cf. Lemma 13). Let us define

$$\Phi_1 = \theta_2\theta_1.$$

The endomorphism Φ_1 has infinite order, its characteristic polynomial $\det(T\mathrm{I_d} - \Phi_1)$ is the product of $(T-1)^{15}$ and the Salem polynomial

$$T^2 + (2-4k^2)T + 1.$$

The aim of this section is to prove the following result:

Theorem 12 *The automorphism Φ_1 extends to an effective Hodge isometry Φ of $H^2(X,\mathbb{Z})$ and there exists an automorphism ι of X which acts on $H^2(X,\mathbb{Z})$ by $\iota^* = \Phi$.*

Let us start by the following Lemma:

Lemma 13 *The morphisms $\theta_1,\ \theta_2,\ \Phi_1$ preserve $\mathrm{NS}(X)$ and are isometries of $\mathrm{NS}(X)$.*

Proof It is simple to check that θ_j preserves the lattice generated by K, L and $v_{17} = \frac{1}{2}(L+\omega_{4d})$. Since for all $1 \leq i,\ j \leq 16$ one has $\theta_j(A_i)\theta_j(A_k) = A_iA_k, \theta_j(L)\theta_j(A_i) = LA_i = 0$, $\theta_j(L)^2 = L^2$, θ_j is an isometry of $\mathrm{NS}(X)$, hence so is $\Phi_1 = \theta_2\theta_1$. □

Let $T_X = \mathrm{NS}(X)^\perp$. We define $\Phi_2 : T_X \to T_X$ as the identity. The map (Φ_1, Φ_2) is an isometry of $\mathrm{NS}(X) \oplus T_X$.

Lemma 14 *The morphism (Φ_1, Φ_2) extends to an isometry Φ of $H^2(X,\mathbb{Z})$.*

Proof Let L_1, L_2 be the lattices $L_1 = \mathrm{NS}(X)$, $L_2 = T_X = \mathrm{NS}(X)^\perp$. Let us denote by

$$q_i : L_i^\vee / L_i \to \mathbb{Q}/2\mathbb{Z}$$

the discriminant form of L_i. By Lemma 11 and its proof, we know the form q_1 on the base w_i.

One has $L_2 = U(2) \oplus U(2) \oplus \langle -4d \rangle$. Let us take the base e_i, $1 \leq i \leq 5$ of L_2 such that the intersection matrix of the e_j's is

$$(e_ie_j)_{1\leq i,j\leq 5} = -\begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4d \end{pmatrix}.$$

The elements $w'_i = \frac{1}{2}e_i$ for $1 \leq i \leq 4$ and $w'_5 = \frac{1}{4d}e_5$ are generators of $L_2^\vee/L_2$. Let

$$\phi : L_2^\vee/L_2 \to L_1^\vee/L_1$$

be the isomorphism (called the gluing map) defined by

$$\phi(w'_i) = w_i.$$

One has $q_1(\phi(\sum a_i w'_i)) = -q_2(\sum a_i w'_i)$ i.e.

$$q_2 = -\phi^* q_1.$$

Since L_1, L_2 are primitive sub-lattices of the even unimodular lattice $H^2(X, \mathbb{Z})$ with $L_2 = L_1^\perp$, the lattice $H^2(X, \mathbb{Z})$ is obtained by gluing L_1 with L_2 by the gluing isomorphism ϕ. In other words $H^2(X, \mathbb{Z})$ is generated by all the lifts in $L_1^\vee \oplus L_2^\vee$ of the elements (w_i, w'_i), $i = 1, \dots, 5$ of the discriminant group of $L_1 \oplus L_2$.

According to general results (see e.g. [16, p. 5]), the element (Φ_1, Φ_2) of the orthogonal group of $L_1 \oplus L_2$ extends to $H^2(X, \mathbb{Z})$ if and only if the gluing map ϕ satisfies $\phi \circ \Phi_2 = \Phi_1 \circ \phi$. A simple computation gives that for $1 \leq i \leq 4$, one has $\theta_j w_i = -w_i = w_i$ (for $j \in \{1, 2\}$), thus $\Phi_1(w_i) = w_i$. Moreover we compute that

$$\theta_j(w_5) = (1 - 2k^2)w_5$$

and since $(1 - 2k^2)^2 = 1$ modulo $4d = 2k(k + 1)$, one gets $\Phi_1(w_5) = \theta_2\theta_1 w_5 = \omega_5$. Since by definition $\Phi_2(w'_i) = w'_i$ for $i = 1, \dots, 5$, we obtain the desired relation $\phi \circ \Phi_2 = \Phi_1 \circ \phi$. □

Remark 15 Because of the relation $\theta_j(w_5) = (1 - 2k^2)w_5$, $j \in \{1, 2\}$ at the end of the proof of Lemma 14, it is not possible to extend the involution θ_j to an isometry, unless $k = 1$. In that case, using the proof of Lemma 17 below, the involution θ_j extends to an effective Hodge isometry (with action by multiplication by -1 on T_X). The resulting non-symplectic involution is in fact known under the name of projection involution, see e.g. [13].

Lemma 16 *The morphism Φ is an Hodge isometry: its $\mathbb{C}$-linear extension $\Phi_\mathbb{C}$: $H^2(X, \mathbb{C}) \to H^2(X, \mathbb{C})$ preserves the Hodge decomposition.*

Proof The map Φ is the identity on the space $T_X \otimes \mathbb{C}$ containing the period. □

Lemma 17 *The Hodge isometry Φ is effective.*

Proof Since X is projective by [4, Proposition 3.11], it is enough to prove that the image by Φ of one ample class is an ample class. Let $m \geq 2$ be an integer. By [9, Proposition 4.3], the divisor $D = mL - \frac{1}{2}\sum_{i \geq 1} A_i$ is ample. The image by θ_1 of D is

$$\theta_1(D) = mL_1 - \frac{1}{2}\left(A'_1 + \sum_{i \geq 2} A_i\right)$$

where by Sect. 2 we have that A_1' is a (-2)-curve, which is disjoint from the $A_j,\ j \geq 2$, and these 16 (-2)-curves have intersection 0 with $L_1 = \theta_1(L)$. There exists an Abelian surface B' such that $X = \mathrm{Km}(B')$ and these 16 (-2)-curves are resolution of the 16 singularities in $B'/[-1]$. Moreover L_1 comes from a polarization M' on B', which clearly generates $\mathrm{NS}(B')$. Thus again by [9, Proposition 4.3], $\theta_1(D)$ is ample.

The analogous proof with $(\theta_2,\ A_2)$ instead of $(\theta_1,\ A_1)$ gives us that $\theta_2(D)$ is also ample. Since $\theta_i,\ i = 1, 2$ are involutions and $\Phi = \theta_2\theta_1$, we conclude that

$$\Phi(\theta_1(D)) = \theta_2(D)$$

is ample, and thus Φ is effective. □

We can now apply the Torelli Theorem for K3 surfaces (see [4, Chap. VIII, Theorem 11.1]): since Φ is an effective Hodge isometry there exists an automorphism $\iota : X \to X$ such that $\iota^* = \Phi$. This finishes the proof of Theorem 12. □

Remark 18 The Lefschetz formula for the fixed locus X^ι of ι on X gives

$$\chi(X^\iota) = \sum_{i=0}^{4}(-1)^i tr(\Phi|H^i(X, \mathbb{R})) = 1 + (4k^2 + 18) + 1 = 20 + 4k^2,$$

(here $\iota^* = \Phi$). If $k = 1$ then $\chi(X^\iota) = 24$ and we can easily see that X^ι contains two rational curves. Indeed in this case as remarked before (Remark 15) θ_i, $i = 1, 2$ can be extended to a non-symplectic involution (still denoted θ_i) of the whole lattice $H^2(X, \mathbb{Z})$. The fixed locus of each θ_i, $i = 1, 2$ are the curves pull-back on X of the six lines in the branching locus of the double cover of $\mathbb{P}^2$ (the θ_i, $i = 1, 2$ are the covering involutions). These curves are different except for the pull-backs ℓ_1 and ℓ_2 of two lines, which are the lines passing through the point of the branching curve corresponding to A_2 if we consider the double cover determined by the involution θ_1, respectively through the point corresponding to A_1 if we consider θ_2. So the infinite order automorphism ι corresponding to $\Phi = \theta_2\theta_1$ fixes the two rational curves ℓ_1 and ℓ_2 on X. By using results of Nikulin on non-symplectic involutions [1] the invariant sublattices $H^2(X, \mathbb{Z})$ for the action of θ_i, $i = 1, 2$ are both isometric to $U \oplus E_8(-1) \oplus \langle -2 \rangle^{\oplus 6}$.

3.2 Action of the automorphism group on Nikulin configurations

The aim of this sub-section is to prove the following result

Theorem 19 *Suppose that $k \geq 2$. There is no automorphism f of X sending the configuration $\mathcal{C} = \sum_{i=1}^{16} A_i$ to the configuration $\mathcal{C}' = A_1' + \sum_{i=2}^{16} A_i$.*

Suppose that such an automorphism f exists. The group of translations by the 2-torsion points on B acts on $X = \mathrm{Km}(B)$ and that action is transitive on the set of curves $A_1, \dots, A_{16}$. Thus up to changing f by $f \circ t$ (where t is such a translation), one can suppose that the image of A_1 is A_1'. Then the automorphism f induces a permutation of the curves $A_2, \dots, A_{16}$. The (-2)-curve $A_1'' = f^2(A_1) = f(A_1')$ is orthogonal to the 15 curves $A_i,\ i > 1$ and therefore its class is in the group generated by L and A_1. By the description of NS(X), the (-2)-class $A_1'' = aA_1 + bL$ has coefficients $a, b \in \mathbb{Z}$. Moreover a, b satisfy the Pell-Fermat equation

$$a^2 - k(k+1)b^2 = 1. \tag{3.1}$$

Let us prove:

Lemma 20 *Let $C = aA_1 + bL$ be an effective (-2)-class. Then there exists $u, v \in \mathbb{N}$ such that $aA_1 + bL = uA_1 + vA_1'$, in particular the only (-2)-curves in the lattice generated by L and A_1 are A_1 and A_1'.*

Proof If (a, b) is a solution of Eq. (3.1), then so are $(\pm a, \pm b)$. We say that a solution is positive if $a \geq 0$ and $b \geq 0$. Let us identify $\mathbb{Z}^2$ with $A = \mathbb{Z}[\sqrt{N}]$ by sending (a, b) to $a + b\sqrt{N}$, where $N = k(k+1)$. The solutions of (3.1) are units of the ring A. According to the Chakravala method solving Eq. (3.1), there exists a solution $\alpha + \beta\sqrt{N}$ (called fundamental) with $\alpha, \beta \in \mathbb{N}^*$ such that the positive solutions are the elements of the form

$$a_m + b_m\sqrt{N} = (\alpha + \beta\sqrt{N})^m,\ m \in \mathbb{N}.$$

The first term of the sequence of convergents of the regular continued fraction for $\sqrt{N}$ is

$$\frac{2k+1}{2},$$

and since $(2k+1, 2)$ is a solution of (3.1), the fundamental solution is $(\alpha, \beta) = (2k+1, 2)$.

An effective (-2)-class $C = aA_1 + bL$ either equals A_1 or satisfies $CL > 0$ and $CA_1 > 0$, therefore $b > 0$ and $a < 0$. Thus if $C \neq A_1$, there exists m such that $C = -a_mA_1 + b_mL$. Since $A_1' = 2L - (2k+1)A_1$, one obtains

$$-a_mA_1 + b_mL = \frac{b_m}{2}A_1' + ((2k+1)\frac{b_m}{2} - a_m)A_1$$

and the Lemma is proved if the coefficients $u_m = \frac{b_m}{2}$ and $v_m = (2k+1)\frac{b_m}{2} - a_m$ are both positive and in $\mathbb{Z}$. Using the relation

$$a_{m+1} + b_{m+1}\sqrt{N} = (2k+1+2\sqrt{N})(a_m + b_m\sqrt{N}),$$

that follows from an easy induction. □

Therefore we conclude that $A_1'' = A_1$ i.e. f permutes A_1 and A_1'. Let us finish the proof of Theorem 19:

Proof The class f^*L is orthogonal to $A'_1, A_2, \ldots, A_{16}$, thus this is a multiple of the class $L' = (2k+1)L - 2k(k+1)A_1$ which has the same property. Since both classes have the same self-intersection and are effective, one gets $f^*L = L'$; by the same reasoning, since $f^*A'_1 = A_1$, one gets $f^*L' = L$. By [9, Proposition 4.3], the divisor

$$D = 2L - \frac{1}{2}\sum_{i\geq 1} A_i$$

is ample, thus $f^*D = 2L' - \frac{1}{2}(A'_1 + \sum_{i\geq 2} A_i)$ is also ample and so is $D + f^*D$. Moreover $D + f^*D$ is preserved by f, thus by [11, Proposition 5.3.3], the automorphism f has finite order. Up to taking a power of it, one can suppose that f has order 2^m for some $m \in \mathbb{N}^*$. Suppose $m = 1$, ie f is an involution. Then

$$\frac{1}{2}(A_1 + A'_1) = L - kA_1$$

is fixed, there are curves A_i, $i > 1$ such that $f(A_i) = A_i$ (say s of such curves; necessarily s is odd) and f permutes the remaining curves A_j by pairs (there are $t = \frac{1}{2}(15 - s)$ such pairs). Let Γ be the lattice generated by the classes A_i fixed by f, by $A_j + f(A_j)$ if $f(A_j) \neq A_j$ and by $L - kA_1$. It is a finite index sub-lattice of $\mathrm{NS}(X)^f$, the fix sub-lattice of the Néron-Severi group. The discriminant group of Γ is

$$\mathbb{Z}/2k\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^s \times (\mathbb{Z}/4\mathbb{Z})^t.$$

Since in $\mathrm{NS}(X)$ there is at most a coefficient $\frac{1}{2}$ on L, the discriminant of $\mathrm{NS}(X)^f$ contains $\mathbb{Z}/k\mathbb{Z}$. If f was non-symplectic, then $\mathcal{M} = \mathrm{NS}(X)^f$ would be a 2-elementary lattice (see [2]; it means that the discriminant group $\mathcal{M}^*/\mathcal{M} \simeq (\mathbb{Z}/2\mathbb{Z})^a$ for some integer positive a). But for $k > 2$ this is impossible, therefore f has to be symplectic.

For $k = 2$, we use the model $Y \hookrightarrow \mathbb{P}^3$ of degree 4 with 15 nodes of X determined by the divisor $L - 2A_1$. Since f preserves $L - kA_1$, the involution on X induces an involution (still denoted f) on $\mathbb{P}^3 = |L - kA_1|$ preserving Y. Up to conjugation, f is $x \to (-x_1 : x_2 : x_3 : x_4)$ or $x \to (-x_1 : -x_2 : x_3 : x_4)$.

Suppose that f is $f : x \to (-x_1 : x_2 : x_3 : x_4)$. The hyperplane $x_1 = 0$ cuts the quartic Y into a quartic plane curve $C_0 \hookrightarrow Y$. The surface Y is a double cover of $\mathbb{P}(2, 1, 1, 1)$ branched over $C_0 \hookrightarrow \mathbb{P}(2, 1, 1, 1)$. The quartic C_0 is irreducible and reduced, since otherwise X would have Picard number > 17. The singularities on C_0 are at most nodes and the corresponding nodes on Y are fixed by f. Let us recall that the number s of fixed nodes is odd.

Suppose that C_0 contains 3 nodes. Its pull back C'_0 on X is a smooth rational curve. The rank of the sub-lattice $\mathrm{NS}(X)^f$ is $1 + s + t = 10$. By [2, Figure 1], the genus of the fixed curve C'_0 must be strictly positive, which is a contradiction.

Suppose that C_0 contains 2 nodes, then the isolated fixed point $(1:0:0:0)$ is also a node; the rank of $\mathrm{NS}(X)^f$ is still 10. One has

$$[\mathrm{NS}(X)^f:\Gamma]^2=\frac{\det\Gamma}{\det\mathrm{NS}(X)^f}=\frac{2^{2+1+2t}}{2^a}=2^{17-a},$$

thus a is odd. However by [2, Figure 1], when $\mathrm{NS}(X)^f$ has rank 10, the integer a is always even, this is a contradiction.

Suppose that C_0 contains 1 node. Its pull back on X is a smooth genus 2 curve. One has $\mathrm{rkNS}(X)^f=9$. By [2, Figure 1], since the fixed curve has genus 2, one has $a=9$, therefore

$$[\mathrm{NS}(X)^f:\Gamma]^2=2^{17-a}=2^8,$$

and there are at most 4-classes which are 2-divisible in the discriminant group

$$\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times(\mathbb{Z}/4\mathbb{Z})^7$$

of Γ. But then the discriminant group of $\mathrm{NS}(X)^f$ would contain a sub-group $\mathbb{Z}/4\mathbb{Z}$, which is a contradiction.

Suppose that f is $f:x\to(-x_1:-x_2:x_3:x_4)$ (observe that we can not exclude immediately this case since Y is singular. If Y would be smooth then such an f would correspond to a symplectic automorphism). The line $x_1=x_2=0$ or $x_3=x_4=0$ cannot be included in Y, otherwise Y would be singular along that line (this is seen using the equation of Y). The number of fixed nodes being odd, there are 1 or 3 fixed nodes of Y on these two lines (the intersection number of each lines with Y being 4).

Suppose that one node is fixed. The corresponding (-2)-curve on X must be stable, moreover $\mathrm{rkNS}(X)^f=9$. But by [2, Figure 1], there is no non-symplectic involution on a K3 surface such that $\mathrm{rkNS}(X)^f=9$ and the fix-locus is a (-2)-curve or is empty. By the same reasoning, one can discard the case of 3 stable rational curves.

We therefore proved that for any $k>1$, f must be symplectic.

A symplectic automorphism acts trivially on the transcendental lattice T_X, which in our situation has rank 5. Therefore the trace of f on $H^2(X,\mathbb{Z})$ equals $6+s>6$. But the trace of a symplectic involution equals 6 (see e.g. [28, Section 1.2]). This is a contradiction, thus f cannot have order 2 and m is larger than 1.

The automorphism $g=f^{2^{m-1}}$ has order 2 and $g(A_1)=A_1$, $g(A_1')=A_1'$, thus $g(L)=L$. There are curves A_i, $i>1$ such that $f(A_i)=A_i$ (say s of such, s is odd since A_1 is fixed) and the remaining curves A_j are permuted 2 by 2 (there are $t=\frac{1}{2}(15-s)$ such pairs). Let similarly as above Γ' be the sub-lattice generated by L, A_1 and the fix classes A_i, $A_j+g(A_j)$. It is a finite index sub-lattice of $\mathrm{NS}(X)^g$ and its discriminant group is

$$\mathbb{Z}/2k(k+1)\mathbb{Z}\times(\mathbb{Z}/2\mathbb{Z})^{s+1}\times(\mathbb{Z}/4\mathbb{Z})^t.$$

By the same reasoning as before, the automorphism g must be symplectic as soon as $k > 1$. But the trace of g is $8 + s > 6$, thus g cannot be symplectic either. Therefore we conclude that such an automorphism f does not exist. □

3.3 Consequences on the Kummer structures on *X*

A Kummer structure on a K3 surface X is an isomorphism class of Abelian surfaces B such that $X \simeq \mathrm{Km}(B)$. The following Proposition is stated in [12]; we give here a proof for completeness:

Proposition 21 *The Kummer structures on X are in one-to-one correspondence with the orbits of Nikulin configurations under the automorphism group* $\mathrm{Aut}(X)$ *of* X.

Proof Let $\mathcal{C}$ be a Nikulin configuration on the K3 surface X. By [19, Theorem 1] of Nikulin, there exists a unique (up to isomorphism) double cover $\tilde{B} \to X$ branched over $\mathcal{C}$. Moreover the minimal model B of $\tilde{B}$ is an Abelian surface, and X is the Kummer surface associated to B, $\mathcal{C}$ being the union of the exceptional curves of the resolution $X = \mathrm{Km}(B) \to B/[-1]$.

Let $\mu : X \to X$ be an automorphism sending a Nikulin configuration $\mathcal{C}$ to $\mathcal{C}'$. Let B, B' be the abelian surfaces such that $\mathcal{C}$ (resp. $\mathcal{C}'$) is the configuration associated to $\mathrm{Km}(B) = X$ (resp. $\mathrm{Km}(B') = X$).

Let $\tilde{B} \to B$ and $\tilde{B}' \to B'$ be the blow-up at the sixteen 2-torsion points of B (resp. B'). Consider the natural map $\tilde{B} \to X \stackrel{\mu}{\to} X$: it is a double cover of X branched over $\mathcal{C}'$ and ramified over the exceptional locus of $\tilde{B} \to B$, thus by the results of Nikulin we just recalled, $\tilde{B}$ is isomorphic to $\tilde{B}'$ and $B \simeq B'$.

Reciprocally, suppose that there is an isomorphism $\phi : B \to B'$. It induces an isomorphism $\tilde{\phi} : \tilde{B} \to \tilde{B}'$ that induces an isomorphism $X = \mathrm{Km}(B) \to \mathrm{Km}(B') = X$ which sends the Nikulin configuration $\mathcal{C}$ corresponding to B to the Kummer structure $\mathcal{C}'$ corresponding to B'. □

According to [12], the number of Kummer structures is finite. If $X = \mathrm{Km}(B)$ and B^* is the dual of B, by result of Gritsenko and Hulek [10] one has also $X \simeq \mathrm{Km}(B^*)$, thus if B is not principally polarized, the number of Kummer structures is at least 2.

When $\mathrm{NS}(B) = \mathbb{Z}M$, by results of Orlov [20] on derived categories, the number of Kummer structures equals 2^s where s is the number of prime divisor of $\frac{1}{2}M^2$. In our situation one has $M^2 = k(k+1)$. By Sect. 3.2 as soon as $k > 2$, there is no automorphism sending the configuration $\mathcal{C} = \sum_{i=1}^{16} A_i$ to $\mathcal{C}' = A_1' + \sum_{i=2}^{16} A_i$, thus

Corollary 22 *Suppose* $k \geq 2$. *The two Nikulin configurations* $\mathcal{C} = \sum_{i=1}^{16} A_i$ *and* $\mathcal{C}' = A_1' + \sum_{i=2}^{16} A_i$ *represent two distinct Kummer structures on* X.

Remark 23 When $k = 2$ then $\frac{k(k+1)}{2} = 3$ is divisible by one prime, thus the configurations $\mathcal{C}$ and $\mathcal{C}'$ are the two representatives of the set of Kummer structures on $X = \mathrm{Km}(B)$. Observe that X is also isomorphic to $\mathrm{Km}(B^*)$, where B^* is the dual of B. Since B is not isomorphic to B^*, the double cover of X branched over $\mathcal{C}'$ is (the blow-up of) B^*.

4 bi-double covers associated to Nikulin configurations

4.1 A hyperelliptic curve with genus $\leq 2k$ and a point of multiplicity $2(2k+1)$ on the Abelian surface B

We keep the notations as above: (B, M) is a polarized Abelian variety with $M^2 = k(k+1)$ and $\text{Pic}(B) = \mathbb{Z}M$. The associated K3 surface $X = \text{Km}(B)$ contains the 17 smooth rational curves

$$A_1, A_1', \ A_2, \ldots, A_{16}$$

such that $A_1, \ldots, A_{16}$ are the 16 disjoint (-2)-curves arising from the Kummer structure, A_1' is a (-2)-curve such that $A_1', \ A_2, \ldots, A_{16}$ is a Nikulin configuration and

$$A_1 A_1' = 4k + 2.$$

Let $\pi : \tilde{B} \to B$ be the blow-up of B at the 16 points of 2-torsion, so that there is a natural double cover $\tilde{B} \to X = \text{Km}(B)$ branched over the 16 exceptional divisors.

Let $\tilde{\Gamma}$ be the pull-back of A_1' on $\tilde{B}$ and let Γ be the image of $\tilde{\Gamma}$ on B. We denote by $E \hookrightarrow \tilde{B}$ the (-1)-curve above A_1. Let us prove the following result

Proposition 24 *The curve $\Gamma \hookrightarrow B$ is hyperelliptic, it has geometric genus $\leq 2k$ and has a unique singularity, which is a point of multiplicity $2(2k+1)$. The curve Γ is in the linear system $|4M|$, in particular $\Gamma^2 = 16k(k+1)$.*

Proof The singularities on a curve that is the union of two smooth curves on a smooth surface are of type

$$\mathfrak{a}_{2m-1}, \ m \geq 1,$$

where an equation of an $\mathfrak{a}_{2m-1}$ singularity is $\{x^{2m} - y^2 = 0\}$. This is well-known by experts but we couldn't find a reference and we therefore sketch a proof. At a singularity p, there are local parameters x, y such that C_1 is given by $y = 0$. By the implicit function theorem, we reduce to the case where the curve C_2 has equation $y = x^m$ for some $m > 0$. Then the singularity has equation $\{y(y - x^m) = 0\}$, which after a variable change becomes $\{x^{2m} - y^2 = 0\}$.

Let us denote by α_m the number of $\mathfrak{a}_{2m-1}$ singularities on the union $A_1 + A_1'$. Since a $\mathfrak{a}_{2m-1}$ singularity contributes to m in the intersection of A_1 and A_1', one has

$$\sum_{m \geq 0} m\alpha_m = 4k + 2.$$

By [4, Table 1, Page 109], the curve $\tilde{\Gamma} \hookrightarrow \tilde{B}$ has a singularity $\mathfrak{a}_{m-1}$ above a singularity $\mathfrak{a}_{2m-1}$ of $A_1 + A_1'$ (by abuse of language a $\mathfrak{a}_0$-singularity means a smooth point). Let Γ' be de normalization of $\tilde{\Gamma}$; a $\mathfrak{a}_{2m-1}$-singularity contributes in the ramification locus of the double cover $\Gamma' \to A_1$ (induced by $\tilde{\Gamma} \to A_1$) by 1 if m is odd and 0 if m is even. Therefore the geometric genus of Γ is

$$2g(\Gamma) - 2 = 2 \cdot (-2) + \sum_{m\, odd} \alpha_m \leq 4k + 2,$$

which gives $g(\Gamma) \leq 2k$. The singularities of $\tilde{\Gamma}$ are at its intersection with E, and since

$$\tilde{\Gamma} E = \frac{1}{2} \pi_1^* A_1 \pi_1^* A_1' = A_1 A_1',$$

we obtain $\tilde{\Gamma} E = 4k + 2$. Since E is contracted by the map $\tilde{B} \to B$, the curve Γ (image of $\tilde{\Gamma}$) has a unique singular point of multiplicity $4k + 2$.

Since $A_1' = 2L - (2k+1)A_1$, its pull back on $\tilde{B}_1$ is $4\tilde{M} - 2(2k+1)\tilde{\Gamma}$ and its image Γ has class $4M$, thus $\Gamma^2 = 16k(k+1)$. □

Remark 25 Let us choose the point of multiplicity $2(2k+1)$ of Γ as the origin 0 of the group B. By construction the curve Γ does not contain any non-trivial 2-torsion point of B_1.

The problem of the intersection of A_1 and A_1'

It is a difficult question to understand how the curves A_1 and A_1' intersect on the Kummer surface $X = \mathrm{Km}(B)$. For $k = 1$ and 2 we know that these curves intersects transversally in $4k + 2$ points, and thus $g(\Gamma) = 2k$. For $k = 1$, it follows from the geometric description of the Jacobian Kummer surface as a double cover of the plane branched over 6 line. For $k = 2$ it is a by-product of [24].

In [6, Section 5, pp. 54–56] Bryan, Oberdieck, Pandharipande and Yin, quoting results of Graber, discuss on a related problem which is about hyperelliptic curves on Abelian surfaces. Let $f : C \to B$ be a degree 1 morphism from a hyperelliptic curve C to an Abelian surface B with image $\bar{C}$, such that the polarization $[\bar{C}]$ is generic. Let $\iota : C \to C$ be the hyperelliptic involution.

Conjecture 26 *(see [6]) Suppose B generic among polarized Abelian surfaces. The differential of f is injective at the Weierstrass points of C, and no non-Weierstrass points p is such that $f(p) = f(\iota(p))$.*

In our situation, that Conjecture means that the rational curves A_1 and A_1' meet transversally. Indeed if they meet at a point tangentially with order $m \geq 2$, then the curve above A_1' has a $\mathfrak{a}_{m-1}$ singularity. If m is even, there is no branch points above that singular point, and thus there are points $p, \iota(p)$ (with p non-Weierstrass) which are mapped to the same point by f. If m is odd and > 1, then the curve C above A_1' has a singularity $\mathfrak{a}_{m-1}$ of type "cusp", the differential of its normalization is 0.

Construction of (nodal or smooth) rational curves on K3 surfaces is an important problem, see e.g. [11, Chapter 13] for a discussion. The existence of two smooth rational curves C_1, C_2 intersecting transversely and such that $C_1 + C_2$ is a multiple nH of a polarization H is also a key point for obtaining the existence of an integer n

such that there exists an integral rational curve in $|nH|$, see [11, Chapter 13, Theorem 1.1] and its proof.

4.2 Invariants of the bidouble covers associated to the special configuration

Let us define

$$D_1 = A'_1,\ D_2 = A_1,\ D_3 = \sum_{i=2}^{16} A_j.$$

By Nikulin results, the divisors $\sum_{i=2}^{16} A_j + A_1$ and $\sum_{i=2}^{16} A_j + A'_1$ are 2-divisible and therefore there exists L_1, L_2, L_3 such that

$$2L_i = D_j + D_k$$

for $\{i, j, k\} = \{1, 2, 3\}$. Each L_i defines a double cover

$$\pi_i : \tilde{B}_i \to X$$

branched over $D_j + D_k$ (here $\tilde{B}_1 = \tilde{B}$). For $i = 1, 2$, above the 16 (-2)-curves of the branch locus of $\pi_i : \tilde{B}_i \to X$ there are 16 (-1)-curves. Let $\tilde{B}_i \to B_i$ be the contraction map, so that the surface B_i $(i = 1, 2)$ is an Abelian surface.

The divisors $D_i, L_i,\ i \in \{1, 2, 3\}$ are the data of a bi-double cover

$$\pi : V \to X$$

which is a $(\mathbb{Z}/2\mathbb{Z})^2$-Galois cover of X branched over the curves $A'_1,\ A_i,\ i \geq 1$. By classical formulas, the surface V has invariants

$$\begin{gathered}\chi(O_V) = 4 \cdot 2 + \tfrac{1}{2} \sum L_i^2 = k \\ K_V^2 = (\sum L_i)^2 = 8k - 30.\end{gathered}$$

The surface V contains 30 (-1)-curves, which are above the 15 curves $A_i,\ i > 1$. The surface V is smooth if and only if the intersection of A_1 and A'_1 is transverse, i.e. if Conjecture 26 holds. Let us suppose that this is indeed the case, then one has moreover the formula

$$p_g(V) = p_g(X) + \sum h^0(X, L_i).$$

The space $H^0(X, L_i)$ is 0 for $i = 1, 2$ because the double covers branched over $D_2 + D_3$ or $D_1 + D_3$ are Abelian surfaces B_i $(i = 1, 2)$ and $1 = p_g(B_i) = p_g(X) + h^0(X, L_i) \geq 1$. It remains to compute $h^0(X, L_3)$. The divisor $L_3 = A_1 + A'_1$ is big and nef (see Sect. 2). By Riemann-Roch, one has

$$\chi(L_3) = \frac{1}{2} L_3^2 + 2 = k + 2.$$

By Serre duality and Mumford vanishing Theorem, $h^1(L_3) = h^1(L_3^{-1}) = 0$. Moreover $h^2(L_3) = h^0(-L_3) = 0$, thus $h^0(L_3) = k + 2$ and therefore $p_g(V) = k + 3$. Let $V \to Z$ be the blow-down map of the 30 (-1)-curves on V which are above the 15 (-2)-curves A_i, $i > 1$ in X. We thus obtain:

Proposition 27 *Suppose that A_1 and A_1' intersect transversally. The surface Z has general type and its invariants are*

$$\chi = k,\ K_Z^2 = 8k,\ p_g(Z) = k + 3,\ \textit{and}\ q = 4.$$

The surface Z is minimal as we see by using the rational map of Z onto the Abelian surface B_1.

Remark 28 The surface Z satisfies

$$c_1^2 = 2c_2 = 8k.$$

Among surfaces with $c_1^2 = 2c_2$ there are surfaces whose universal covers is the bi-disk $\mathbb{H} \times \mathbb{H}$. For $k = 1$, it turns out that Z is the product of two genus 2 curves, thus its universal cover is $\mathbb{H} \times \mathbb{H}$. For $k = 2$, we obtain the so-called Schoen surfaces, whose universal cover is not $\mathbb{H} \times \mathbb{H}$ (see [7], [24]).

Let (W, ω) be a smooth projective algebraic variety of dimension $2n$ over $\mathbb{C}$ equipped with a holomorphic $(2, 0)$-form of maximal rank $2n$. Let us recall that a n dimensional subvariety $Z \subset W$ is called Lagrangian if the restriction of ω to Z is trivial. We remark that

Proposition 29 *The surface Z is a Lagrangian surface in $B_1 \times B_2$.*

Proof In [5], Bogomolov and Tschinkel associate a Lagrangian surface to the data of Kummer surfaces $S_1 = \mathrm{Km}(A_1)$, $S_2 = \mathrm{Km}(A_2)$ and a $K3$ surface S such that there is a rational map $S \to S_i$, $i = 1, 2$.

In our situation, we take $S_1 = S_2 = S = \mathrm{Km}(B)$, we consider the Kummer structure $\mathrm{Km}(B_1)$ for S_1 and the Kummer structure $\mathrm{Km}(B_2)$ (see also Remark 9) for S_2, and the identity map for $S \to S_i$.

According to [5, Section 3], the bi-double cover Z is a sub-variety of $B_1 \times B_2$ which is Lagrangian. □

Let us now discuss what is happens if we do not make assumption on the transversality of the intersection of A_1 and A_1'. Let us denote by $\mathbb{A}_m$ a surface singularity with germ

$$\{x^{m+1} = y^2 + z^2\}$$

and by $\mathfrak{a}_m$ a curve singularity with germ $\{x^{m+1} = y^2\}$.

Since A_1, A_1' are smooth, the singularities of $A_1 + A_1'$ are of type $\mathfrak{a}_{2m-1}$, $m > 0$. Let s be a $\mathfrak{a}_{2m-1}$-singularity of $A_1 + A_1'$. Recall that $\tilde{B}_1$ is the cover of X branched over $\sum_{i=1}^{16} A_i$. The curve singularity above s in $\pi_1^* A_1' \subset \tilde{B}_1$ is a $\mathfrak{a}_{m-1}$ singularity (see e.g. [4, Table 1, P. 109]).

Thus above the singularity s of type $\mathfrak{a}_{2m-1}$ of $A_1 + A_1'$, the surface V has a singularity of type $\mathbb{A}_{m-1}$, (where in fact a $\mathbb{A}_0$ (resp. $\mathfrak{a}_0$) point is a smooth point).

The singularities $\mathbb{A}_m$ are ADE singularities and by the Theorem of Brieskorn on simultaneous resolution of singularities, they do not change the values of K^2, χ and p_g of the surface $\tilde{V}$ which is the minimal resolution of V (we consider the two successive double covers $V \to \tilde{B}_1$ and $\tilde{B}_1 \to X$).

Thus the surface Z obtained by taking the minimal desingularisation of V and the contraction of the 30 exceptional curves has the same invariants $\chi(Z)$, K_Z^2 and $p_g(Z)$ as if the intersection of A_1 and A_1' was transverse. We observe that the image of the natural map $Z \to B_1 \times B_2$ is also a Lagrangian surface by [5, Section 3].

Let α_m be the number of $\mathfrak{a}_{2m-1}$ singularities on $A_1 + A_1'$. Using Miyaoka's bound on the number of quotient singularities on a surface of general type (here to be the surface B_3, the double cover of X branched over $A_1 + A_1'$), one gets:

$$\sum \left(n - \frac{1}{n} \right) \alpha_n \leq \frac{4}{3} k.$$

For $k = 1$, a configuration of $6\mathfrak{a}_1$ singularities on $A_1 + A_1'$ is the only possibility. For $k = 2$, the possibilities are

$$10\mathfrak{a}_1,\ 8\mathfrak{a}_1 + \mathfrak{a}_3,\ 7\mathfrak{a}_1 + \mathfrak{a}_5,$$

but we know from explicit computations in [24] that for a generic Abelian surface polarized by M with $M^2 = 6$, the singularities of $A_1 + A_1'$ are $10\mathfrak{a}_1$. For $k = 3$ the possibilities are

$$14\mathfrak{a}_1,\ 12\mathfrak{a}_1 + \mathfrak{a}_3,\ 10\mathfrak{a}_1 + 2\mathfrak{a}_3,\ 11\mathfrak{a}_1 + \mathfrak{a}_5,\ 10\mathfrak{a}_1 + \mathfrak{a}_7.$$

4.3 The *H*-constant of the curve Γ

Let X be a surface, $\mathcal{P}$ be a non-empty finite set points on X and let $\bar{X} \to X$ be the blow-up of X at $\mathcal{P}$. For a curve C let $\bar{C}_{\mathcal{P}}$ be the strict transform of C on $\bar{X}$. The H-constant of C is defined by

$$H(C) = \inf_{\mathcal{P}} \frac{(\bar{C}_{\mathcal{P}})^2}{\#\mathcal{P}}$$

and the H-constant of X is $H(X) = \inf_C H(C)$, where the infimum is taken over reduced curves. The H-constants have been introduced for studying the bounded negativity Conjecture, which predicts that there exists a bound b_X such that for any reduced curve C on X, one has $C^2 \geq b_X$.

Let A be the generic Abelian surface polarized by M with $M^2 = k(k+1)$ and let Γ be the curve with a unique singularity which is of multiplicity $4k+2$ and is in the numerical equivalence class of $4M$. One computes immediately

$$H(\Gamma) = \Gamma^2 - (4k + 2)^2 = -4.$$

For the moment, one do not know curves on Abelian surfaces which have H-constants lower than -4. We use these curves in a more thorough study of curves with low H-constants in [26].

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