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On the Cohomology of the Stover Surface

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ABSTRACT
We study a surface discovered by Stover which is the surface with minimal Euler number and maximal automorphism group among smooth arithmetic ball quotient surfaces. We study the natural map $\wedge^2 H^1(S,\mathbb{C}) \to H^2(S,\mathbb{C})$ and discuss the problem related to the so-called Lagrangian surfaces. We obtain that this surface S has maximal Picard number and has no higher genus fibrations. We compute that its Albanese variety A is isomorphic to $(\mathbb{C}/\mathbb{Z}[\alpha])^7$, for $\alpha = e^{2i\pi/3}$.

KEYWORDS
ball quotient surfaces; Lagrangian surfaces; Hurwitz ball quotients

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1. Introduction

By the recent work of Stover [Stover 14], the number of automorphisms of a smooth compact arithmetic ball quotient surface $X = \Gamma\backslash\mathbb{B}_2$ is bounded by $288 \cdot e(X)$, where $e(X)$ denotes the topological Euler number of X.

Furthermore, Stover characterizes the arithmetic ball quotient surfaces X whose automorphism groups attain this bound, which by analogy with Hurwitz curves, he calls *Hurwitz ball quotients*. All such surfaces are finite Galois coverings of the Deligne–Mostow orbifold $\Lambda\backslash\mathbb{B}_2$ corresponding to the quintuple $(2/12, 2/12, 2/12, 7/12, 11/12)$ (see [Mostow 88, Stover 14]).

In [Stover 14], Stover constructs a Hurwitz ball quotient S with Euler number $e(S) = 63$ and automorphism group $\mathrm{Aut}(S)$ isomorphic to $U_3(3) \times \mathbb{Z}/3\mathbb{Z}$, of order $18,144 = 2^5 3^4 7$. He shows that S is the unique Hurwitz ball quotient with Euler number $e \le 63$. Having this property the surface S can be seen as the two-dimensional analog of the Klein's quartic which is the unique curve (uniformized by the ball $\mathbb{B}_1$) with minimal genus and maximal possible automorphism group.

The surface grew up out of the list of maximal arithmetic lattices in $PU(2, 1)$ studied by Prasad–Yeung and Cartwright–Steger in connection with fake projective planes, [Prasad and Yeung 07, Cartwright and Steger 10]. The strategy for finding fake projective planes there was to list all maximal arithmetic lattices of small covolume first and then to search for subgroups of suitable index. One of these maximal groups is the lattice Λ above (denoted by $\mathcal{C}_{11}$ in [Prasad and Yeung 07]).

Determining an explicit presentation of Λ in terms of generators and relations, and using MAGMA algorithm LowIndexSubgroups, Cartwright and Steger showed that Λ does not contain any lattice which is isomorphic to the fundamental group of a fake projective plane. But after showing that Λ has the smallest covolume among the arithmetic ball quotients, the MAGMA procedure LowIndexNormalSubgroups is used to discover the smallest normal and torsion-free subgroup, leading to the surface S. There is another remarkable surface associated with Λ, namely the Cartwright–Steger surface X (see [Cartwright and Steger 10]), the unique ball quotient surface with $e(X) = 3$ and first Betti number 2 (see [Cartwright et al.]). The surface S is a covering of X.

Our aim is to study more closely the cohomology of this particular surface S, which we will call the *Stover surface* in the following. This surface S has the following numerical invariants (see [Stover 14]):

$e(S)$	$H_1(S,\mathbb{Z})$	q	$p_g = h^{2,0}$	$h^{1,1}$	$b_2(S)$
63	$\mathbb{Z}^{14}$	7	27	35	89

Let V be a vector space. Let us recall that a two-vector $w \in \wedge^2 V$ has *rank 1* or is *decomposable* if there are vectors $w_1, w_2 \in V$ with $w = w_1 \wedge w_2$. A vector $w \in \wedge^2 V$ has *rank 2* if there exist linearly independent vectors $w_i \in V$, $i = 1, \dots, 4$ such that $w = w_1 \wedge w_2 + w_3 \wedge w_4$.

Let B be an Abelian fourfold and let $p : S \to B$ be a map such that $p(S)$ generates the group B. We say that S is *Lagrangian with respect to* p if there exists a basis $w_1, \dots, w_4$ of $p^*H^0(B, \Omega_B)$ such that the rank-2 vector

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$w = w_1 \wedge w_2 + w_3 \wedge w_4$ is in the kernel of the natural map $\phi^{2,0} : \wedge^2 H^0(S, \Omega_S) \to H^0(S, K_S)$.

Theorem 1.

(i) The natural map

$$\phi^{1,1} : H^0(S, \Omega_S) \otimes H^1(S, \mathcal{O}_S) \to H^1(S, \Omega_S)$$

is surjective with a 14-dimensional kernel. The kernel of the map

$$\phi^{2,0} : \wedge^2 H^0(S, \Omega_S) \to H^0(S, K_S)$$

is seven-dimensional and contains no decomposable elements. The set of rank-2 vectors in $Ker(\phi^{2,0})$ *is a quadric hypersurface.*

(ii) There exist an infinite number (up to isogeny) of maps $p : S \to B$ *(where B is an Abelian fourfold) such that S is Lagrangian with respect to p.*

(iii) The Albanese variety of S is isomorphic to $(\mathbb{C}/\mathbb{Z}[\alpha])^7$*, for* $\alpha = e^{2i\pi/3}$*.*

(iv) The surface S has maximal Picard number.

Using the Castelnuovo–De Franchis theorem, the fact that there are no decomposable elements in the kernel of $\phi^{2,0}$ means that S has no fibration $f : S \to C$ onto a curve of genus $g > 1$. Moreover, Theorem 1 implies that S has the remarkable feature that both maps

$$\phi^{2,0} : \wedge^2 H^{1,0}(S) \to H^{2,0}(S)$$
$$\phi^{1,1} : H^{1,0}(S) \otimes H^{0,1}(S) \to H^{1,1}(S)$$

have a non-trivial kernel. After Schoen surfaces (see [Ciliberto et al. 15, Remark 2.6]), this is the second example of surfaces enjoying such properties. For a detailed description of this subject, see [Amorós et al. 96, Barja 07, Bastianelli et al. 10, Bogomolov and Tschinkel 00, Campana 95, Causin and Pirola 06] and beginning of Section 4.

For the motivation and a historic account on surfaces with maximal Picard number, we refer to [Beauville 14].

2. The second lower central quotient of the fundamental group of *S*

Let $\Pi := \pi_1(X)$ be the fundamental group of a manifold X. The group $H_1(X, \mathbb{Z})$ is the abelianization of Π: $H_1(X, \mathbb{Z}) = \Pi/\Delta$, where $\Delta := [\Pi, \Pi]$ is the derived subgroup of Π, i.e., the subgroup generated by all elements $[h, g] = h^{-1}g^{-1}gh$, $h, g \in \Pi$.

The second group in the lower central series $[\Delta, \Pi]$ is the group generated by commutators $[h, g]$, with $h \in \Delta$, $g \in \Pi$. It is a normal subgroup of the commutator group Δ. According to [Beauville], we have the following results:

Proposition 2.

(1) (Sullivan) Let X be a compact connected Kähler manifold. There exists an exact sequence

$$0 \to Hom(\Delta/[\Delta, \Pi], \mathbb{R}) \to \wedge^2 H^1(X, \mathbb{R}) \to H^2(X, \mathbb{R}).$$

(2) (Beauville) Suppose $H_1(X, \mathbb{Z})$ *is torsion free. Then the group* $\Delta/[\Delta, \Pi]$ *is canonically isomorphic to the cokernel of the map*

$$\mu : H_2(X, \mathbb{Z}) \to Alt^2(H^1(X, \mathbb{Z}))$$
$$\text{given by } \mu(\sigma)(a, b) = \sigma \cap (a \wedge b),$$

where $Alt^2(H^1(X, \mathbb{Z}))$ *is the group of skew-symmetric integral bilinear forms on* $H^1(X, \mathbb{Z})$*.*

In the case of the Stover surface, computer calculations give us the following result:

Theorem 3. *Let* $\Pi = \pi_1(S)$ *be the fundamental group of the Stover surface and* $\Delta = [\Pi, \Pi]$*. The group* $\Delta/[\Delta, \Pi]$ *is isomorphic to* $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}^{28}$*.*

Proof. By the construction of S [Stover 14], the fundamental group Π is isomorphic to the kernel $ker(\varphi)$ of the unique epimorphism $\varphi : \Lambda \longrightarrow G$ from the Deligne–Mostow lattice Λ corresponding to the quintuple (2/12, 2/12, 2/12, 7/12, 11/12) onto the finite group $G = U_3(3) \times \mathbb{Z}/3\mathbb{Z}$. The lattice Λ is described by Mostow in [Mostow 88] as a complex reflection group, and by generators and relation by Cartwright and Steger in [Cartwright and Steger 10]. This lattice has the following presentation:

$$\Lambda = \langle j, u, v, b | u^4, v^8, [u, j], [v, j], j^{-3}v^2, uvuv^{-1}uv^{-1}, (bj)^2(vu^2)^{-1}, [b, vu^2], b^3, (bvu^3)^3 \rangle.$$

MAGMA command `LowIndexSubgroups` is used to identify the unique subgroup $\Gamma \lhd \Lambda$ of index 3, which is $\Gamma = \langle u, jb, bj \rangle$. Using the primitive permutation representation of $U_3(3)$ of degree 28, MAGMA is able to identify an homomorphism φ from Γ onto $U_3(3)$ induced from the assignment:

$$u \mapsto (3, 8, 23, 20)(4, 24, 6, 12)(7, 9, 14, 22) \times (10, 19, 11, 13)(15, 16, 21, 18)(17, 26, 27, 25)$$

$$jb \mapsto (1, 9, 20, 12, 19, 23, 6, 16)(2, 27, 14, 17, 13, 26, 15, 25)(3, 24)(4, 5, 10, 21, 7, 11, 28, 8)$$

$$bj \mapsto (1, 13, 20, 15, 19, 2, 6, 14)(4, 9, 10, 12, 7, 23, 28, 16)(5, 27, 21, 17, 11, 26, 8, 25)(22, 24).$$

This homomorphism extends to an homomorphism φ from Λ onto G such that $\Pi = ker(\varphi)$ is a torsion-free

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normal subgroup in Λ; it is the fundamental group of S (see [Stover 14]). Let $\Delta = [\Pi, \Pi]$ and $\Delta_2 = [\Delta, \Pi]$. It is easy to check that Δ_2 is a distinguished subgroup in Π. The image of Δ under the quotient map $\Pi \longrightarrow \Pi/\Delta_2$ is Δ/Δ_2, but we observe that it is also equal the commutator subgroup $[\Pi/\Delta_2, \Pi/\Delta_2]$, and therefore, the computation of Δ/Δ_2 is reduced to one of the derived groups $[\Pi/\Delta_2, \Pi/\Delta_2]$.

The MAGMA command `g:=Rewrite(G,g)` is used to obtain generators and relations of both subgroups $\Gamma < \Lambda$ and $\Pi < \Gamma$. The command `NilpotentQuotient(.,2)` applied to Π describes Π/Δ_2 in terms of a polycyclic presentation. The derived subgroup $[\Pi/\Delta_2, \Pi/\Delta_2]$ is obtained with `DerivedGroup(.)` applied to Π/Δ_2. Finally, applying the MAGMA function `AQInvariants` to $[\Pi/\Delta_2, \Pi/\Delta_2]$, MAGMA computes that the structure of Δ/Δ_2 is $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}^{28}$. □

Corollary 4. *The dimension of the kernel of* $\wedge^2 H^1(S, \mathbb{R}) \to H^2(S, \mathbb{R})$ *is 28.*

3. Computation of the map $\wedge^2 H^1(S, \mathbb{C}) \to H^2(S, \mathbb{C})$

Let A be the Albanese variety of the Stover surface S. The invariants are as follows:

$$H_1(A, \mathbb{Z}) = H_1(S, \mathbb{Z}) = \mathbb{Z}^{14},$$
$$H_2(A, \mathbb{Z}) = \wedge^2 H_1(A, \mathbb{Z}),\ H^{2,0}(A) = \wedge^2 H^{1,0}(S)$$
$$H^{1,1}(A) = H^{1,0}(S) \otimes H^{0,1}(S),\ H^{0,2}(A) = \wedge^2 H^{0,1}(S),$$

and

$H_1(A, \mathbb{Z})$	q	$h^{2,0}(A)$	$h^{1,1}(A)$	$b_2(A)$
$\mathbb{Z}^{14}$	7	21	49	91

We have a map respecting Hodge decompositions

$$\begin{array}{ccccc} H^{2,0}(A) & \oplus & H^{1,1}(A) & \oplus & H^{0,2}(A) \\ \downarrow & & \downarrow & & \downarrow \\ H^{2,0}(S) & \oplus & H^{1,1}(S) & \oplus & H^{0,2}(S) \end{array}$$

which is an equivariant map of $\mathrm{Aut}(S)$-modules. By Corollary 4, the kernel of that map is 28-dimensional; it is an $\mathrm{Aut}(S)$-module.

In order to state Theorem 5, one needs to recall some properties of the group $U_3(3)$ and state some notations. According to the Atlas tables [Conway et al. 85], the group $U_3(3)$ has 14 irreducible representations χ_i, $1 \le i \le 14$ of respective dimension 1,6,7,7,7,14,21,21,21,27,28,28,32,32.

The irreducible representations of $\mathrm{Aut}(S) = U_3(3) \times \mathbb{Z}/3\mathbb{Z}$ are the χ_i^t, $i = 1, \dots, 14$, $t = 0, 1, 2$ where $(g, s) \in U_3(3) \times \mathbb{Z}/3\mathbb{Z}$ acts on the same space as χ_i with action $(g, s) \cdot v = \alpha^s g(v)$ with $\alpha = e^{2i\pi/3}$ a primitive third root of unity.

Theorem 5.

(1) The image of S by the Albanese map $\vartheta : S \to A$ is two-dimensional.

(2) The map $H^{1,1}(A) \to H^{1,1}(S)$ is surjective, with a 14-dimensional kernel isomorphic to χ_6^0 as an $\mathrm{Aut}(S)$*-module.*

(3) We have $H^1(S, \mathbb{Z}) = \chi_3^1 \oplus \chi_3^2$ and $H^{1,1}(S) = \chi_1^0 \oplus \chi_3^0 \oplus \chi_{10}^0$, as $\mathrm{Aut}(S)$*-modules.*

(4) The kernel of the natural map $\wedge^2 H^0(S, \Omega_S) \to H^0(S, K_S)$ is seven-dimensional, isomorphic to χ_3^0 as a $\mathrm{Aut}(S)$*-module.*

Proof. Suppose that the image of S in A is one-dimensional. Then there exists a smooth curve C of genus 7 and a fibration $f : S \to C$; the map $\wedge^2 H^0(S, \Omega_S) \to H^0(S, K_S)$ is the zero map and the kernel of $\wedge^2 H^1(S, \mathbb{C}) \to H^2(S, \mathbb{C})$ would be at least 42-dimensional, which is impossible.

According to the Atlas character table [Conway et al. 85], the possibilities for the $U_3(3)$-module $H_1(S, \mathbb{Z}) = H_1(A, \mathbb{Z}) = \mathbb{Z}^{14}$ are as follows:

$$\chi_3^{\oplus 2},\ \mathcal{R}_{\mathbb{Z}}(\chi_4) = \mathcal{R}_{\mathbb{Z}}(\chi_5) = \chi_4 \oplus \chi_5,\ \chi_4^{\oplus 2},\ \chi_5^{\oplus 2} \text{ or } \chi_6$$

where $\mathcal{R}_{\mathbb{Z}}(\chi_j)$ is the restriction to $\mathbb{Z}$ of the seven-dimensional complex representation χ_j defined over $\mathbb{Z}[i]$. It cannot be $\chi_4^{\oplus 2}$ nor $\chi_5^{\oplus 2}$ because these are not defined over $\mathbb{Z}$ (some traces of elements are in $\mathbb{Z}[i] \setminus \mathbb{Z}$). We cannot have $H^1(S, \mathbb{Z}) = \chi_6$ since χ_6 remains irreducible, but $H^1(S, \mathbb{Z}) \otimes \mathbb{C} = H^{1,0} \oplus H^{0,1}$ is a Hodge decomposition on which the representation of $U_3(3)$ splits.

By duality, the kernel of $H^{2,0}(A) \to H^{2,0}(S)$ has the same dimension d as the kernel of $H^{0,2}(A) \to H^{0,2}(S)$. Let k be the dimension of the kernel of the $U_3(3)$-equivariant map $H^{1,1}(A) \to H^{1,1}(S)$. We have $28 = k + 2d$. Moreover, since $h^{1,1}(S) = 35$ and $h^{1,1}(A) = 49$, we obtain $28 \ge k \ge 14$.

Let us suppose that $H^1(S, \mathbb{Z}) = \chi_4 \oplus \chi_5$. Then the representation $H^{1,1}(A)$ equals $\chi_4 \otimes \chi_5 = \chi_1 + \chi_7 + \chi_{10}$ (of dimension $1 + 21 + 27$). An Abelian variety on which a finite group G acts possesses a G-invariant polarization (for example $\sum_{g \in G} g^* L$, where L is some polarization). Therefore, the one-dimensional $\mathrm{Aut}(S)$-invariant space of $H^{1,1}(A)$ is generated by the class of an ample divisor and the natural map $\vartheta^* : H^{1,1}(A) \to H^{1,1}(S)$ is injective on that subspace. Thus, the map ϑ^* has a kernel of dimension $k = 21$, 27 or 48. This is impossible because $k + 2d$ equals 28.

We conclude that $H^1(S, \mathbb{Z}) = \chi_3^{\oplus 2}$. Thus, we have

$$H^{2,0}(A) = \wedge^2 \chi_3 = \chi_3 \oplus \chi_6$$

(the dimensions are $21 = 7 + 14$) and

$$H^{1,1}(A) = \chi_3^{\otimes 2} = \chi_1 \oplus \chi_3 \oplus \chi_6 \oplus \chi_{10}$$

($49 = 1 + 7 + 14 + 27$). By checking the possibilities, we obtain $k = 14$, $H^{1,1}(S) = \chi_1 \oplus \chi_3 \oplus \chi_{10}$, and the map $H^{1,1}(A) \to H^{1,1}(S)$ is surjective. The kernel of the map $H^{2,0}(A) \to H^{2,0}(S)$ is isomorphic to χ_3, of dimension 7; the action of $U_3(3)$ on $H^{2,0}(S)$ is then $H^{2,0}(S) = \chi_6 \oplus \chi$, where χ is a 13-dimensional representation. □

Proposition 6.

a) The Albanese variety A of S is isomorphic to $(\mathbb{C}/\mathbb{Z}[\alpha])^7$, *for* $\alpha = e^{2i\pi/3}$.

b) The surface S has maximal Picard number.

Remark 7. Since A is CM, it follows that S is Albanese standard [Schoen 06], i.e., the class of its image inside its Albanese variety A sits in the subring of $H^*(A, \mathbb{Q})$ generated by the divisor classes. This is in contrast with the above-mentioned Schoen surfaces (see [Ciliberto et al. 15]).

Proof. Let $\sigma \in \mathbb{Z}/3\mathbb{Z}$ be a generator of the center of $Aut(S) = U_3(3) \times \mathbb{Z}/3\mathbb{Z}$. It corresponds to an element $\sigma' \in \Lambda$ normalizing Π in Λ, such that the larger group Π' generated by Π and σ' contains Π with index 3. Using MAGMA, one finds that we can choose $\sigma' = j^4$, where j is the order 12 element described in the proof of Theorem 3.

The quotient surface S/σ of S by σ is equal to $\mathbb{B}_2/\Pi'$. The fundamental group of S' is Π'/Π'_{tors}, where Π'_{tors} is the subgroup of Π' generated by torsion elements. Using MAGMA, one finds that Π' has a set of eight generators, in which seven are considered to be torsion elements. Using these elements, we readily compute that Π'/Π'_{tors} is trivial. Therefore, the space of one-forms on S that are invariant by σ is 0.

Using the symmetries of $U_3(3)$, we see that the action of σ on the tangent space $H^0(S, \Omega_S)^*$ is the multiplication by α or α^2. After possible permutation of σ and σ^2, we can suppose it is α and we see that the representation of $Aut(S)$ on $H_1(S, \mathbb{Z})$ is $\chi_3^1 \oplus \chi_3^2$. The lattice $H_1(S, \mathbb{Z}) \subset H^0(S, \Omega_S)^*$ is a $\mathbb{Z}[\alpha]$-module. The ring $\mathbb{Z}[\alpha]$ is a principal ideal domain. Therefore, $H_1(S, \mathbb{Z}) = \mathbb{Z}[\alpha]^7$ (for the choice of a certain basis) and A is isomorphic to $(\mathbb{C}/\mathbb{Z}[\alpha])^7$.

Therefore, A has maximal Picard number and all the classes of $H^{1,1}(A)$ are algebraic. These classes remain of course algebraic under the map $H^{1,1}(A) \to H^{1,1}(S)$, which is surjective. Thus, S is a surface with maximal Picard number. □

4. Lagrangian surfaces and the Stover surface

Let X be a smooth projective surface. The cohomology group $H^1(X, \mathbb{Q})$ and the homomorphism

$$\phi^2 : \wedge^2 H^1(X, \mathbb{C}) \to H^2(X, \mathbb{C})$$

determine important properties of the fundamental group of X (its nilpotent completion, see [Amorós et al. 96]). One is interested to understand the $(2, 0)$-part

$$\phi^{2,0} : \wedge^2 H^0(X, \Omega_X) \to H^0(X, K_X)$$

of the map ϕ^2, and specifically by the kernel of $\phi^{2,0}$.

Let us recall that a vector $w \in \wedge^2 H^0(X, \Omega_X)$ has rank n if the minimal number k of 1-forms w_i, $i = 1 \dots k$ such that $w = w_1 \wedge w_2 + \cdots + w_{k-1} \wedge w_k$ equals $2n$. The theorem of Castelnuovo–De Franchis is the following result : a rank-1 form $w = w_1 \wedge w_2$ is in the kernel of $\phi^{2,0}$ if and only if there exists a fibration $f : X \to C$, where C is a smooth curve of genus > 1 such that the forms w_1, w_2 are the pull-back of two 1-forms on C.

In the case of the existence of a rank-1 vector in $\ker \phi^{2,0}$, one thus has a geometric interpretation of it. Moreover, in that case, the fundamental group of X surjects onto the fundamental group of the base curve of the fibration. Hence, $\pi_1(X)$ has an infinite nilpotent tower.

In general, the non-triviality of $\ker \phi^{2,0}$ implies that the fundamental group is either nilpotent of class ≥ 2 or has an infinite nilpotent tower. Construction of surfaces with no fibrations onto curves of genus ≥ 2 and a non trivial kernel $\ker \phi^2$ is usually difficult (see [Bastianelli et al. 10]), and we do not know if there exists a surface with a nilpotent fundamental group of class ≥ 3 (see [Campana 95]).

Let B be an Abelian fourfold and let $p : S \to X$ be a map such that $p(X)$ generates B (as an Abelian group). Let us recall that according to [Bastianelli et al. 10], the surface X is called Lagrangian (with respect to p) if there exists a basis $w_1, \dots, w_4$ of $p^* H^0(B, \Omega_B)$ such that the rank-2 vector $w = w_1 \wedge w_2 + w_3 \wedge w_4$ is in the kernel of the natural map $\phi^{2,0} : \wedge^2 H^0(X, \Omega_X) \to H^0(X, K_X)$. In that case, the image of X in B is a Lagrangian subvariety of B.

Let us prove

Theorem 8.

(1) The seven-dimensional space $Ker(\phi^{2,0})$ *contains rank-1 element. The algebraic set of rank-2 vectors in* $Ker(\phi^{2,0})$ *is a quadric* $\tilde{Q} \subset Ker(\phi^{2,0})$.

(2) There exists an infinite number (up to isogeny) of maps $p : S \to B$ *where B is an Abelian fourfold such that S is Lagrangian with respect to p.*

(3) There exists an infinite number (up to isogeny) of maps $p : S \to B$, *where B is an Abelian fourfold such that*

$$\tilde{Q} \cap p^* H^0(B, \wedge^2 \Omega_B) = \{0\},$$

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and for some of them we even have $Ker(\phi^{2,0}) \cap p^*H^0(B, \wedge^2\Omega_B) = \{0\}$.

(4) *The generic rank-2 element* w *in* $\tilde{Q} \subset Ker(\phi^{2,0})$ *does not correspond to any morphism to an Abelian fourfold.*

Proof. We proved in Theorem 5 that

$$H^{2,0}(A) = \wedge^2\chi_3 = \chi_3 \oplus \chi_6$$

and the kernel of $\phi^{2,0} : H^{2,0}(A) \to H^{2,0}(S)$ is the seven-dimensional subspace with representation χ_3. In a basis $\gamma = (e_1, \ldots, e_7)$ of $\chi_3 = H^0(S, \Omega_S) = H^{1,0}(S)$, the following two matrices A, B are generators of the group $U_3(3)$:

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Using the basis $\beta = (e_{ij})_{1\le i<j\le 7}$ of $\wedge^2\chi_3$ ($e_{ij} = e_i \wedge e_j$) with order $e_{ij} \le e_{st}$ if $i < s$ or $i = s$ and $j \le t$, one computes that the subspace $Ker(\phi^{2,0}) = \chi_3 \subset \wedge^2\chi_3$ is generated by the columns of the matrix $M \in M_{21,7}$, where ${}^tM = (N, 2I_7)$, for

$$N = \begin{pmatrix} 0 & 0 & 2 & -2 & -2 & -2 & 0 & 2 & 2 & -2 & 2 & 2 & 2 & 2 \\ -1 & 0 & 0 & 2 & 4 & 0 & 1 & -3 & -3 & 1 & -3 & -4 & -2 & -4 \\ 0 & -2 & 0 & -2 & -2 & 0 & -2 & 2 & 2 & 0 & 0 & 2 & 2 & 2 \\ -1 & -2 & 2 & 0 & -2 & 0 & -1 & 1 & 3 & 1 & 1 & 0 & 0 & 2 \\ -1 & 1 & -1 & 3 & 1 & 3 & 0 & -4 & -2 & 2 & 0 & -4 & -2 & -2 \\ 0 & 3 & -3 & 1 & -1 & 1 & 1 & -3 & -3 & -1 & 1 & 0 & -2 & -2 \\ 1 & 1 & 1 & 3 & 3 & 1 & 2 & -2 & 0 & 0 & 0 & -2 & 0 & -2 \end{pmatrix} \in M_{7,14}$$

and I_7 the 7×7 identity matrix. Thus, we obtain the ideal I_V of the algebraic set V of pairs $(w_1, w_2) \in \chi_3 \oplus \chi_3$ such that $w_1 \wedge w_2 \in Ker(\phi^{2,0}) \subset \wedge^2\chi_3$. That ideal is generated by 14 homogeneous quadratic polynomials in the variables $x_1, \ldots, x_{14}$. Let W be the algebraic set of pairs $(w_1, w_2) \in \chi_3 \oplus \chi_3$ such that $w_1 \wedge w_2 = 0 \in \wedge^2\chi_3$. The ideal I_W of W is generated by the 2×2 minors of the matrix

$$L = \begin{pmatrix} x_1 & \ldots & x_7 \\ x_8 & \ldots & x_{14} \end{pmatrix}.$$

Since $W \subset V$, we have $Rad(I_V) \subset Rad(I_W)$, where $Rad(I)$ is the radical of an ideal I. On the other hand, using Maple, one can check that the 21 minors of L are in $Rad(I_V)$. Hence, $Rad(I_W) \subset Rad(I_V)$, and $V = W$. We therefore conclude that the kernel of $\phi^{2,0}$ contains no decomposable elements.

A two-vector w over a characteristic 0 field can be expressed uniquely as $w = \sum_{i,j} a_{ij} e_i \wedge e_j$, where $a_{ij} = -a_{ji}$. The rank of the vector w is half the rank of the (skew-symmetric) coefficient matrix $A_w := (a_{ij})_{1\le i,j\le 7}$ of w [Bryant et al. 91, Thm 1.7 & Remark p. 13]. Thus, the two-vector $w = a_1v_1 + \ldots + a_7v_7$ in $Ker(\phi^{2,0})$ (where the v_i, $i = 1 \ldots 7$ are the vectors corresponding to the columns of the matrix M) is a rank-2 vector if and only if the 49 6×6 minors of the matrix A_w are 0. The radical of the ideal generated by these minors is principal, generated by a homogeneous quadric in $a_1, \ldots, a_7$ whose associated symmetric matrix is

$$Q = \begin{pmatrix} 7 & 3 & 3 & 1 & -3 & -3 & -5 \\ 3 & 7 & 3 & 3 & 1 & -3 & -3 \\ 3 & 3 & 7 & 3 & 3 & 1 & -3 \\ 1 & 3 & 3 & 7 & 3 & 3 & 1 \\ -3 & 1 & 3 & 3 & 7 & 3 & 3 \\ -3 & -3 & 1 & 3 & 3 & 7 & 3 \\ -5 & -3 & -3 & 1 & 3 & 3 & 7 \end{pmatrix}.$$

Therefore, $w \in Ker(\phi^{2,0})$ has rank 2 if and only if $(a_1, \ldots, a_7)Q^t(a_1, \ldots, a_7) = 0$.

The point $(10 + 8\alpha, -7, 0, 0, 7, 0, 0)$ lies on the associated smooth quadric $\tilde{Q}$, and therefore $\tilde{Q}(\mathbb{Q}[\alpha])$ is infinite. Let be w be a two-vector in $\tilde{Q}(\mathbb{Q}[\alpha])$. The decomposable vector $\wedge^2 w \neq 0$ has coordinates in $\mathbb{Q}[\alpha]$ in the basis $(e_{i1} \wedge \ldots . \wedge e_{i4})$ of $\wedge^4 H^0(S, \Omega_S)$. The corresponding four-dimensional vector space W is therefore generated by 4 vectors $w_1, \ldots, w_4$ with coordinates over $\mathbb{Q}[\alpha]$ in the basis $\gamma = (e_1, \ldots, e_7)$ of $H^0(S, \Omega_S)$.

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One computes that the image of $\mathbb{Q}[U_3(3) \times \mathbb{Z}/3\mathbb{Z}]$ in $M_7(\mathbb{Q}[\alpha])$ is 49-dimensional over $\mathbb{Q}[\alpha]$. Thus,

$$\mathbb{Q}[U_3(3) \times \mathbb{Z}/3\mathbb{Z}] = M_7(\mathbb{Q}(\alpha)) \; (= End(A) \otimes \mathbb{Q})$$

in the basis γ ($H_1(S, \mathbb{Q}[\alpha])^* \subset H^0(S, \Omega_S)$, is the $\mathbb{Q}[\alpha]$-vector space generated by $e_1, \ldots, e_7$) and therefore there exists a morphism $p : S \to E^4 = B$ (where $E = \mathbb{C}/\mathbb{Z}[\alpha]$) such that $W = p^* H^0(B, \Omega_B)$. By the hypothesis the image $p(S)$ generates B. By construction,

$$\wedge^2 p^* H^0(B, \Omega_B) \cap Ker(\phi^{2,0})$$

is at least one-dimensional since it contains w, and therefore S is Lagrangian for p.

Conversely, the trace of an order 2 automorphism $\sigma \in \mathrm{Aut}(S) \subset \mathrm{Aut}(A)$ acting on the tangent space of A at 0 equals to -1. Therefore, the image B' of the endomorphism $p : \sigma - 1_A$, where 1_A is the identity of A as an Abelian fourfold. Using Maple, one computes that

$$\wedge^2 p^* H^0(B, \Omega_B) \cap Ker(f) = \{0\}.$$

Let $\vartheta : S \to A$ be the Albanese map of S, and let $q : A \to A$ be an endomorphism with a four-dimensional image and a representation in $M_7(\mathbb{Q}) \subset M_7(\mathbb{Q}(\alpha))$ in the basis γ. Since the matrix Q is positive definite, we have

$$\wedge^2 p^* H^0(B, \Omega_B) \cap \tilde{Q} = \{0\},$$

where p is the map $p = q \circ \vartheta : S \to B$. Therefore, S is not Lagrangian with respect to p. □

Remark 9. Let X be a surface and let $\phi^{2,0} : \wedge^2 H^0(X, \Omega_X) \to H^0(X, K_X)$ be the natural map. Let $d = \dim Ker(\phi^{2,0})$ and $q = \dim H^0(X, \Omega_X)$. In the proof of Theorem 8, we saw that the set of rank-k vectors in $Ker(\phi^{2,0})$ is a determinantal variety: the intersection of minors of size $\geq 2k + 1$ of some anti-symmetric matrix of size $q \times q$ with linear entries in d variables. It is quite remarkable that for Stover's surface the set of rank-2 vectors (obtained as the zero set of 49 6×6 minors of a size $q = 7$ matrix) is a hypersurface in $Ker(\phi^{2,0})$. This hypersurface is the unique $U_3(3)$-invariant quadric in $Ker(\phi^{2,0})$.

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