Genus 2 curves configurations on Fano surfaces

by

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Abstract. We study the configurations of genus 2 curves on the Fano surfaces of cubic threefolds. We establish a link between some involutive automorphisms acting on such a surface S and genus 2 curves on S. We give a partial classification of the Fano surfaces according to the automorphism group generated by these involutions and determine the configurations of their genus 2 curves. We study the Fano surface of the Klein cubic threefold for which the 55 genus 2 curves generate a rank $25 = h^{1,1}$ index 2 subgroup of the Néon-Severi group.

1. Introduction

Let S be a smooth surface which verifies the following Hypothesis:

HYPOTHESIS 1. The variety S is a smooth complex surface of general type. The cotangent sheaf Ω_S of S is generated by its space $H^0(\Omega_S)$ of global sections and the irregularity $q = \dim H^0(\Omega_S)$ satisfies $q > 3$.

Let T_S be the tangent sheaf, $\pi : \mathbb{P}(T_S) \to S$ be the projection and let

$$\psi : \mathbb{P}(T_S) \to \mathbb{P}(H^o(\Omega_S)^*) = \mathbb{P}^{q-1}$$

be the cotangent map of S defined by: $\pi_* \psi^* \mathcal{O}_{\mathbb{P}q-1}(1) = \Omega_S$.
In [9] a curve $C \hookrightarrow S$ is called non-ample if there is a section

$$t : C \to \mathbb{P}(T_S)$$

such that $\psi(t(C))$ is a point. The cotangent sheaf of S is ample (i.e. $\psi^* \mathcal{O}_{\mathbb{P}q-1}(1)$ is ample) if and only if S does not contain non-ample curves [9]. In other words, the non-ample curves are the obstruction for the cotangent sheaf to be ample. A natural example of a non-ample curve is given by a smooth curve $C \hookrightarrow S$ of genus 1, where a section $t : C \to \mathbb{P}(T_S)$ such that $\psi(t(C))$ is a point is given by the natural quotient:

$$\Omega_S \otimes \mathcal{O}_C \to \Omega_C \,.$$

A step further after the study of ampleness of $\psi^* \mathcal{O}_{\mathbb{P}q-1}(1)$ is the study of the very-ampleness. In projective geometry, the simplest object after points are lines. An obstruction

for invertible sheaf $\psi^* \mathcal{O}_{\mathbb{P}^{q-1}}(1)$ to be very ample is that the surface contains a curve C for which there is a section $t : C \to \mathbb{P}(T_S)$ such that $\psi(t(C))$ is a line. We say that such a curve C satisfies property ($*$). A natural example of such a curve is given by a smooth curve $C \hookrightarrow S$ genus 2, where the section $t : C \to \mathbb{P}(T_S)$ is given by the natural quotient:

$$\Omega_S \otimes \mathcal{O}_C \to \Omega_C .$$

In the present paper, we classify the curves that satisfy property ($*$) on Fano surfaces and we focus our attention on the genus 2 curves.

By definition, a Fano surface parametrizes the lines of a smooth cubic threefold $F \hookrightarrow \mathbb{P}^4$. This scheme is a surface S that verifies Hypothesis 1 and has irregularity $q = 5$. By the Tangent Bundle Theorem [4] 12.37, the image F' of the cotangent map $\psi : \mathbb{P}(T_S) \to \mathbb{P}(H^0(\Omega_S)^*)$ of S is a hypersurface of $\mathbb{P}(H^0(\Omega_S)^*) \simeq \mathbb{P}^4$ that is isomorphic to the original cubic F. Moreover, when we identify F and F', the triple $(\mathbb{P}(T_S), \pi, \psi)$ is the universal family of lines on F.

By a generic point of the cubic threefold F goes 6 lines. For s a point of S the Fano surface of lines on F, we denote by L_s the corresponding line on F and we denote by C_s the incidence divisor:

$$C_s = \overline{\{t/t \neq s, \ L_t \text{ cuts } L_s\}} .$$

This divisor is smooth and has genus 11 for s generic.

Let $D \hookrightarrow S$ be a curve such that there is a section $D \to \mathbb{P}(T_S)$ mapped onto a line by ψ (property ($*$)). There exists a point t of S such that this line is L_t. As all the lines of $\psi_* \pi^* D$ goes through L_t, there exists a residual divisor R such that:

$$C_t = D + R .$$

THEOREM 2. *A curve D on S that verifies property ($*$) is a non-genus 1 irreducible component of an incidence divisor C_t.*

If an incidence divisor C_t is not irreducible, then it can split only in the following way:
a) $C_t = E + R$ where E is an elliptic curve, R verifies property ($*$) and has genus 7, it is an irreducible fiber of a fibration of S and $RE = 4$.
b) $C_t = E + R + E'$ where E, E' are two smooth curves of genus 1. The curve R verifies property ($*$) and has arithmetical genus 4.
c) $C_t = D + R$ where D is a smooth genus 2 curve, the curve R has arithmetical genus 4, $D^2 = -4$, $DR = 6$ and $R^2 = -3$. The curves D and R satisfy property ($*$).
d) If $D \neq D' \hookrightarrow S$ are two smooth curves of genus 2 on S, then $C_s D = 2$ and:

$$DD' \in \{0, 1, 2\} .$$

The involutive automorphisms of a Fano surface S can be classified into two type, say I and II. We prove in [9] that there is a bijection between the set of elliptic curves E on S and the set of involutions σ_E of type I, in such a way that:

$$(\sigma_E \sigma_{E'})^{EE'} = 1 ,$$

where EE' is the intersection number of E and E'. This formula and the full classification of groups generated by involutions of type I, enable us to determine all the configurations of genus 1 curves on Fano surfaces. We give here an analogous result for automorphisms of type II and genus 2 curves:

THEOREM 3. *To each involution g of type II in* $\mathrm{Aut}(S)$, *there corresponds a curve D_g of genus 2 on S.*

Let G be one of the groups: $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{D}_n$, $n \in \{2, 3, 5, 6\}$ *(dihedral group of order 2n),* $\mathbb{A}_5$ *(alternating group) and* $PSL_2(\mathbb{F}_{11})$. *There exists a Fano surface S with the following properties*:

A) *We can identify G to a sub-group of* $\mathrm{Aut}(S)$. *By this identification, each involution of G has type II.*

B) *The number N of genus 2 curves on S is as follows*:

Group G	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{D}_2$	$\mathbb{D}_3$	$\mathbb{D}_5$	$\mathbb{D}_6$	$\mathbb{A}_5$	$PSL_2(\mathbb{F}_{11})$
N	1	3	3	5	7	15	55

These curves are smooth for S generic.

C) *The intersection number of the genus 2 curves D_g, D_h is given by the formula*:

$$D_g D_h = \begin{cases} -4 & if & g = h \\ 0 & if & o(gh) = 2 \text{ or } 6 \\ 2 & if & o(gh) = 3 \\ 1 & if & o(gh) = 5 \end{cases}$$

where $o(f)$ is the order of the element f.

D) *For $G = \mathbb{D}_3$ (resp. $G = \mathbb{D}_5$), let D be the sum of the 3 (resp. 5) genus 2 curves D_g (g involution of G). The divisor D is a fiber of a fibration of S.*

E) *For $G = \mathbb{A}_5$, the 15 genus 2 curves generates a sub-lattice Λ of* $\mathrm{NS}(S)$ *of rank 15, signature $(1, 14)$ and discriminant $2^{24}3^6$. For S generic, Λ has finite index inside* $\mathrm{NS}(S)$. *There exist an infinite number of such surfaces with maximal Picard number $h^{1,1} = 25$.*

F) *For $G = PSL_2(\mathbb{F}_{11})$, S is the Fano surface of the Klein cubic*

$$x_1 x_5^2 + x_5 x_3^2 + x_3 x_4^2 + x_4 x_2^2 + x_2 x_1^2 = 0.$$

The sublattice Λ' of the Néon-Severi group $\mathrm{NS}(S)$ *generated by the 55 smooth genus 2 curves has rank $25 = h^{1,1}$ and discriminant $2^2 11^{10}$. The group* $\mathrm{NS}(S)$ *is generated by Λ' and the class of an incidence divisor.*

We want to mention that Edge in [6] was the first to give a classification of cubics threefolds according to subgroups of $PSL_2(\mathbb{F}_{11})$, in order to understand the geometry of a genus 26 curve embedded in $\mathbb{P}^4$, with automorphism group $PSL_2(\mathbb{F}_{11})$.

After proving the above classification Theorem, we discuss on its completeness. We finish this paper by a conjecture about the existence of a surface of special type in $\mathbb{P}^4$ with a particular configuration of 55 (-2)-curves. The conjecture comes naturally after studying the configuration of the 55 genus 2 curves on the Fano surface of the Klein cubic.

2.   Genus 2 curves on Fano surfaces

Let S be the Fano surface of lines on a smooth cubic threefold $F \hookrightarrow \mathbb{P}^4$. We denote by L_s the line on F that corresponds to a point s of S. Let C_s be the divisor that parametrizes the lines going through L_s.

Let $D \hookrightarrow S$ be a curve such that there is a section $D \to \mathbb{P}(T_S)$ mapped onto a line by the cotangent map ψ (property $(*)$). There exists a point t of S such that this line is L_t. As all the lines of $\psi_* \pi^* D$ goes through L_t, there exists a residual divisor R such that:

$$C_t = D + R \,.$$

Let us prove Theorem 2. We first need to recall some well-known facts on the Fano surfaces, the main reference used here is [3].

Let s be a point of S and r be a generic point of C_s. The plane X that contains the lines L_s and L_r cuts the cubic F along the lines L_s, L_r and a third one denoted by $L_{j_s(r)}$ such that:

$$XF = L_s + L_r + L_{j_s(r)} \,.$$

The rational map $j_s : C_s \to C_s$ extends to an automorphism of C_s. Moreover the quotient Γ_s of C_s by j_s parametrizes the planes X that contain L_s and that cut F along three lines. The scheme Γ_s is a plane quintic, and:

LEMMA 4 ([3], Lemma 2).   *The quotient $j_s : C_s \to \Gamma_s$ is ramified over the singular points of Γ_s. These singular points are ordinary double points.*

We need also the following facts:

LEMMA 5 ([4], Lemma 10.4, Proposition 10.3 formula 10.11).   *For a point s on S, the incidence divisor C_s is ample, verifies $C_s^2 = 5$ and $3C_s$ is numerically equivalent to the canonical divisor of S.*
([9], *Proposition 10, Theorem 13*). *Let s be a point of an elliptic curve $E \hookrightarrow S$. There exists an effective divisor R such that $C_s = E + R$. The curve R is a fiber of a fibration of S onto E and satisfies $RE = 4$. Moreover, $E^2 = -3$ and $C_t E = 1$.*

Let us suppose that the quintic Γ_s decomposes as $\Gamma_s = L + U$ where L is a line. As $LU = 4$, the component of C_s over L is smooth and ramified over 4 points: it has genus 1. Conversely, suppose that $C_t = E + R$ where E is an elliptic curve. Then $ER = E(C_s - E) = 4$ and j_s restricted to E is a degree 2 morphism ramified above 4 points of L. Hence, the image L of E by j_s is a rational curve that cuts the image of R into 4 points. Thus L is a line.

Now, we proved in [9] that an incidence divisor C_t can have at most 2 smooth genus 1 irreducible components. Hence there is at most 2 lines that are components of the divisor Γ_s. The case a) and b) of Theorem 2 occur (see [9]).

Suppose now that the quintic Γ_t splits as $\Gamma_t = Q + T$ with Q an irreducible quadric. Then the irreducible component D of C_t over Q is branched over 6 points of Q, thus this is a smooth curve of genus 2.

Let us recall that the Albanese map $S \to \mathrm{Alb}(S)$ is an embedding. We consider S as a subvariety of $\mathrm{Alb}(S)$.

LEMMA 6. *Let D be a smooth curve of genus 2 on S, let $L_t \hookrightarrow F$ be the image by ψ of the section obtained by the quotient $\Omega_S \otimes \mathcal{O}_D \to \Omega_D$ and let $J(D)$ be the Jacobian of D. Consider the natural map : $J(D) \hookrightarrow \mathrm{Alb}(S)$. The tangent space*

$$T J(D) \hookrightarrow T \mathrm{Alb}(S) = H^0(\Omega_S)^*$$

is the subjacent space to the line $L_t \hookrightarrow \mathbb{P}(H^0(\Omega_S)^*) = \mathbb{P}^4$, i.e. $H^0(L_t, \mathcal{O}_{L_t}(1))^* = T J(D)$.

Proof. This is consequence of the definition of the cotangent map and the fact that the section $D \to \mathbb{P}(T_S)$ given by the surjection

$$\Omega_S \otimes \mathcal{O}_D \to \Omega_D$$

is mapped onto the line L_t by the cotangent map $\psi : \mathbb{P}(T_S) \to F$. $\square$

Let us suppose that the quintic Γ_t decomposes as $\Gamma_t = Q + Q' + L$ with Q, Q' irreducible quadrics and L a line. Then there exist 2 smooth curves of genus 2 D, D' and an elliptic curve E such that:

$$C_t = D + D' + E$$

Let $\vartheta : S \to \mathrm{Alb}(S)$ be an Albanese morphism of S. By the previous Lemma, the 2 curves $\vartheta(D)$, $\vartheta(D')$ are contained on the same abelian surface $J(D) = J(D') \hookrightarrow \mathrm{Alb}(S)$. Let B be the quotient of A by the smallest abelian variety containing $J(D)$ and E in $\mathrm{Alb}(S)$. The morphism $S \to \mathrm{Alb}(S) \to B$ contracts the ample divisor C_t onto a point and its image generates B of dimension > 0: it is impossible.

Thus if Γ_t is reducible: $\Gamma_t = Q + T$ with Q a smooth quadric, the other component T is an irreducible cubic.

Let R be the residual divisor of D in the incidence divisor: $C_t = D + R$. We have:

$$5 = C_t^2 = D^2 + R^2 + 2DR.$$

We know that $DR = 6$ because the quadric Q and the cubic T such that $\Gamma_t = Q + T$ cut each others in 6 points, thus:

$$D^2 + R^2 = -7.$$

Moreover, $R^2 = (C_t - D)^2 = D^2 - 2C_t D + 5$. Thus $2D^2 - 2C_t D + 5 = -7$ and

$$D^2 - C_t D = -6$$

The divisor $3C_t$ is linearly equivalent to a canonical divisor of S, thus $D^2 + 3C_t D = 2$, and we obtain $C_t D = 2$ and $D^2 = -4$ and then $C_t R = 3$, $R^2 = -3$, R has genus 4.

Let $D' \neq D$ be a second smooth curve of genus 2. As $DC_s = 2$, we obtain $D'(D + R) = 2$. The intersection numbers $D'R$ and DD' are positive, hence $0 \leq DD' \leq 2$.

3. Link between genus 2 curves and automorphisms, the Klein cubic

3.1. Involutive automorphisms and genus 2 curves on Fano surfaces

Let t be a point of S. We can suppose that the corresponding line L_t on the cubic is given by $x_1 = x_2 = x_3 = 0$, then this cubic has equation:

$$\{C + 2x_4 Q_1 + 2x_5 Q_2 + x_4^2 x_1 + 2x_4 x_5 \ell + x_5^2 x_3 = 0\}$$

where C, Q_1, Q_2, ℓ are forms in the variables x_1, x_2, x_3. Let Γ_t be the scheme that parametrizes the planes containing L_t and such that their intersection with F is 3 lines. This scheme Γ_t is the quintic given by the equation:

$$(x_1 x_3 - \ell^2)C - Q_1^2 x_3 + 2Q_1 Q_2 \ell - Q_2^2 x_1 = 0$$

on the plane $\mathbb{P}(x_1 : x_2 : x_3)$ (see [3], equation (6)).

DEFINITION 7. An automorphism conjugated to the involutive automorphism:

$$f : x \to (x_1 : x_2 : x_3 : -x_4 : -x_5)$$

in $PGL_5(\mathbb{C})$ is called a harmonic inversion of lines and planes.

The harmonic inversion f acts on the cubic if and only if $Q_1 = Q_2 = 0$. In that case, a plane model of Γ_t is

$$\Gamma_t = \{(x_1 x_3 - \ell^2)C = 0\}$$

and the cubic has equation:

$$F_2 = \{C + x_4^2 x_1 + 2x_4 x_5 \ell + x_5^2 x_3 = 0\}.$$

If the conic $\mathcal{Q} = \{x_1 x_3 - \ell^2 = 0\}$ is smooth, the divisor C_t, which is the double cover of Γ_t branched over the singularities of Γ_t, splits as follows:

$$C_t = D + R$$

with D a smooth curve of genus 2. If this conic $\mathcal{Q}$ is not smooth, then C_s splits as follows:

$$C_s = E + E' + R$$

with E and E' two elliptic curves. Note that in the last case, we can suppose $\ell = 0$ and we see immediately that two other automorphisms act on the cubic threefold. The divisor $E + E'$ has also genus 2. We proved:

COROLLARY 8. *To each harmonic inversion acting on the cubic threefold F, there corresponds a curve of arithmetical genus 2 on the Fano surface of F. Such a curve is smooth or sum of 2 elliptic curves.*

Let g be an harmonic inversion acting on F. It acts also on S and $H^0(\Omega_S)$.

LEMMA 9. *The trace of the action of g on $H^0(\Omega_S)$ is equal to 1.*

Proof. We can suppose that the cubic is:

$$F_2 = \{C + x_4^2 x_1 + 2x_4 x_5 \ell + x_5^2 x_3 = 0\}$$

and that $g : x \to (x_1 : x_2 : x_3 : -x_4 : -x_5)$. By the Tangent Bundle Theorem [4], Theorem 12.37, we can consider the homogeneous coordinates $x_1, \ldots, x_5$ as a basis of $H^0(\Omega_S)$. Thus, we see that the action of g on S has trace 1 or -1.
The line $x_1 = x_2 = x_3 = 0$ lies in the cubic and correspond to a fixed point t of f. The action of f on the tangent space $TS_t = H^0(L_t, \mathcal{O}(1)) \hookrightarrow H^0(\Omega_S)^*$ is thus the identity or the multiplication by -1. As t is an isolated fixed point, it is the multiplication by -1 and the action of g on $H^0(\Omega_S)$ has trace 1. $\qquad\square$

REMARK 10. A) As a genus 2 curve has self-intersection number -4, there is a finite number of such curves on a Fano surface.

B) It is equivalent to consider couples (Γ, M) where Γ is a plane quintic and M is an odd theta characteristic of Γ or to consider couples (S, t) where S a Fano surface and t a point on S (for these facts, see by example [5], Theorem 4.1).

The moduli space of Fano surfaces with a smooth genus 2 curve is thus equal to the moduli of reducible plane quintics $\Gamma_t = Q + C$, (Q smooth quadric, C irreducible cubic), plus a theta characteristic.

The moduli space of $Q \simeq \mathbb{P}^1$ plus 6 points $p_1, \ldots, p_6$ has dimension 3. We can suppose that Q is the quadric $\{x^2 + y^2 + z^2 = 0\}$ inside $\mathbb{P}^2$. The cubics through $p_1, \ldots, p_6$ form a 3 dimensional linear system. Thus the moduli space of Fano surfaces that contains a genus 2 curve is 6 dimensional.

C) Given a reducible quintic curve $Q+C$ (with only simple singularities and Q smooth of degree 2, C irreducible), there are 32 choices ([7], Corollary 2.7) of odd theta characteristics giving non-isomorphic Fano surfaces containing a point t such that $\Gamma_t \simeq Q + C$. For only one of these Fano surfaces, the genus 2 curve corresponds to an order 2 automorphism, namely the Fano surface of the cubic:

$$F_2 = \{C + x_4^2 x_1 + 2x_2 x_4 x_5 + x_5^2 x_3 = 0\}.$$

D) As a genus 2 curve is a cover of $\mathbb{P}^1$ branched over 6 points, any genus 2 curve can be embedded inside a Fano surface, in ∞^3 ways.

E) We have not found a geometric interpretation of the action of an harmonic inversion on a Fano surface. That explains perhaps why we do not know if our classification of group generated by harmonic inversions acting on smooth cubic threefolds is complete or not.

3.2. Partial classification of configurations of genus 2 curves, the Fano surface of the Klein cubic threefold.

Let G be a group generated by order 2 elements acting (faithfully) on a 5 dimensional space V such that the trace of an order 2 element is equal to 1 (as in Lemma 9).

We say that a Fano surface S (resp. the cubic threefold F corresponding to S) has type G if G acts on S and the representation of G on $H^0(\Omega_S)$ is isomorphic to the one on V.

For the groups $\mathbb{D}_n$, $n \in \{2, 3, 5, 6\}$, $\mathbb{A}_5$ and $PSL_2(\mathbb{F}_{11})$, we take the unique 5 dimensional representation such that the elements have the following trace according to their order:

order	2	3	5	6	11
trace	1	-1	0	1	$1/2(1 + i\sqrt{11})$

The aim of this paragraph is to prove the two following theorems:

THEOREM 11. A) *The number N of curves of genus 2 on a Fano surface of type G is given by the following table*:

Group G	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{D}_2$	$\mathbb{D}_3$	$\mathbb{D}_5$	$\mathbb{D}_6$	$\mathbb{A}_5$	$PSL_2(\mathbb{F}_{11})$
N	*1*	*3*	*3*	*5*	*7*	*15*	*55*

For S generic these genus 2 *curves are smooth.*

B)   *The* 3 *genus* 2 *curves of a surface of type* $\mathbb{D}_2$ *are disjoint.*

C)   *A surface S of type* $\mathbb{D}_3$ *contains* 3 *curves* D_1, D_2, D_3 *of genus* 2, *such that* $D_i D_j = 2$ *if* $i \neq j$.

There exists a fibration $\gamma : S \to E$ *onto an elliptic curve such that* $D_1 + D_2 + D_3$ *is a fiber of* γ.

D)   *A surface S of type* $\mathbb{D}_5$ *contains* 5 *curves* $D_1, \dots, D_5$ *of genus* 2, *such that* $D_i D_j = 1$ *if* $i \neq j$.

There exists a fibration $\gamma : S \to E$ *onto an elliptic curve such that* $D_1 + \cdots + D_5$ *is a fiber of* γ.

E)   *Let S be a surface of type* $\mathbb{D}_6$. *There exists a fibration* $\gamma : S \to E$ *onto an elliptic curve and genus* 2 *curves* $D_1, \dots, D_7$ *such that* $F_1 = D_1 + D_2 + D_3$ *and* $F_2 = D_4 + D_5 + D_6$ *are* 2 *fibers of* γ *and* D_7 *is contained inside a third fiber. By C), the fibers* F_1 *and* F_2 *corresponds to the* 2 *subgroups* $\mathbb{D}_3$ *of* $\mathbb{D}_6$.

F)   *A Fano surface S of type* $\mathbb{A}_5$ *contains* 15 *smooth curves of genus* 2. *They generates a sub-lattice* Λ *of* $\mathrm{NS}(S)$ *of rank* 15, *signature* (1, 14) *and discriminant* $2^{24}3^6$. *For S generic* Λ *has finite index inside* $\mathrm{NS}(S)$. *There exist an infinite number of surfaces of type* $\mathbb{A}_5$ *with maximal Picard number* $h^{1,1} = 25$.

REMARK 12.   In [10], we give Fano surfaces of type of some groups (by example the symmetric group Σ_5), but the genus 2 curves we obtain in that way are sum of elliptic curves, and we are interested by smooth genus 2 curves.

The Klein cubic threefold:

$$F_{Kl} = \{x_1 x_5^2 + x_5 x_3^2 + x_3 x_4^2 + x_4 x_2^2 + x_2 x_1^2 = 0\}$$

is the only one cubic of type $PSL_2(\mathbb{F}_{11})$ and this group is its full automorphism group [1]. It contains 55 involutions. Let S_{Kl} be the Fano surface of lines of F_{Kl}. To each involution g of $PSL_2(\mathbb{F}_{11})$, we denote by $D_g \hookrightarrow S_{Kl}$ the corresponding curve of arithmetical genus 2 on S (see Corollary 8).

THEOREM 13.   *The* 55 *genus* 2 *curves* D_g *are smooth. Their configuration is as follows*:

$$(3.1) \qquad\qquad D_g D_h = \begin{cases} -4 & if & g = h \\ 0 & if & o(gh) = 2 \ or \ 6 \\ 2 & if & o(gh) = 3 \\ 1 & if & o(gh) = 5 \end{cases}$$

where we denote by $o(g)$ *the order of an automorphism* g.

The sublattice Λ' *of the Néon-Severi group* $\mathrm{NS}(S_{Kl})$ *generated by the* 55 *genus* 2 *curves has rank* $25 = h^{1,1}(S_{Kl})$ *and discriminant* $2^2 11^{10}$.

The group $\mathrm{NS}(S_{Kl})$ *is generated by* Λ' *and the class of an incidence divisor.*

Let us prove Theorems 11 and 13.

Let D_g, D_h be 2 genus 2 curves on a Fano surface S corresponding (Corollary 8) to involutions g, h and let n be the order of gh.

LEMMA 14. *Suppose that $n \in \{2, 3, 5, 6\}$ and S has type $\mathbb{D}_n$. The intersection number $D_h D_g$ is independent of the Fano surface of type $\mathbb{D}_n$.*

Proof. The family V of cubics forms $F_{eq} \in \mathbb{P}(H^0(\mathbb{P}^4, \mathcal{O}(3)))$ such that the cubic $\{F_{eq} = 0\}$ is smooth is open.

The group $\mathbb{D}_n$ acts naturally on $H^0(\mathbb{P}^4, \mathcal{O}(3))$, the third symmetric power of $H^0(\mathbb{P}^4, \mathcal{O}(1))$. For a cubic F of type $\mathbb{D}_n$, there is a cubic form F_{eq} in V such that $F = \{F_{eq} = 0\}$ and there exists a character $\chi : \mathbb{D}_n \to \mathbb{C}^*$ such that $F_{eq} \circ g = \chi(g) F_{eq}$. For $n \in \{2, 3, 5, 6\}$, the smoothness condition on F implies that χ is trivial (we used a computer).

Therefore, the family V_n of cubic of type $\mathbb{D}_n$ is an open set of a projective space. Now it suffice to consider a smooth curve inside V_n between two points. That gives a flat family of smooth cubic threefolds and of Fano surfaces. On each Fano surfaces we get two genus 2 curves and the two families of genus 2 curves are flat (see [8], III, 9.7). The intersection number of the genus 2 curves is therefore constant. □

Let S be a Fano surface of type $\mathbb{D}_5$ and let D_g and D_h be as in the previous lemma.

LEMMA 15. *We have*: $D_g D_h = 1$.

Proof. Let S_0 be the Fano surface of the cubic $F_0 = \{x_1^3 + \cdots + x_5^3 = 0\}$. Let σ be an element of the permutation group Σ_5 ; σ acts on $\mathbb{C}^5$ by $x \to (x_{\sigma 1}, \ldots, x_{\sigma 5})$ and acts on F by taking the projectivisation.

The involutions $g = (1, 3)(4, 5)$ and $h = (1, 2)(3, 5)$ have trace 1, their product is an order five element with trace equal to 0. In [9], we prove that the corresponding genus 2 curves are $D = E_1 + E_2$ and $D' = E'_1 + E'_2$ with E_1, E_2, E'_1, E'_2 elliptic curves on S_0 such that $DD' = 1$. We now apply Lemma 14 to S of type $\mathbb{D}_5$. □

Let S_{Kl} be the Fano surface of the Klein cubic. Let x, y, z, w be elements of $\{0, 1, 2\}$ and let $\Lambda_{x,y,z,w}$ be the lattice generated by 55 generators L_g with intersection numbers:

$$L_g L_h = \begin{cases} -4 & if & g = h \\ x & if & o(gh) = 2 \\ y & if & o(gh) = 3 \\ z & if & o(gh) = 5 \\ w & if & o(gh) = 6 \end{cases}.$$

By Theorem 2 and Lemma 14, the lattice generated by the 55 genus 2 curves on S_{Kl} is isomorphic to one of the lattices $\Lambda_{x,y,z,w}$. The lattices $\Lambda_{0,2,1,0}$ and $\Lambda_{0,0,0,2}$ are the only ones of rank less or equal to $25 = h^{1,1}(S)$. By Lemma 15, the lattice $\Lambda_{0,0,0,2}$ cannot be the lattice generated by the 55 genus 2 curves on S_{Kl}.

The lattice $\Lambda_{0,2,1,0}$ has rank 25, discriminant $2^2 11^{10}$, signature (1, 24). In [11], we computed a basis of $\mathrm{NS}(S_{Kl})$: it has rank 25 and discriminant 11^{10}. The lattice $\Lambda_{0,1,2,0}$ has thus index 2 in $\mathrm{NS}(S_{Kl})$ and, as we know that $C_s D_g = 2$, we can check that the incidence divisor C_s is not in this lattice.

By Lemma 14:

COROLLARY 16. *The formula 3.1 giving the intersection number of genus 2 curves on S_{Kl} holds also for the Fano surfaces of type $\mathbb{D}_n$, $n \in \{2, 3, 5, 6\}$ and $\mathbb{A}_5$.*

Proof. The groups $\mathbb{D}_n$, $n \in \{2, 3, 5, 6\}$ are subgroups of $\mathbb{A}_5$ and $PSL_2(\mathbb{F}_{11})$.   □

Let S be a Fano surface of type $\mathbb{D}_3$. Let us prove that:

LEMMA 17. *There exists a fibration $\gamma : S \to E$ with E an elliptic curve such that $D_1 + D_2 + D_3$ is a fiber of γ.*

Proof. We have: $(D_1 + D_2 + D_3)^2 = 0$.
The dihedral group $\mathbb{D}_3$ acts on $H^0(\Omega_S)$ by two copies of a representation of degree 2 plus the trivial representation of degree 1 (see also the next paragraph).
Let g be an involution of $\mathbb{D}_3$. We know its action on $H^0(\Omega_S)^*$, in particular, we know the eigenspace $T(g)$ with eigenvalues -1. By Lemma 6, $T(g)$ is the tangent space of a genus 2 curve. We can check that the sub-space W of $H^0(\Omega_S)^*$ generated by the tangent space coming from the 3 genus 2 curves has dimension 4.
The space W is the tangent space of a 4 dimensional variety B inside the Albanese variety of S. Thus there exist a fibration $q : \mathrm{Alb}(S) \to E$ where E is an elliptic curve. The 3 genus 2 curves are contracted by the composition of the Albanese map and q.   □

The similar assertions D) and E) of Theorem 11 relative to the fibrations of surfaces of type $\mathbb{D}_5$ and $\mathbb{D}_6$ are proved in the same way.

LEMMA 18. *For S generic of type $G = \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{D}_n$, $n \in \{2, 3, 5, 6\}$, the genus 2 curves D_g are smooth.*

Proof. It is enough to check that the conic $\mathcal{Q}$ defined in paragraph 3.1 is smooth if the cubic of type G is generic (we give the equation of such cubic in the next paragraph).   □

Let us prove part F) of Theorem 11. Let S be a Fano surface of type $\mathbb{A}_5$. By Corollary 16, we know the intersection numbers of the 15 genus 2 curves D_g corresponding to the 15 involutions of $\mathbb{A}_5$. The lattice Λ generated by these 15 curves has signature $(1, 14)$ and discriminant $2^{24}3^6$.

LEMMA 19. *The genus 2 curves D_g on a Fano surface S of type $\mathbb{A}_5$ and $PSL_2(\mathbb{F}_{11})$ are smooth.*

Proof. In [9], we proved that the elliptic curves E on a Fano surface S correspond bijectively with automorphisms $\sigma_E \in \mathrm{Aut}(S)$ such that the trace of σ_E acting on $H^0(\Omega)$ is -3 (involutions of type I). We classified all automorphism groups generated by involutions of type I.
Moreover, by [9], Theorem 13, for 2 elliptic curves $E \neq E'$ on S, we have $(\sigma_E \sigma_{E'})^2 = 1$ if and only if $EE' = 1$ i.e. if and only if $E + E'$ is a genus 2 curve on S. In that case, the trace of the involution $\sigma_E \sigma_{E'}$ on $H^0(\Omega)$ is 1 (involution of type II).
Suppose that one genus 2 curve on a Fano surface of type $\mathbb{A}_5$ is the sum of 2 elliptic curves. By transitivity, the 15 genus 2 curves are also sum of genus 2 curves and $\mathbb{A}_5$ (group generated by automorphisms of type II), is an automorphism sub-group of the group generated by involutions of type I.
By the classification of automorphism groups generated by involutions of type I ([9], Theorem 26), $\mathbb{A}_5$ is a subgroup of the symmetric group Σ_5 or of the reflection group $G(3, 3, 5)$

acting on $\mathbb{C}^5$. Thus it must exists elements a, b of Σ_5 or $G(3,3,5)$ such that:

$$a^2 = b^3 = (ab)^5 = 1$$

(relations defining the alternating group of degree 5) and $Tr(a) = 1$, $Tr(b) = -1$. But it is easy to check that no such elements exist in Σ_5 nor in $G(3,3,5)$. Therefore the 15 genus 2 curves on a Fano surface of type $\mathbb{A}_5$ are smooth.

As an involution in $PSL_2(\mathbb{F}_{11})$ in contained inside a group $\mathbb{A}_5$, the 55 genus 2 curves on S_{Kl} are smooth. $\square$

Let a, b be the generators a, b of $\mathbb{A}_5$ such that:

$$a^2 = b^3 = (ab)^5 = 1\,,$$

with a the diagonal matrix with diagonal elements $-1, -1, 1, 1, 1$ and:

$$b = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & -2 & -2 & -1 \end{pmatrix}.$$

A cubic threefold of type $\mathbb{A}_5$ has equation:

$$a(x_4^3 + x_4(x_1^2 - x_2^2 + x_3^2) + x_3(-x_2^2 + 3x_4^2 + x_5^2) + 2x_3^2 x_5$$
$$+2x_1 x_2(x_3 + x_4 + x_5) + 2x_3 x_4 x_5) + b(-x_3^3 + x_3(x_1^2 - x_2^2 - x_4^2)$$
$$+x_4(x_1^2 - 3x_3^2 - x_5^2) - 2x_4^2 x_5 + 2x_1 x_2(x_3 + x_4 + x_5) - 2x_3 x_4 x_5) = 0\,.$$

Let t be the point on S corresponding to the line $\{x_3 = x_4 = x_5 = 0\}$. The Quintic Γ_t is $\Gamma_t = Q + C$ for C the cubic:

$$a(x_4^3 + x_4 x_3^2 + x_3(3x_4^2 + x_5^2) + 2x_3^2 x_5 + 2x_3 x_4 x_5)$$
$$+b(-x_3^3 - x_3 x_4^2 - x_4(3x_3^2 + x_5^2) - 2x_4^2 x_5 - 2x_3 x_4 x_5) = 0\,.$$

This define a pencil of elliptic curves and by Remark 10, the cubics of type $\mathbb{A}_5$ form a 1 dimensional family of cubics.

Let $\mathrm{Alb}(S)$ be the Albanese variety of S. By the symmetries of $\mathbb{A}_5$ acting on $\mathrm{Alb}(S)$, this Abelian variety is isogenous to E^5 for some elliptic curve E. If E has no complex multiplication, then the Picard number of $\mathrm{Alb}(S)$ is 15, otherwise it is 25. By [12], the Picard number of a Fano surface S and of its Albanese variety $\mathrm{Alb}(S)$ are equal. Thus the Picard number of S is 15 or 25. As the curve E varies, the case E with CM occurs.

REMARK 20.   The number of elliptic curves on a Fano surface is bounded by 30 and the Fano surface of the Fermat cubic threefold is the only one to contain 30 elliptic curves. It is tempting to think that 55 is the bound for the number of smooth genus 2 curves on a Fano surface and that S_{Kl} is the only one to reach this bound.

3.3. Construction of Fano surfaces of a given type, on the completeness of the groups classification.

In order to get a classification of automorphism groups of Fano surfaces generated by involutions $g, h, \ldots$, it is natural to study their products i.e. to look at the dihedral group generated by two elements g, h. In [11], we prove that the order of $Aut(S)$ is prime to 7

and that an automorphism σ of S preserve a 5 dimensional principally polarized Abelian variety. That implies ([2], Proposition 13.2.5 and Theorem 13.2.8) that the Euler number of the order n of σ is less or equal to 10, therefore:

$$n \in \{2, \ldots, 6, 8, 9, 10, 11, 12, 15, 16, 18, 20, 22, 24, 30\} .$$

Hence, it is wise to study representations of dihedral group of order $2n$ with n in the above set, such that the order 2 elements have trace equal to 1. Our method is a case by case check; we have results for the cubic threefold of type $\mathbb{D}_n$ with $n \in \{2, 3, 4, 5, 6, 8, 11, 12, 16, 20, 22, 24\}$.

The $V_{\frac{k}{n}}$, $0 < k < n$ representation of the dihedral group of order $2n$ (generated by a, b such that $a^n = b^2 = 1$, $bab = a^{-1}$) is given by the matrices:

$$a = \left(\begin{array}{cc} \cos \frac{2k\pi}{n} & -\sin \frac{2k\pi}{n} \\ \sin \frac{2k\pi}{n} & \cos \frac{2k\pi}{n} \end{array} \right), \, b = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right).$$

There is also the trivial representation T, the linear representation $L : a \to 1, b \to -1$, and if n is even, the representations $L_1 : a \to -1, b \to 1$ and $L_2 : a \to -1, b \to -1$. The representation $V_{\frac{k}{n}}$ is faithful if and only if k is prime to n; it is irreducible if and only if $k \neq \frac{n}{2}$. The representations $V_{\frac{k}{n}}$ and $V_{\frac{n-k}{n}}$ are equivalent.

⊙ A surface of type $\mathbb{Z}/2\mathbb{Z}$ is the Fano surface of the cubic threefold F_2 given in paragraph 3.1.

⊙ The group $\mathbb{D}_2$ is given by the representation $L + L_1 + L_2 + 2T$. An invariant cubic F_4 has equation:

$$F_4 = \{(ax_1^2 + bx_2^2 + cx_3^2)x_4 + (dx_1^2 + ex_2^2 + fx_3^2)x_5 + gx_4^3 + hx_5^3 + kx_1x_2x_3 = 0\}$$

where $a, \ldots, k$ are constants.

⊙ The representation $\mathbb{D}_3$ is given by $2V_{\frac{1}{3}} + T$. A cubic threefold of type $\mathbb{D}_3$ has equation:

$$\begin{aligned} x_5^3 + (x_1^2x_5 + x_2^2x_5) + (x_3^2x_5 + x_4^2x_5) + a(x_1^3 - 3x_2^2x_1) + b(x_3^3 - 3x_4^2x_3) + c(x_1x_3x_5 \\ + x_2x_4x_5) + d(x_3^2x_1 - x_4^2x_1 - 2x_2x_3x_4) + e(x_1^2x_3 - x_2^2x_3 - 2x_1x_1x_4) . \end{aligned}$$

Let us denote by $\mathbb{D}_3'$ the dihedral group of order 6 such that an element of order 2 (resp. 3) has trace equals to 1 (resp 2). Its representation is $V_{\frac{1}{2}} + L + 2U$. A cubic threefold F of type $\mathbb{D}_3'$ has equation:

$$ax_4^3 + bx_5^3 + (x_1^2 + x_2^2)(ux_4 + vx_5) + c(x_1^3 - 3x_2^2x_1) + dx_5^2x_4 + ex_4^2x_5 + fx_3^2x_5 + gx_3^2x_4 = 0$$

The involutions $x \to (x_1, \pm x_2, \pm x_3, x_4, x_5)$ act on F. This gives a genus 2 curve on the Fano surface that is sum of two elliptic curves: this is not interesting.

There is a third representation of the dihedral group of order 6 such that the trace of the order 2 elements is 1, but it is not faithfull.

⊙ The representation $\mathbb{D}_5$ is given by $V_{\frac{1}{5}} + V_{\frac{2}{5}} + T$. A cubic threefold of type $\mathbb{D}_5$ has equation:

$$\begin{aligned} x_5^3 + c(x_1^2x_5 + x_2^2x_5) + d(x_3^2x_5 + x_4^2x_5) + a(x_1^2x_3 - x_2^2x_3 \\ + 2x_1x_2x_4) + b(-x_3^2x_1 + x_4^2x_1 + 2x_2x_3x_4) = 0 . \end{aligned}$$

◎ The representation $\mathbb{D}_6$ is given by $V\frac{1}{6} + V\frac{2}{6} + T$. A cubic threefold of type $\mathbb{D}_6$ has equation:

$$ax_5^3 + b(x_1^2 x_5 + x_2^2 x_5) + c(x_3^2 x_5 + x_4^2 x_5) + d(x_3^3 - 3x_4^2 x_3) + e(x_1^2 x_3 - x_2^2 x_3 + 2x_1 x_2 x_4) = 0.$$

The dihedral group of order 12 contain the dihedral group of order 6. For this last group, the only interesting representation is $\mathbb{D}_3$ (such that the trace of an order 3 element equals -1). With that point in mind, we can check that the only interesting representation of the dihedral group of order 12 is $\mathbb{D}_6$.

◎ There is another 5 dimensional representation of the alternating group of degree 5 such that the trace of an order 2 element is 1, but the trace of an order 3 element is 2 as for $\mathbb{D}'_3$ and that implies that the corresponding genus 2 are sum of 2 elliptic curves: this is not interesting.

◎ Let us now prove the following

PROPOSITION 21. *There do not exist a Fano surface of type the dihedral group of order $2n$ with $n \in \{4, 8, 12, 16, 20, 24\}$.*

Proof. Let a, b be generators of the dihedral group of order 8 such that $a^4 = 1, b^2 = 1, bab = a^3$. We are looking for representations such that the trace of the order 2 elements b, ab, a^2b, a^3b is 1. For the traces of a and a^2, the possibilities are $Tr(a) = -1, Tr(a^2) = 1$ or $Tr(a) = 3, Tr(a^2) = 1$ or $Tr(a) = 1, Tr(a^2) = -3$. In each cases, we computed the spaces V_χ of cubics such that $F_{eq} \circ g = \chi(g)F_{eq}$ for character χ. That gives no smooth cubic threefolds.

The dihedral groups of order 16, 24, 32, 40, 48 contain the dihedral group of order 8, thus they cannot be type of a cubic. □

3.4. A conjecture

The lattice $\Lambda_{0,0,0,2}$ of the proof of Theorem 13 has rank 21, discriminant 11.2^{22} and signature $(1, 20)$. It is remarkable that this lattice has the right signature to be the Néon-Severi group of a surface.

There is a natural representation of the group $PSL_2(\mathbb{F}_{11})$ on $\mathbb{C}^5$ (for which the Klein cubic generates the space invariant cubics). On the other hand classification of surfaces not of general type in $\mathbb{P}^4 = \mathbb{P}(\mathbb{C}^5)$ is an old unsolved problem.

The author's opinion is that we should consider the lattice $\frac{1}{2}\Lambda_{0,0,2,0}$ of rank 21 given by the generators $\{L_g, g$ involution$\}$ and the relations:

$$L_g L_h = \begin{cases} -2 & if & g = h \\ 1 & if & o(gh) = 6 \\ 0 & & otherwise \end{cases}$$

as the lattice generated by a configuration of 55 lines on a surface $S^?$ in $\mathbb{P}^4$. That surface $S^? \hookrightarrow \mathbb{P}^4$ is expected as the (may be non-complete) intersection of invariants of the group $PSL_2(\mathbb{F}_{11})$ acting on $\mathbb{P}^4$, in such a way that $PSL_2(\mathbb{F}_{11})$ acts on it.

On a surface of general type, the lattice generated by (-2)-curves is negative definite. As the lattice $\frac{1}{2}\Lambda_{0,0,2,0}$ is generated by (-2)-curves and has signature $(1, 20)$, the surface $S^?$ can not be of general type.

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