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The Fano surface of the Klein cubic threefold

By

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Abstract

We prove that the Klein cubic threefold F is the only smooth cubic threefold which has an automorphism of order 11. We compute the period lattice of the intermediate Jacobian of F and study its Fano surface S. We compute also the set of fibrations of S onto a curve of positive genus and the intersection between the fibres of these fibrations. These fibres generate an index 2 sub-group of the Néron-Severi group and we obtain a set of generators of this group. The Néron-Severi group of S has rank $25 = h^{1,1}$ and discriminant 11^{10}.

1. Introduction.

Let $F \hookrightarrow \mathbf{P}^4$ be a smooth cubic threefold. Its intermediate Jacobian

$$J(F) := H^{2,1}(F, \mathbf{C})^*/H_3(F, \mathbf{Z})$$

is a 5 dimensional principally polarized abelian variety $(J(F), \Theta)$ that plays in the analysis of curves on F a similar role to the one played by the Jacobian variety in the study of divisors on a curve.

The set of lines on F is parametrized by the so-called Fano surface of F which is a smooth surface of general type that we will denote by S. The Abel-Jacobi map $\vartheta : S \to J(F)$ is an embedding that induces an isomorphism $\mathrm{Alb}(S) \to J(F)$ where

$$\mathrm{Alb}(S) := H^0(\Omega_S)^*/H_1(S, \mathbf{Z})$$

is the Albanese variety of S, Ω_S is the cotangent bundle and $H^0(\Omega_S) := H^0(S, \Omega_S)$ (see [7] 0.6, 0.8 for details).

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The tangent bundle theorem ([7] Theorem 12.37) enables to recover the cubic F from its Fano surface. Moreover it gives a natural isomorphism between the spaces $H^0(\Omega_S)$ and $H^0(F,\mathcal{O}_F(1)) = H^0(\mathbf{P}^4,\mathcal{O}_{\mathbf{P}^4}(1))$. As we mainly work with the Fano surface, we will identify the basis $x_1,\dots,x_5$ of $H^0(\mathbf{P}^4,\mathcal{O}_{\mathbf{P}^4}(1))$ with elements of $H^0(\Omega_S)$ (for more explanations about these facts, see the discussion after Proposition 2.1). We will also identify the abelian variety $J(F)$ with $\mathrm{Alb}(S)$.

In [13], we give the classification of elliptic curve configurations on a Fano surface. It is proved that this classification is equivalent to the classification of the automorphism sub-groups of S that are generated by certain involutions. Moreover, it is also proved that the automorphism groups of a cubic and its Fano surface are isomorphic.

In the present paper, we pursue the study of these groups. By Lemma 2.1 below, the order of the automorphism group $\mathrm{Aut}(S)$ of the Fano surface divides $11\cdot 7\cdot 5^2 3^6 2^{23}$.

This legitimates the study of the Fano surfaces which have automorphisms of order 7 or 11. A. Adler [1] has proved that the automorphism group of the Klein cubic:

$$F:\ x_1x_5^2+x_5x_3^2+x_3x_4^2+x_4x_2^2+x_2x_1^2=0$$

is isomorphic to $PSL_2(\mathbf{F}_{11})$. We prove that:

Proposition 1.1. *A smooth cubic threefold has no automorphism of order* 7.
The Klein cubic is the only one smooth cubic threefold which has an automorphism of order 11.

If a curve admits a sufficiently large group of automorphisms, Bolza has given a method to compute a period matrix of its Jacobian (see [6], 11.7). As for the case of curves, we use the fact that the Klein cubic F admits a large group of automorphisms to compute the period lattice of its intermediate Jacobian $J(F)$ or, what is the same thing, the period lattice $H_1(S,\mathbf{Z})\subset H^0(\Omega_S)^*$ of the two dimensional variety S.

To state the main result of this work, we need some notations:

Let be $\nu=\frac{-1+i\sqrt{11}}{2}$ where $i\in\mathbf{C}$, $i^2=-1$ and let $\mathbf{E}$ be the elliptic curve $\mathbf{E}=\mathbf{C}/\mathbf{Z}[\nu]$. Let us denote by $e_1,\dots,e_5\in H^0(\Omega_S)^*$ the dual basis of $x_1,\dots,x_5$. Let be $\xi=e^{\frac{2i\pi}{11}}$ and for $k\in\mathbf{Z}/11\mathbf{Z}$, let $v_k\in H^0(\Omega_S)^*$ be :

$$v_k=\xi^k e_1+\xi^{9k}e_2+\xi^{3k}e_3+\xi^{4k}e_4+\xi^{5k}e_5.$$

Theorem 1.1. *1) The period lattice $H_1(S,\mathbf{Z})\subset H^0(\Omega_S)^*$ of the Fano surface of the Klein cubic is equal to:*

$$\Lambda=\frac{\mathbf{Z}[\nu]}{1+2\nu}(v_0-3v_1+3v_2-v_3)+\frac{\mathbf{Z}[\nu]}{1+2\nu}(v_1-3v_2+3v_3-v_4)+\bigoplus_{k=0}^{2}\mathbf{Z}[\nu]v_k$$

and the first Chern class of the Theta divisor is: $c_1(\Theta)=\frac{i}{\sqrt{11}}\sum_{j=1}^{j=5}dx_j\wedge d\bar{x}_j$.
2) The Néron-Severi group $\mathrm{NS}(S)$ *of* S *has rank* $25=h^{1,1}(S)$ *and discriminant*

11^{10}.
3) Let $\vartheta : S \to \mathrm{Alb}(S)$ *be an Albanese morphism and* $\mathrm{NS}(\mathrm{Alb}(S))$ *be the Néron-Severi group of* $\mathrm{Alb}(S)$. *The set of numerical classes of fibres of connected fibrations of* S *onto a curve of positive genus is in natural bijection with* $\mathbf{P}^4(\mathbf{Q}(\nu))$ *and the fibres of these fibrations generate rank* 25 *sub-lattice* $\vartheta^*\mathrm{NS}(\mathrm{Alb}(S))$.

Actually, the period lattice is equal to $c\Lambda$ where $c \in \mathbf{C}$ is a constant, but we can normalize $e_1, \dots, e_5$ in such a way that $c = 1$ (see Remark 1).

A set of generators of $\mathrm{NS}(S)$ and a formula for their intersection numbers are given in Theorem 4.1.

We remark that $J(F) \simeq \mathrm{Alb}(S)$ is isomorphic to $\mathbf{E}^5$ but by [7] (0.12), this isomorphism is not an isomorphism of principally polarized abelian varieties (p.p.a.v.). The fact that $J(F)$ is isomorphic to $\mathbf{E}^5$ is proved in [2] in a different way.

The main properties used to prove Theorem 1.1 are the fact that the action of the group $\mathrm{Aut}(S)$ on $\mathrm{Alb}(S)$ preserves the polarization Θ and the fact that the class of $S \hookrightarrow \mathrm{Alb}(S)$ is equal to $\frac{1}{3!}\Theta^3$. We use also the knowledge of the analytic representation of the automorphisms of the p.p.a.v. $(\mathrm{Alb}(S), \Theta)$.

To close this introduction, let us mention two known facts on this cubic: (1) the cotangent sheaf of its Fano surface is ample [13], (2) as the plane Klein quartic, the Klein cubic threefold has a modular interpretation [10] (about the analogy with the Klein quartic, see also Remark 2).

2. The unique cubic with an automorphism of order 11.

Let us recall some facts proved in [13] and fix the notations and conventions:

Definition 2.1. A morphism between two abelian varieties $f : A \to B$ is the composition of a homomorphism of Abelian varieties $g : A \to B$ and a translation. We call g the *homomorphism part* of f. The differential $dg : T_A(0) \to T_B(0)$ at the point 0 is called the *analytic representation* of both f and g, where $T_A(0)$ and $T_B(0)$ denote the tangent spaces of A and B at 0.

An automorphism f of a smooth cubic $F \hookrightarrow \mathbf{P}^4$ preserves the lines on F and induces an automorphism $\rho(f)$ of the Fano surface S.
An automorphism σ of S induces an automorphism σ' of the Albanese variety of S such that the following diagram:

$$\begin{array}{ccc} S & \stackrel{\vartheta}{\to} & \mathrm{Alb}(S) \\ \sigma \downarrow & & \sigma' \downarrow \\ S & \stackrel{\vartheta}{\to} & \mathrm{Alb}(S) \end{array}$$

is commutative (where $\vartheta : S \to \mathrm{Alb}(S)$ is a fixed Albanese morphism). The tangent space of the Albanese variety $\mathrm{Alb}(S)$ is $H^0(\Omega_S)^*$. We denote by $M_\sigma \in GL(H^0(\Omega_S)^*)$ the analytic representation of σ' : it is the dual of the pull-back $\sigma^* : H^0(\Omega_S) \to H^0(\Omega_S)$. We denote by

$$q : GL(H^0(\Omega_S)^*) \to PGL(H^0(\Omega_S)^*)$$

the natural quotient map.

Theorem 2.1. *1) For $\sigma \in \mathrm{Aut}(S)$, the homomorphism part of the automorphism σ' is an automorphism of the p.p.a.v. $(\mathrm{Alb}(S), \Theta)$.*
2) Let M be the analytic representation of an automorphism of $(\mathrm{Alb}(S), \Theta)$, then $q(M)$ is an automorphism of $F \hookrightarrow \mathbf{P}^4 = \mathbf{P}(H^0(\Omega_S)^*)$.
3) The morphism $\rho : \mathrm{Aut}(F) \to \mathrm{Aut}(S)$ is an isomorphism and its inverse is the morphism : $\mathrm{Aut}(F) \to \mathrm{Aut}(S)$; $\sigma \mapsto q(M_\sigma)$.
4) The group $\mathrm{Aut}(S)$ is a sub-group of $\mathrm{Aut}(\mathrm{Alb}(S), \Theta)$. If S is a generic Fano surface, then:

$$\mathrm{Aut}(F) \simeq \mathrm{Aut}(S) \simeq \mathrm{Aut}(\mathrm{Alb}(S), \Theta)/\left\langle [-1] \right\rangle .$$

When C is an non-hyperelliptic curve and $(J(C), \Theta)$ is its jacobian, there is an isomorphism $\mathrm{Aut}(C) \simeq \mathrm{Aut}(J(C), \Theta)/\left\langle [-1] \right\rangle$ ([6] Chap.11, exercise 19). Result 4) of Theorem 2.1 is thus the analogue for a cubic and its intermediate Jacobian.

Proof. Part 1) and 3) are proved in [13], they imply that the morphism:

$$\begin{array}{ccc} \mathrm{Aut}(S) & \to & \mathrm{Aut}(\mathrm{Alb}(S), \Theta) \\ \sigma & \mapsto & \sigma'' \end{array}$$

is injective, where σ'' is the homomorphism part of σ'.
Let us denote by $B_x X$ the blow-up at the point x of a variety X. By [5], the point 0 of $\mathrm{Alb}(S)$ is the unique singularity of the divisor Θ : it is thus preserved by any automorphism τ of the p.p.a.v. $(\mathrm{Alb}(S), \Theta)$.
The automorphism τ induces an automorphism of $B_0\Theta$ and $B_0\mathrm{Alb}(S)$. Let M denotes the analytic representation of τ. Let E be the exceptional divisor of $B_0\Theta$. The exceptional divisor of $B_0\mathrm{Alb}(S)$ is $\mathbf{P}(H^0(\Omega_S)^*)$ and we consider the following diagram:

$$\begin{array}{ccc} E & \to & B_0\Theta \\ \downarrow & & \downarrow \\ \mathbf{P}(H^0(\Omega_S)^*) & \to & B_0\mathrm{Alb}(S) \end{array}$$

where all the maps are embeddings. The action on E of τ is obtained by restriction of the action of τ on $\mathbf{P}(H^0(\Omega_S)^*)$ (that is the space of tangent directions to the point 0 of $\mathrm{Alb}(S)$) ; this last action is given by the automorphism $q(M)$ that is the projectivization of the differential of the automorphism τ at 0.
Now, by [5] théorème p. 190, the exceptional divisor $E \hookrightarrow \mathbf{P}(H^0(\Omega_S)^*)$ is the cubic $F \hookrightarrow \mathbf{P}(H^0(\Omega_S)^*)$ itself, thus property 2) holds.
Suppose that $q(M)$ is the identity. There exists a root of unity λ such that M is the multiplication by λ. By [6], Corollary 13.3.5, the order of λ is $1, 2, 3, 4$ or 6, and if the order d is $3, 4$ or 6, then $\mathrm{Alb}(S)$ is isomorphic to E^5 where E is the unique elliptic curve with an automorphism of order d.
On the other hand, the divisor Θ is symmetric : $[-1]^*\Theta = \Theta$ (see [7]), hence if the cubic threefold is generic, $q(M)$ is the identity if and only if τ is equal to $[1]$ or $[-1]$ and: $\mathrm{Aut}(F) \simeq \mathrm{Aut}(S) \simeq \mathrm{Aut}(\mathrm{Alb}(S), \Theta)/\left\langle [-1] \right\rangle$. □

Lemma 2.1. *The order of the group* $\mathrm{Aut}(S)$ *divides* $11 \cdot 7 \cdot 5^2 3^6 2^{23}$.

Proof. The automorphism group of a p.p.a.v. acts faithfully on the group of n-torsion points for $n \geq 3$ ([6], Corollary 5.1.10). Thus the order of the group of automorphisms of a 5 dimensional p.p.a.v. must divides $a_n = \# GL_{10}(\mathbf{Z}/n\mathbf{Z})$ for $n \geq 3$. In particular, it divides

$$11 \cdot 7 \cdot 5^2 3^6 2^{23} = gcd(a_3, a_5, a_7, a_{11}).$$

Theorem 2.1 implies then that the order of $\mathrm{Aut}(S)$ divides $11 \cdot 7 \cdot 5^2 3^6 2^{23}$. $\square$

Let us prove the following:

Proposition 2.1. *The Klein cubic is the only one smooth cubic threefold which has an automorphism of order* 11*. There is no smooth cubic threefold which possesses an automorphism of order* 7*.*

Let $F \hookrightarrow \mathbf{P}^4$ be a smooth cubic threefold and S be its Fano surface. In the Introduction, we mention that the cubic threefold F can be recovered knowing only the surface S. This important, but non-trivial, result is called the Tangent Bundle Theorem and is due to Fano, Clemens-Griffiths and Tyurin (Beauville also gives another proof in [5]). We give more explanation about that result ; it will explain how we identify the spaces $H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4})$ and $H^0(\Omega_S)$ and the main ideas of the proof of Proposition 2.1:
Let us consider the natural morphism of vector spaces:

$$Ev : \oplus_{n \in \mathbf{N}} \mathrm{S}^n H^0(\Omega_S) \to \oplus_{n \in \mathbf{N}} H^0(S, \mathrm{S}^n \Omega_S)$$

given by the natural maps on each pieces of the graduation. The Tangent Bundle Theorem can be formulated as follows:
The kernel of Ev is an ideal of the ring $\oplus_{n \in \mathbf{N}} \mathrm{S}^n H^0(\Omega_S)$ generated by a cubic $F_{eq} \in \mathrm{S}^3 H^0(\Omega_S)$ and the cubic threefold $\{F_{eq} = 0\} \hookrightarrow \mathbf{P}(H^0(\Omega_S)^*)$ is (isomorphic to) the original cubic $F \hookrightarrow \mathbf{P}^4$.
By this Theorem, the homogenous coordinates $x_1, \dots, x_5 \in H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(1))$ of $\mathbf{P}^4$ are also a basis of the global holomorphic 1-forms of S.

Proof. (of Proposition 2.1). The intermediate Jacobian of a smooth cubic threefold with an order 11 automorphism is a 5 dimensional p.p.a.v. that possesses an automorphism of order 11. By the theory of complex multiplication there are only four such principally abelian varieties, they are denoted by $X_1, \dots, X_4$ in [9].

By [7], an intermediate Jacobian is not a Jacobian of a curve, but by Theorem 2 of [9], the p.p.a.v. X_1 and X_2 are Jacobians of curves, thus we can eliminate X_1 and X_2.

The abelian variety X_3 has an automorphism τ' of order 11 such that the eigenvalues of its analytic representation M are $\{\xi, \xi^2, \xi^3, \xi^5, \xi^7\}$, $\xi = e^{\frac{2i\pi}{11}}$.
Suppose that X_3 is the intermediate Jacobian of a cubic threefold $F \hookrightarrow \mathbf{P}^4$. By Theorem 2.1, the morphism ${}^tM$ acts on $H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(1))$ and the projectivized

morphism $q(M)$ is an automorphism of $F \hookrightarrow \mathbf{P}^4$.
Let $\mathrm{S}^3({}^tM)$ be the action of ${}^tM$ on

$$H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(3)) = \mathrm{S}^3 H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(1)).$$

An equation of F is an eigenvector for $\mathrm{S}^3({}^tM)$. We can easily compute the eigenspaces of $\mathrm{S}^3({}^tM)$ and check that no one contains the equation of a smooth cubic threefold, thus X_3 cannot be an intermediate Jacobian.

The p.p.a.v. X_1, X_2 and X_3 are not intermediate Jacobians and by the Torelli Theorem [7], the association $F \to (J(F), \Theta)$ is injective, hence the p.p.a.v. X_4 is the intermediate Jacobian of the Klein cubic and this cubic is (up to isomorphism) the only one smooth cubic which has an order 11 automorphism.

Let us prove that there are no smooth cubic threefolds with an automorphism of order 7.
By the Theorems 13.1.2. and 13.2.8. and Proposition 13.2.5. of [6], the differential of an automorphism of order 7 of a five dimensional Abelian variety has eigenvalues $1, 1, \mu, \mu^a, \mu^b$ where μ is a primitive 7-th root of unity and a, b are integers such that $\{\mu, \mu^a, \mu^b\}$ is a set containing three pairwise non-complex conjugate primitive roots of unity.
Up to the change of μ by a power, there are two possibilities:
The first case is $a = 2$ and $b = 3$. For $c \in \mathbf{Z}/7\mathbf{Z}$, let us denote by χ_c the representation $\mathbf{C} \to \mathbf{C};\ t \mapsto \mu^c t$ of $\mathbf{Z}/7\mathbf{Z}$. The third symmetric product of the representation:

$$\begin{array}{ccc} H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(1)) & \to & H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(1)) \\ x_1, x_2, x_3, x_4, x_5 & \mapsto & \mu x_1, \mu^2 x_2, \mu^3 x_3, x_4, x_5 \end{array}$$

decomposes as the direct sum:

$$6\chi_0 + 4\chi_1 + 6\chi_2 + 6\chi_3 + 5\chi_4 + 4\chi_5 + 4\chi_6.$$

By example, the space corresponding to $6\chi_0$ is generated by the forms x_4^3, $x_4^2 x_5$, $x_4 x_5^2$, x_5^3, $x_2^2 x_3$, $x_3^2 x_1$. No element of this space is an equation of a smooth cubic threefold. In the similar way, we can check that the other factors do not give a smooth cubic.
The second case is $a = 2$ and $b = 4$. In the same manner, we can check that we do not obtain a smooth cubic in that case. $\square$

3. The period lattice of the intermediate Jacobian of the Klein cubic.

Let F be the Klein cubic:

$$x_1 x_5^2 + x_5 x_3^2 + x_3 x_4^2 + x_4 x_2^2 + x_2 x_1^2 = 0$$

and let S be its Fano surface. Let $\vartheta : S \to \mathrm{Alb}(S)$ be a fixed Albanese morphism; it is an embedding. Let us compute the period lattice $H_1(S, \mathbf{Z}) \subset H^0(\Omega_S)^*$ of the Albanese variety of S.

The order 5 automorphism:

$$g:\,(z_1:z_2:z_3:z_4:z_5)\mapsto(z_5:z_1:z_4:z_2:z_3)$$

acts on F. Let $\sigma=\rho(g)$ be the automorphism of S defined in paragraph 2. By Theorem 2.1, there exists a 5-th root of unity θ such that $M_\sigma\in GL(H^0(\Omega_S)^*)$ is equal to:

$$M_\sigma:(z_1,z_2,z_3,z_4,z_5)\mapsto\theta(z_5,z_1,z_4,z_2,z_3)$$

in the basis $e_1,\dots,e_5$ dual to $x_1,\dots,x_5$. Since the Klein cubic F and g are defined over $\mathbf{Q}$ and the Fano surface contains points defined over $\mathbf{Q}$, we deduce that the morphism M_σ (that is the differential at a point defined over $\mathbf{Q}$ of an automorphism defined over $\mathbf{Q}$) is defined over $\mathbf{Q}$, thus: $\theta=1$.
Let be $\xi=e^{\frac{2i\pi}{11}}$. The equation of the Klein cubic is chosen in such a way that it is easy to check that the automorphism

$$f:\,(z_1:z_2:z_3:z_4:z_5)\mapsto(\xi z_1:\xi^9z_2:\xi^3z_3:\xi^4z_4:\xi^5z_5)$$

acts on it. Let be $\tau=\rho(f)$. By Theorem 2.1, the analytic representation of the homomorphism part of τ' is:

$$M_\tau:(z_1,z_2,z_3,z_4,z_5)\mapsto\xi^j(\xi z_1,\xi^9z_2,\xi^3z_3,\xi^4z_4,\xi^5z_5),$$

where $j\in\mathbf{Z}/11\mathbf{Z}$. Proposition 13.2.5. of [6] implies that $j=0$.

Notation 3.1. *For $k\in\mathbf{Z}/11\mathbf{Z}$, let v_k be the vector:*

$$\begin{aligned}v_k&=\xi^ke_1+\xi^{9k}e_2+\xi^{3k}e_3+\xi^{4k}e_4+\xi^{5k}e_5\\&=(M_\tau)^kv_0\in H^0(\Omega_S)^*\end{aligned}$$

and let be $\ell_k=\xi^kx_1+\xi^{9k}x_2+\xi^{3k}x_3+\xi^{4k}x_4+\xi^{5k}x_5\in H^0(\Omega_S)$.

Let us construct a sub-lattice of $H_1(S,\mathbf{Z})$.
Let q_1 be the homomorphism part of

$$\sum_{k=0}^{k=4}(\sigma')^k$$

(where $\sigma'\circ\vartheta=\vartheta\circ\sigma$). Its analytic representation is:

$$\begin{array}{cccc}dq_1: H^0(\Omega_S)^* & \to & H^0(\Omega_S)^*\\ z & \mapsto & \ell_0(z)v_0.\end{array}$$

and its image is an elliptic curve which we denote by $\mathbf{E}$. The restriction of the homomorphism part of $q_1\circ\tau':\mathrm{Alb}(S)\to\mathbf{E}$ to $\mathbf{E}$ is the multiplication by:

$$\nu=\xi+\xi^9+\xi^3+\xi^4+\xi^5=\frac{-1+i\sqrt{11}}{2},$$

thus the curve $\mathbf{E}$ has complex multiplication by the principal ideal domain $\mathbf{Z}[\nu]$ and there is a constant $c\in\mathbf{C}^*$ such that :

$$H_1(S,\mathbf{Z})\cap\mathbf{C}v_0=\mathbf{Z}[\nu]cv_0.$$

Remark 1. Up to a normalization of the basis $e_1,\dots,e_5$ by a multiplication by the scalar $\frac{1}{c}$, we suppose that $c=1$. Under this normalization, the Klein cubic remains the same :

$$F=\{x_1x_5^2+x_5x_3^2+x_3x_4^2+x_4x_2^2+x_2x_1^2=0\}.$$

Let $\Lambda_0\subset H^0(\Omega_S)^*$ be the $\mathbf{Z}$-module generated by the v_k, $k\in\mathbf{Z}/11\mathbf{Z}$. The group Λ_0 is stable under the action of M_τ and $\Lambda_0\subset H_1(S,\mathbf{Z})$.

Lemma 3.1. *The $\mathbf{Z}$-module $\Lambda_0\subset H_1(S,\mathbf{Z})$ is equal to the lattice:*

$$R_0=\mathbf{Z}[\nu]v_0+\mathbf{Z}[\nu]v_1+\mathbf{Z}[\nu]v_2+\mathbf{Z}[\nu]v_3+\mathbf{Z}[\nu]v_4.$$

Proof. We have:

$$\nu v_0=v_1+v_3+v_4+v_5+v_9,$$

hence νv_0 is an element of Λ_0. This implies that the vectors $\nu v_k=(M_\tau)^k\nu v_0$ are elements of Λ_0 for all k, hence: $R_0\subset\Lambda_0$. Conversely, we have:

$$v_5=v_0+(1+\nu)v_1-v_2+v_3+\nu v_4.$$

This proves that the lattice R_0 contains the vectors $v_k=(M_\tau)^k v_0$ generating Λ_0, thus: $R_0=\Lambda_0$. □

Let us compute the first Chern class $c_1(\Theta)$ of the Theta divisor Θ of $\mathrm{Alb}(S)$:

Lemma 3.2. *Let H be the matrix of the Hermitian form of the polarization Θ in the basis $e_1,\dots,e_5$. There exists a positive integer a such that:*

$$H=a\frac{2}{\sqrt{11}}I_5$$

where I_5 is the size 5 identity matrix.

Proof. By Theorem 2.1, the homomorphism part of τ' preserves the polarization Θ. This implies that:

$${}^tM_\tau H\bar{M}_\tau=H$$

where $\bar{M}_\tau$ is the matrix in the basis $e_1,..,e_5$ whose coefficients are conjugated of those of M_τ. The only Hermitian matrices that verify this equality are the diagonal matrices. By the same reasoning with σ instead of τ, we obtain that these diagonal coefficients are equal and:

$$H=a\frac{2}{\sqrt{11}}I_5$$

where a is a positive real (H is a positive definite Hermitian form). As H is a polarization, the alternating form $c_1(\Theta)=\Im m(H)$ take integer values on $H_1(S,\mathbf{Z})$, hence $a=\Im m({}^tv_2H\bar{v}_1)$ is an integer. □

Now, we construct a lattice that contains $H_1(S,\mathbf{Z})$:
Let be $k\in\mathbf{Z}/11\mathbf{Z}$. The analytic representation of the morphism $q_1\circ((\tau')^k)$ is:

$$\begin{array}{ccc} H^0(\Omega_S)^* & \to & H^0(\Omega_S)^* \\ z & \mapsto & \ell_k(z)v_0. \end{array}$$

Let be $\lambda\in H_1(S,\mathbf{Z})$. As

$$H_1(S,\mathbf{Z})\cap\mathbf{C}v_0=\mathbf{Z}[\nu]v_0,$$

the scalar $\ell_k(\lambda)$ is an element of $\mathbf{Z}[\nu]$. Let us define:

$$\Lambda_4=\{z\in H^0(\Omega_S)^*/\ell_k(z)\in\mathbf{Z}[\nu],\,0\leq k\leq 4\}.$$

Lemma 3.3. *The* $\mathbf{Z}$*-module* $\Lambda_4\supset H_1(S,\mathbf{Z})$ *is equal to the lattice:*

$$R_1=\sum_{k=0}^{k=3}\frac{\mathbf{Z}[\nu]}{1+2\nu}(v_k-v_{k+1})+\mathbf{Z}[\nu]v_0.$$

Moreover M_τ *stabilizes* Λ_4.

Proof. Let be $\ell_1^*,\dots,\ell_5^*\in H^0(\Omega_S)^*$ be the dual basis of $\ell_1,\dots,\ell_5$ (see Notations 3.1). Then $\Lambda_4=\bigoplus_{j=1}^5\mathbf{Z}[\nu]\ell_j^*$. Since $(e_1,\dots,e_5)=(\ell_1^*,\dots,\ell_5^*)A$ and $(v_0,\dots,v_4)=(e_1,\dots,e_5)^tA$ for the matrix:

$$A=\begin{pmatrix} 1&1&1&1&1\\ \xi&\xi^9&\xi^3&\xi^4&\xi^5\\ \xi^2&\xi^7&\xi^6&\xi^8&\xi^{10}\\ \xi^3&\xi^5&\xi^9&\xi&\xi^4\\ \xi^4&\xi^3&\xi&\xi^5&\xi^9 \end{pmatrix},$$

we have $(\ell_1^*,\dots,\ell_5^*)=(v_0,\dots,v_4)^tA^{-1}A$. Moreover:

$${}^tA^{-1}A=\frac{1}{1+2\nu}\begin{pmatrix} -1&-\nu&0&-1&1-\nu\\ -\nu&2&0&-\nu&3+\nu\\ 0&0&0&1&-1\\ -1&-\nu&1&-2&1-\nu\\ 1-\nu&3+\nu&-1&1-\nu&2+2\nu \end{pmatrix}.$$

Let B be the matrix:

$$B=\begin{pmatrix} -\nu-1&1&-1&0&5\\ 1&-1&0&0&\nu\\ -1&0&0&1&-1-\nu\\ 0&0&1&0&\nu\\ 0&1&0&0&\nu \end{pmatrix}\in SL_5(\mathbf{Z}[\nu]).$$

We have $(\ell_1^*,\dots,\ell_5^*)B=(v_0,\dots,v_4)^tA^{-1}AB=(\frac{v_0-v_1}{1+2\nu},\frac{v_1-v_2}{1+2\nu},\frac{v_2-v_3}{1+2\nu},\frac{v_3-v_4}{1+2\nu},v_0)$, thus $\Lambda_4=R_1$. By using the equality:

$$v_5=v_0+(1+\nu)v_1-v_2+v_3+\nu v_4,$$

we easily check that the vector $M_\tau(\frac{1}{1+2\nu}(v_3-v_4))\in H^0(\Omega_S)^*$ is in Λ_4, hence Λ_4 is stable by M_τ. $\square$

Now, using the action of M_τ, we determine the lattice $H_1(S,\mathbf{Z})$ among lattices Λ such that $\Lambda_0\subset\Lambda\subset\Lambda_4$.
We denote by $\phi:\Lambda_4\to\Lambda_4/\Lambda_0$ the quotient map. The ring $\mathbf{Z}[\nu]/(1+2\nu)$ is the finite field with 11 elements. The quotient Λ_4/Λ_0 is a $\mathbf{Z}[\nu]/(1+2\nu)$-vector space with basis:

$$\begin{array}{ll} t_1=\frac{1}{1+2\nu}(v_0-v_1)+\Lambda_0, & t_2=\frac{1}{1+2\nu}(v_1-v_2)+\Lambda_0,\\ t_3=\frac{1}{1+2\nu}(v_2-v_3)+\Lambda_0, & t_4=\frac{1}{1+2\nu}(v_3-v_4)+\Lambda_0.\end{array}$$

Let R be a lattice such that : $\Lambda_0\subset R\subset\Lambda_4$. The group $\phi(R)$ is a sub-vector space of Λ_4/Λ_0 and:

$$\phi^{-1}\phi(R)=R+\Lambda_0=R.$$

The set of lattices R such that $\Lambda_0\subset R\subset\Lambda_4$ is thus in bijection with the set of sub-vector spaces of Λ_4/Λ_0 and these lattices are also $\mathbf{Z}[\nu]$-modules.
Because M_τ preserves Λ_0, the morphism M_τ induces a morphism $\widehat{M_\tau}$ on the quotient Λ_4/Λ_0 such that $\phi\circ M_\tau=\widehat{M_\tau}\circ\phi$. As M_τ stabilizes $H_1(S,\mathbf{Z})$, the vector space $\phi(H_1(S,\mathbf{Z}))$ is stable by $\widehat{M_\tau}$. Let be:

$$\begin{aligned} w_1&=-t_1+3t_2-3t_3+t_4\\ w_2&=t_1-2t_2+t_3\\ w_3&=-t_1+t_2\\ w_4&=t_1.\end{aligned}$$

The matrix of $\widehat{M_\tau}$ is

$$\begin{pmatrix} 0&0&0&-1\\ 1&0&0&4\\ 0&1&0&5\\ 0&0&1&4\end{pmatrix}$$

in the basis $t_1,\dots,t_4$ of Λ_4/Λ and the matrix of $\widehat{M_\tau}$ is:

$$\begin{pmatrix} 1&1&0&0\\ 0&1&1&0\\ 0&0&1&1\\ 0&0&0&1\end{pmatrix}$$

in the basis $w_1,\dots,w_4$. The sub-spaces stable by $\widehat{M_\tau}$ are the space $W_0=\{0\}$ and the spaces W_j, $1\leq j\leq 4$ generated by $w_1,..,w_j$. Let Λ_j be the lattice $\phi^{-1}W_j$, then:

Theorem 3.2. *The lattice $H_1(S,\mathbf{Z})$ is equal to Λ_2, and Λ_2 is equal to*

$$R_2=\frac{\mathbf{Z}[\nu]}{1+2\nu}(v_0-3v_1+3v_2-v_3)+\frac{\mathbf{Z}[\nu]}{1+2\nu}(v_1-3v_2+3v_3-v_4)+\bigoplus_{k=0}^{2}\mathbf{Z}[\nu]v_k.$$

Moreover, the Hermitian matrix associated to Θ *is equal to* $\frac{2}{\sqrt{11}}I_5$ *in the basis* $e_1,\dots,e_5$ *and* $c_1(\Theta)=\frac{i}{\sqrt{11}}\sum_{k=1}^{5}dx_k\wedge d\bar{x}_k$.

Proof. Let $c_1(\Theta)=\Im m(H)=i\frac{a}{\sqrt{11}}\sum dx_k\wedge d\bar{x}_k$ be the alternating form of the principal polarization Θ. Let $\lambda_1,\dots,\lambda_{10}$ be a basis of a lattice Λ. By definition, the square of the Pfaffian $Pf_\Theta(\Lambda)$ of Λ is the determinant of the matrix

$$(c_1(\Theta)(\lambda_j,\lambda_k))_{1\leq j,k\leq 10}\,.$$

Since Θ is a principal polarization, we have $Pf_\Theta(H_1(S,\mathbf{Z}))=1$.
It is easy to find a basis of Λ_j ($j\in\{0,\dots,4\}$). For example, the space W_2 is generated by $w_2=t_1-2t_2+t_3$ and $w_1+w_2=t_2-2t_3+t_4$ and as

$$\begin{array}{c}\phi(\frac{1}{1+2\nu}(v_0-3v_1+3v_2-v_3))=w_2,\\ \phi(\frac{1}{1+2\nu}(v_1-3v_2+3v_3-v_4))=w_1+w_2,\end{array}$$

the lattice R_2 (that contains Λ_0) is equal to Λ_2.
Then, with the help of a computer, we can calculate the square of the Pfaffian P_j of the lattice Λ_j and verify that it is equal to:

$$a^{10}11^{4-2j}$$

where a is the integer of Lemma 3.2. As a is a positive integer, the only possibility that P_j equals 1 is $j=2$ and $a=1$. $\square$

Remark 2. Let C be the Klein quartic curve : this curve is canonically embedded into $\mathbf{P}^2=\mathbf{P}(H^0(C,\Omega_C)^*)$ and there exists a basis x_1,x_2,x_3 of $H^0(C,\Omega_C)$ such that $C=\{x_1^3x_2+x_2^3x_3+x_3^3x_1=0\}$. The automorphism group of C is $PSL_2(\mathbf{F}_7)$. By taking exactly the same arguments as the Klein cubic threefold, it is possible to compute the period lattice $H_1(C,\mathbf{Z})\subset H^0(C,\Omega_C)^*$.

4. The Néron-Severi group of the Fano surface of the Klein cubic.

Let us define:

$$\begin{array}{c}u_1=\frac{1}{1+2\nu}(v_0-3v_1+3v_2-v_3),\ u_2=\frac{1}{1+2\nu}(v_1-3v_2+3v_3-v_4),\\ u_3=v_0,\ u_4=v_1,\ u_5=v_2,\end{array}$$

and let $y_1,\dots,y_5\in H^0(\Omega_S)$ be the dual basis of $u_1,\dots,u_5$.
Let be k, $1\leq k\leq 5$. The image of $H_1(S,\mathbf{Z})$ by $y_k\in H^0(\Omega_S)$ is $\mathbf{Z}[\nu]$, and this form is the analytic representation of a morphism of Abelian varieties

$$r_k:\mathrm{Alb}(S)\to\mathbf{E}=\mathbf{C}/\mathbf{Z}[\nu].$$

By Theorem 3.2, the morphisms $r_1,\dots,r_5$ form a basis of the $\mathbf{Z}[\nu]$-module of rank 5 of homomorphisms between $\mathrm{Alb}(S)$ and $\mathbf{E}$.

We denote by Λ_A^* the free $\mathbf{Z}[\nu]$-module of rank 5 generated by $y_1,\dots,y_5$ and for $\ell\in\Lambda_A^*\setminus\{0\}$, we denote by $\Gamma_\ell:\mathrm{Alb}(S)\to\mathbf{E}$ the morphism whose

analytic representation is $\ell : H^0(\Omega_S)^* \to \mathbf{C}$.
Let $\vartheta : S \to \mathrm{Alb}(S)$ be a fixed Albanese morphism. We denote by $\gamma_\ell : S \to \mathbf{E}$ the morphism $\gamma_\ell = \Gamma_\ell \circ \vartheta$ and we denote by F_ℓ the numerical equivalence class of a fibre of γ_ℓ (this class is independent of the choice of ϑ).
We define the scalar product of two forms $\ell, \ell' \in \Lambda_A^*$ by:

$$\langle \ell, \ell' \rangle = \sum_{k=1}^{k=5} \ell(e_k)\overline{\ell'(e_k)}$$

and the norm of ℓ by:

$$\|\ell\| = \sqrt{\langle \ell, \ell \rangle}.$$

We denote by $\mathrm{NS}(X)$ the Néron-Severi group of a variety X. For a point s of S, we denote by C_s the incidence divisor that parametrizes the lines on F that cut the line corresponding to the point s. The aim of this section is to prove the following result:

Theorem 4.1. *1) Let ℓ, ℓ' be non-zero elements of Λ_A^*. The fibre F_ℓ has arithmetic genus:*

$$g(F_\ell) = 1 + 3\|\ell\|^2,$$

satisfies $C_s F_\ell = 2\|\ell\|^2$ and :

$$F_\ell F_{\ell'} = \|\ell\|^2 \|\ell'\|^2 - \langle \ell, \ell' \rangle \langle \ell', \ell \rangle .$$

2) The image of the morphism $\vartheta^* : \mathrm{NS}(\mathrm{Alb}(S)) \to \mathrm{NS}(S)$ is a rank 25 sub-lattice of discriminant $2^2 11^{10}$.
3) The following 25 fibres

$$\left\{ \begin{array}{cc} F_{y_k} & k \in \{1, \dots, 5\}, \\ F_{y_k + y_l} & 1 \le k < l \le 5, \\ F_{y_k + \nu y_l} & 1 \le k < l \le 5, \end{array} \right.$$

form a $\mathbf{Z}$-basis of $\vartheta^* \mathrm{NS}(\mathrm{Alb}(S))$ and together with the class of the incident divisor C_s ($s \in S$), they generate $\mathrm{NS}(S)$. The lattice $\mathrm{NS}(S)$ has discriminant 11^{10}.

We identify elements of the Néron-Severi group of $\mathrm{Alb}(S)$ with alternating forms.

Lemma 4.1. *The Néron-Severi group of $\mathrm{Alb}(S)$ is generated by the 25 forms:*

$$\left\{ \begin{array}{cc} \frac{i}{\sqrt{11}} dy_k \wedge d\bar{y}_k & k \in \{1, \dots, 5\}, \\ \frac{i}{\sqrt{11}} (dy_k \wedge d\bar{y}_l + dy_l \wedge d\bar{y}_k) & 1 \le k < l \le 5, \\ \frac{i}{\sqrt{11}} (\nu dy_k \wedge d\bar{y}_l + \bar{\nu} dy_l \wedge d\bar{y}_k), & 1 \le k < l \le 5. \end{array} \right.$$

Proof. The Hermitian form $H' = \frac{2}{\sqrt{11}} I_5$ in the basis $u_1, \dots, u_5$ defines a principal polarization of $\mathrm{Alb}(S)$. Let $\mathrm{End}^s(\mathrm{Alb}(S))$ be the group of symmetrical morphisms for the Rosati involution associated to H'. An endomorphism of $\mathrm{Alb}(S)$ can be represented by a matrix $A \in M_5(\mathbf{Z}[\nu])$ in the basis $u_1, \dots, u_5$. The symmetrical endomorphisms satisfy ${}^tAH' = H'\bar{A}$ i.e. ${}^tA = \bar{A}$. A basis $\mathcal{B}$ of the group of symmetrical elements is :

$$\left\{ \begin{array}{cc} e_{kk} & k \in \{1, \dots, 5\}, \\ e_{kl} + e_{lk} & 1 \leq k < l \leq 5, \\ \nu e_{kl} + \overline{\nu} e_{lk} & 1 \leq k < l \leq 5, \end{array} \right.$$

where e_{kl} is the matrix with entry 1 in the intersection of line k and row l and 0 elsewhere.
By [6], Proposition 5.2.1 and Remark 5.2.2., the map:

$$\begin{array}{cccc} \phi_{H'} : \mathrm{End}^s(\mathrm{Alb}(S)) & \rightarrow & \mathrm{NS}(\mathrm{Alb}(S)) \\ A & \mapsto & \Im m(\cdot {}^tAH' \bar{\cdot}) \end{array}$$

is an isomorphism of groups. We obtain the base of the lemma by taking the image by $\phi_{H'}$ of the base $\mathcal{B}$. □

The Néron-Severi group of the curve $\mathbf{E} = \mathbf{C}/\mathbf{Z}[\nu]$ is the $\mathbf{Z}$-module generated by the form $\eta = \frac{i}{\sqrt{11}} dz \wedge d\bar{z}$. Let be $\ell \in \Lambda_A^* \setminus \{0\}$. We have:

$$\Gamma_\ell^* \eta = \frac{i}{\sqrt{11}} d\ell \wedge d\bar{\ell}$$

and this form is the Chern class of the divisor $\Gamma_\ell^* 0$.

Lemma 4.2. *The* 25 *forms:*

$$\left\{ \begin{array}{cc} \eta_k = \Gamma_{y_k}^* \eta & k \in \{1, \dots, 5\}, \\ \eta_{k,l}^1 = \Gamma_{y_k + y_l}^* \eta & 1 \leq k < l \leq 5, \\ \eta_{k,l}^\nu = \Gamma_{y_k + \nu y_l}^* \eta & 1 \leq k < l \leq 5. \end{array} \right.$$

are a basis of the Néron-Severi group of $\mathrm{Alb}(S)$.

Proof. Let $1 \leq k \leq 5$ be an integer. The element $\Gamma_{y_k}^* \eta = \frac{i}{\sqrt{11}} dy_k \wedge d\bar{y}_k$ lies in the basis of Lemma 4.1. Let $1 \leq l < k \leq 5$ be integers, let be $a \in \{1, \nu\}$, and $\ell = y_k + a y_l$. We have:

$$\Gamma_\ell^* \eta = \frac{i}{\sqrt{11}} (dy_k \wedge d\bar{y}_k + \bar{a} dy_k \wedge d\bar{y}_l + a dy_l \wedge d\bar{y}_k + a\bar{a} dy_l \wedge d\bar{y}_l),$$

this proves, when we take $a = 1$ and next $a = \nu$, that the forms of the basis of Lemma 4.1 are $\mathbf{Z}$-linear combinations of the forms $\eta_k, \eta_{k,l}^1, \eta_{k,l}^\nu$, $1 \leq k, l \leq 5$. □

Let us prove Theorem 4.1.

Proof. By [7], the homology class of S in $\mathrm{Alb}(S)$ is equal to $\frac{\Theta^3}{3!}$, thus the intersection of the fibres F_ℓ and $F_{\ell'}$ is equal to:

$$\int_A \frac{1}{3!}\wedge^3 c_1(\Theta)\wedge\Gamma_\ell^*\eta\wedge\Gamma_{\ell'}^*\eta.$$

Write ℓ in the basis $x_1,..,x_5$: $\ell = a_1x_1+\cdots+a_5x_5$ and $\ell' = b_1x_1+\cdots+b_5x_5$, then:

$$\frac{1}{3!}(\frac{i}{\sqrt{11}})^2 d\ell\wedge d\bar{\ell}\wedge d\ell'\wedge d\bar{\ell'}\wedge(\wedge^3 c_1(\Theta))$$

is equal to:

$$\begin{array}{c}(\frac{i}{\sqrt{11}})^5(\sum a_j dx_j)\wedge(\sum \bar{a}_j d\bar{x}_j)\wedge(\sum b_j dx_j)\wedge(\sum \bar{b}_j d\bar{x}_j)\\ \wedge\sum_{h<j<k} dx_h\wedge d\bar{x}_h\wedge dx_j\wedge d\bar{x}_j\wedge dx_k\wedge d\bar{x}_k\end{array}$$

that is equal to:

$$(\sum_{k\neq j}(a_k\bar{a}_k b_j\bar{b}_j - a_k\bar{a}_j b_j\bar{b}_k))\frac{1}{5!}\wedge^5 c_1(\Theta).$$

But : $\int_A \frac{1}{5!}\wedge^5 c_1(\Theta)=1$ because Θ is a principal polarization of $\mathrm{Alb}(S)$, hence:

$$\begin{aligned}F_\ell F_{\ell'} = \int_A \tfrac{1}{3!}\wedge^3 c_1(\Theta)\wedge\Gamma_\ell^*\eta\wedge\Gamma_{\ell'}^*\eta \quad &= \textstyle\sum_{k\neq j}(a_k\bar{a}_k b_j\bar{b}_j - a_k\bar{a}_j b_j\bar{b}_k)\\ &= \|\ell\|^2\|\ell'\|^2 - \langle\ell,\ell'\rangle\langle\ell',\ell\rangle.\end{aligned}$$

By [7] (10.9) and Lemma 11.27, $\frac{3}{2}\vartheta^* c_1(\Theta)$ is the Poincaré dual of a canonical divisor K of S, hence:

$$KF_\ell = \frac{3}{2}\vartheta^* c_1(\Theta)\vartheta^*\Gamma_\ell^*\eta = \frac{3}{2}\int_A \frac{1}{3!}\wedge^4 c_1(\Theta)\wedge\Gamma_\ell^*\eta$$

and:

$$KF_\ell = \int_A 6(\frac{i}{\sqrt{11}})^5(\sum a_j dx_j)\wedge(\sum \bar{a}_j d\bar{x}_j)\wedge\sum_{1\leq k\leq 5}(\bigwedge_{j\neq k}(dx_j\wedge d\bar{x}_j))$$

so $KF_\ell = 6\sum_{k=1}^{k=5} a_k\bar{a}_k = 6\|\ell\|^2$. Thus we have: $g(F_\ell) = (KF_\ell+0)/2+1 = 3\|\ell\|^2+1$. By [7], $3C_s$ is numerically equivalent to K, hence $C_sF_\ell = 2\|\ell\|^2$.

Lemma 4.2 gives us a basis $\eta_1,...,\eta_{25}$ of $\mathrm{NS}(\mathrm{Alb}(S))$ and we know the intersections $\vartheta^*\eta_k\vartheta^*\eta_l$ in the Fano surface. With the help of a computer, we can verify that the determinant of the intersection matrix:

$$(\vartheta^*\eta_k\vartheta^*\eta_l)_{1\leq k,l\leq 25}$$

is equal to $2^2 11^{10}$. By general results of [13], the index of $\vartheta^*\mathrm{NS}(\mathrm{Alb}(S))\subset\mathrm{NS}(S)$ is 2 and $\mathrm{NS}(S)$ is generated by $\vartheta^*\mathrm{NS}(\mathrm{Alb}(S))$ and the class of an incidence divisor C_s. $\square$

Corollary 4.1. *1) Let C be a smooth curve of genus > 0 and let $\gamma : S \to C$ be a fibration with connected fibres. Then there exists an isomorphism $j : \mathbf{E} \to C$ and a form $\ell \in \Lambda_A^*$ such that $\gamma = j \circ \gamma_\ell$.*
2) The set of numerical classes of fibres of connected fibrations of S onto a curve of positive genus is in natural bijection with $\mathbf{P}^4(\mathbf{Q}(\nu))$.
3) The fibres of these fibrations generate $\vartheta^*\mathrm{NS}(\mathrm{Alb}(S))$.

To prove Corollary 4.1, we need the following Lemma:

Lemma 4.3. *1) Let be $\ell \in \Lambda_A^* \setminus \{0\}$, $\ell = t_1 y_1 + \cdots + t_5 y_5$. The fibration Γ_ℓ has connected fibres if and only if $t_1, \ldots, t_5$ generate $\mathbf{Z}[\nu]$.*
2) Let $\Gamma : \mathrm{Alb}(S) \to C$ be a morphism with connected fibres onto an elliptic curve C. Then $C = \mathbf{E}$ and there exists $\ell \in \Lambda_A^*$ such that $\Gamma = \Gamma_\ell$.

Proof. Let $\mathbf{t} \subset \mathbf{Z}[\nu]$ be the ideal of $\mathbf{Z}[\nu]$ generated by $t_1, \ldots, t_5$. This ideal satisfies

$$d\Gamma_\ell(H_1(S, \mathbf{Z})) = \mathbf{t}.$$

The morphism Γ_ℓ factorizes through the natural morphisms : $\mathrm{Alb}(S) \to \mathbf{C}/\mathbf{t}$ and $\mathbf{C}/\mathbf{t} \to \mathbf{E} = \mathbf{C}/\mathbf{Z}[\nu]$.
If $\mathbf{t} \neq \mathbf{Z}[\nu]$, then the fibres of Γ_ℓ are not connected because the fibres of $\mathbf{C}/\mathbf{t} \to \mathbf{E}$ are not connected.
Let us recall that $\mathbf{Z}[\nu]$ is a principal ideal domain : there exist a generator g of $\mathbf{t}$ and $\ell' \in \Lambda_A^*$ such that $\ell = g\ell'$. If we replace ℓ by ℓ', we are now reduced to the case where $\mathbf{t} = \mathbf{Z}[\nu]$, $g = 1$.
In that case, there exist $a_1, \ldots, a_5 \in \mathbf{Z}[\nu]$ such that $\sum a_i t_i = 1$. The homomorphism $\mathbf{E} \to \mathrm{Alb}(S)$ whose analytic representation is:

$$\begin{array}{ccc} \mathbf{C} & \to & H^0(\Omega_S)^* \\ z & \mapsto & \sum a_i u_i \end{array}$$

is a section of Γ_ℓ, hence : $\mathrm{Alb}(S) \simeq \mathbf{E} \times \mathrm{Ker}(\Gamma_\ell)$, and since $\mathrm{Alb}(S)$ is connected, that implies that the fibre $\mathrm{Ker}(\Gamma_\ell)$ is connected ; thus Γ_ℓ has connected fibres.
Now let be Γ as in part 2). The curve C is isogenous to $\mathbf{E}$; let $j : C \to \mathbf{E}$ be an isogeny. There exists ℓ such that $j \circ \Gamma = \Gamma_\ell$. Moreover, we proved that there exists ℓ' such that $\Gamma_{\ell'}$ is the Stein factorization of $\Gamma_\ell = j \circ \Gamma$. As Γ and $\Gamma_{\ell'}$ have the same fibres, we see that

$$C = \mathrm{Alb}(S)/\mathrm{Ker}(\Gamma) = \mathrm{Alb}(S)/\mathrm{Ker}(\Gamma_{\ell'}) = \mathbf{E}$$

and $\Gamma = \Gamma_{\ell'}$. □

Let us prove Corollary 4.1:

Proof. Let $\gamma : S \to C$ be a fibration onto a curve of genus > 0. Since the natural morphism $\wedge^2 H^0(\Omega_S) \to H^0(S, \wedge^2 \Omega_S)$ is an isomorphism [7], the Castelnuovo Lemma implies that the curve C has genus 1.
Let $\Gamma : \mathrm{Alb}(S) \to C$ be the morphism such that $\gamma = \Gamma \circ \vartheta$. The fibres of Γ are

connected, otherwise their trace on $S \hookrightarrow \mathrm{Alb}(S)$ would be disconnected fibres of γ.
Lemma 4.3 2) implies that $C = \mathbf{E}$ and there exists a ℓ such that $\Gamma = \Gamma_\ell$. Moreover, this ℓ satisfies $\ell(H_1(S,\mathbf{Z})) = \mathbf{Z}[\nu]$ and thus defines a point in $\mathbf{P}^4_{\mathbf{Z}}(\mathbf{Z}[\nu])$; this last set is canonically identified with $\mathbf{P}^4(\mathbf{Q}(\nu))$. Thus, we proved that to the numerical class of a fibre of a connected fibration γ, we can associate a point of $\mathbf{P}^4(\mathbf{Q}(\nu))$ (the numerical class of a fibre determine the fibration).
Conversely, by Lemma 4.3 1), to a point of $\mathbf{P}^4(\mathbf{Q}(\nu))$, there corresponds a form $\ell \in \Lambda_A^*$ (up to a sign) such that Γ_ℓ has connected fibres. Let γ be the Stein factorization of $\Gamma_\ell \circ \vartheta$. We proved that there is a form ℓ' such that $\gamma = \Gamma_{\ell'} \circ \vartheta$ and $\Gamma_{\ell'}$ has connected fibres. Thus $\ell = \ell'$, $\Gamma_\ell \circ \vartheta$ has connected fibres and to ℓ we associate the fibre $F_\ell \in \mathrm{NS}(S)$. This class F_ℓ is independent of the choice of $\pm\ell$. That ends the proof of parts 1) and 2).
The point 3) is a reformulation of Lemma 4.2. $\square$

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