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ON GENERALIZED KUMMER SURFACES AND THE ORBIFOLD BOGOMOLOV-MIYAOKA-YAU INEQUALITY

XAVIER ROULLEAU

ABSTRACT. A generalized Kummer surface $X = \mathrm{Km}(T, G)$ is the resolution of a quotient of a torus T by a finite group of symplectic automorphisms G. We complete the classification of generalized Kummer surfaces by studying the last two groups which have not been yet studied. For these surfaces we compute the associated Kummer lattice K_G, which is the minimal primitive sub-lattice containing the exceptional curves of the resolution $X \to T/G$. We then prove that a K3 surface is a generalized Kummer surface of type $\mathrm{Km}(T, G)$ if and only if its Néron-Severi group contains K_G.

For smooth-orbifold surfaces $\mathcal{X}$ of Kodaira dimension ≥ 0, Kobayashi proved the orbifold Bogomolov-Miyaoka-Yau inequality $c_1^2(\mathcal{X}) \leq 3c_2(\mathcal{X})$. For Kodaira dimension 2, the case of equality is characterized as $\mathcal{X}$ being uniformized by the complex 2-ball $\mathbb{B}_2$. For smooth-orbifold K3 and Enriques surfaces we characterize the case of equality as being uniformized by $\mathbb{C}^2$.

1. INTRODUCTION

A K3 surface X is called a *generalized Kummer surface*, and we write $X = \mathrm{Km}(T, G)$ if it is the resolution of a quotient T/G where T is a torus and G is a finite group of automorphisms of T. Let $X = \mathrm{Km}(T, G)$ be a generalized Kummer surface and let F_G be the sub-lattice of the Néron-Severi group $\mathrm{NS}(X)$ generated by the exceptional divisors $\mathcal{C}_G$ of the resolution $X \to T/G$. The minimal primitive sub-lattice K_G of $\mathrm{NS}(X)$ containing the lattice F_G is called the *Kummer lattice* of G.

In [17] Nikulin computes the Kummer lattice $K_{\mathbb{Z}/2\mathbb{Z}}$ and obtains the famous result that for a K3 surface X, it is equivalent to being a Kummer or to containing 16 disjoint (-2)-curves or that there exists a primitive embedding of the lattice $K_{\mathbb{Z}/2\mathbb{Z}}$ in the Néron-Severi group of X.

That result linking the primitive embedding of a lattice K_G contained in $\mathrm{NS}(X)$ to a geometric description of X has then been extended by Bertin [2] and Garbagnati [7] to the 7 other symplectic automorphism groups G acting on some torus T such that the action of G preserves the origin of T.

It turns out that there are symplectic groups G which have not yet been studied: when G has no global fixed points on the torus T. Up to taking quotient of G by its translation sub-group, one can suppose that G contains no translations. Fujiki has described and classified such pairs (T, G): then G is isomorphic to $\hat{Q}_8$ (the quaternion group) or $\hat{T}_{24}$ (the binary tetrahedral group of order 24). We compute

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that the singularities of the quotient $T/\hat{Q}_8$ are $\mathcal{C}_{\hat{Q}_8} = 6A_3 + A_1$ and the ones of $T/\hat{T}_{24}$ are $\mathcal{C}_{\hat{T}_{24}} = 4A_2 + 2A_3 + A_5$. Let $G = \hat{Q}_8$ or $\hat{T}_{24}$. In sub-sections 3.2 and 3.3, we describe the minimal primitive sub-lattice K_G containing the lattice F_G generated by the exceptional curves $\mathcal{C}_G$ of the minimal resolution of T/G, and we obtain the following result.

Theorem 1. *Let X be a K3 surface and let G be the group $\hat{Q}_8$ or $\hat{T}_{24}$. The following conditions are equivalent:*

(i) *X is a generalized Kummer surface $X = \mathrm{Km}(T, G)$.*
(ii) *The Kummer lattice K_G is primitively embedded in $\mathrm{NS}(X)$.*
(iii) *X contains a configuration of ADE curves $\mathcal{C}_G$.*

Then we turn our attention to a related question, which was our initial motivation. Let $\mathcal{C}$ be a configuration of disjoint ADE curves on a smooth surface X and let $X \to \mathcal{X}$ be the contraction of the connected components of $\mathcal{C}$. To the singular surface $\mathcal{X}$ one can associate its orbifold Chern numbers, denoted by $c_1^2(\mathcal{X})$, $c_2(\mathcal{X}) \in \mathbb{Q}$, which depend on the Chern numbers $c_1^2(X)$, $c_2(X)$ of X and on the number and type of the ADE singularities of $\mathcal{X}$. These orbifold Chern numbers have the following property.

Theorem 2 (Orbifold Bogomolov-Miyaoka-Yau inequality [10,13,14,16]).[1] *Suppose that X is a minimal algebraic surface of Kodaira dimension ≥ 0. Then:*

(A) *One has*

$$c_1^2(\mathcal{X}) \leq 3c_2(\mathcal{X}). \tag{1.1}$$

(B) *Suppose X has general type. Equality holds in (1.1) if and only if there exists a discrete cocompact lattice Γ in $PU(2,1)$ such that $\mathcal{X} = \mathbb{B}_2/\Gamma$. In other words, one has equality if and only if $\mathcal{X}$ is uniformizable by the unit ball $\mathbb{B}_2$.*

Here a discrete cocompact lattice means a sub-group which is discrete in $PU(2,1)$ such that the points with non-trivial isotropy are isolated. These isotropy groups are finite, and the quotient $\mathbb{B}_2/\Gamma$ is compact. A consequence of Theorem 2 is that in case of equality in (1.1), there always exists a finite uniformization of $\mathcal{X}$, i.e., a smooth ball quotient surface Z having a finite group of automorphisms G such that $\mathcal{X} = Z/G$.

It is now natural to ask if there is an analog of part (B) of Theorem 2 for surfaces of Kodaira dimension 0 and 1. In this paper we study that problem for surfaces having Kodaira dimension $\kappa = 0$, for which equality $c_1^2(\mathcal{X}) = 3c_2(\mathcal{X})$ is in fact equivalent to $c_2(\mathcal{X}) = 0$. Let X be a K3 surface, let $\mathcal{C}$ be a configuration of ADE curves on X, and let $X \to \mathcal{X}$ be the contraction of the curves in $\mathcal{C}$. We obtain the following result.

Theorem 3. *The equality $c_1^2(\mathcal{X}) = 3c_2(\mathcal{X})$ holds if and only if there exists a discrete cocompact lattice Γ in the affine linear group $\mathbb{C}^2 \rtimes GL_2(\mathbb{C})$ such that $\mathcal{X} = \mathbb{C}^2/\Gamma$.*

Using the now complete classification of generalized Kummer surfaces, we will in fact see that in case of equality $c_1^2(\mathcal{X}) = 3c_2(\mathcal{X})$, the K3 X is a generalized Kummer surface, which result implies Theorem 3.

[1] Note that there exist stronger versions of Theorem 2, in particular with other quotient singularities. But for surfaces of Kodaira dimension 0, which is the case of interest for us, the only quotient singularities one can obtain are ADE.

We then obtain the same result as Theorem 3 for Enriques surfaces: for an Enriques surface X and $X \to \mathcal{X}$ the contraction of a configuration $\mathcal{C}$ of ADE curves, one has $c_1^2(\mathcal{X}) = 3c_2(\mathcal{X})$ if and only if $\mathcal{C}$ is the union of 8 disjoint (-2)-curves. We moreover construct the Enriques surfaces containing such a configuration.

Among algebraic surfaces with Kodaira dimension 0, there remain the Abelian and bi-elliptic surfaces, which satisfy $c_1^2 = 3c_2 = 0$. The universal cover of these surfaces is $\mathbb{C}^2$, and they do not contain rational curves. Therefore the question is closed for surfaces with $\kappa = 0$.

The paper is organized as follows: in section 2 we recall the notation and the main results we will need, which are mainly the results of Garbagnati [7]. In section 3, we recall Fujiki's beautiful classification of automorphism groups of 2-dimensional tori and we give a more detailed account of the previous work on generalized Kummer surfaces. Then we describe the Kummer lattices K_G for $G = \hat{Q}_8, \hat{T}_{24}$ and prove that if a Kummer surface contains a configuration $\mathcal{C}_G$, then it is a generalized Kummer surface $\mathrm{Km}(T, G)$. In section 4, we prove Theorem 3 on K3 and Enriques surfaces.

2. Preliminaries

2.1. **Notation.** $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ is the cyclic group of order n.

Q_8 is the quaternion group (order 8, has a unique involution ι, $Q_8/\iota \simeq (\mathbb{Z}_2)^2$).

D_{12} is the binary dihedral group (order 12).

T_{24} is the binary tetrahedral group (order 24, isomorphic to $\mathrm{SL}_2(\mathbb{F}_3)$; Q_8 is a normal sub-group of it).

For $n \in \mathbb{Z}$, $[n] : T \to T$ is the multiplication by n maps on a torus T.

For more on the problem of generalized Kummer surfaces, we recommend the paper of Garbagnati [7], from which we tried to follow the notation.

2.2. **Lattices, divisible sets.** For a lattice L, we denote by $L^\vee$ its dual. The *length* of L is the minimal number of generators of its *discriminant group* $L^\vee/L$. A sub-lattice M of L is said to be *primitive* if L/M is torsion free.

Proposition 4 ([18, Proposition 1.6.1]). *Let L be a unimodular lattice, let M be a primitive sublattice of L, and let $M^\perp$ be the orthogonal to M in L. The discriminant group of M is isomorphic to the discriminant group of $M^\perp$. In particular, since the length of a lattice is at most the rank of the lattice, $l(M) = l(M^\perp) \leq \min(\mathrm{rk}(M), \mathrm{rk}(M^\perp))$.*

A set of disjoint smooth rational curves $(C_i)_{i\in I}$ on a surface X is called *even* if there exists an invertible sheaf $\mathcal{L}$ such that $\mathcal{O}(\sum_i C_i) = \mathcal{L}^{\otimes 2}$. On a K3 surface, an even set contains 8 or 16 disjoint curves.

Let C_i^j, $1 \leq i \leq n$, $j \in \{1, 2\}$, be a set of n disjoint A_2 configurations (so that $C_i^1 C_i^2 = 1$) on a surface. The divisor $D = \sum_{i=1}^n C_i^1 + 2C_i^2$ is called 3-*divisible* if $\mathcal{O}(D) = \mathcal{L}^{\otimes 3}$ where $\mathcal{L}$ is an invertible sheaf. On a K3 surface, the support of a 3-divisible divisor contains 6 or 9 disjoint A_2 configurations.

We will use repeatedly the following consequence of Proposition 4.

Lemma 5. *Let X be a K3 surface containing 12 disjoint (-2)-curves. Then there exists an even set supported on 8 of these curves.*

Let X be a K3 surface containing 13 disjoint (-2)-curves. Then there exist two linearly independent even sets of curves, supported on 12 of these curves.

Proof. This is well known; see e.g. [8, Remark 8.10]. The discriminant group of the lattice generated by the 12 curves is $(\mathbb{Z}_2)^{12}$, and it has length $12 > \min(12, 22-12)$. Thus there exists at least one non-trivial divisible class.

The second part follows e.g. from the first: there is an even set, then remove one curve from that even set, there exists still a set of 12 disjoint curves, thus another even set, with a different support. □

In the present paper an *ADE configuration* $\mathcal{C}$ on a surface will have a polysemic meaning. It could mean a set of ADE singularities on a surface $\mathcal{X}$ or the set of the exceptional curves of its minimal resolution $X \to \mathcal{X}$.

For numbers $\alpha_n, \delta_n, \varepsilon_n \in \mathbb{N}^*$ with $\delta_i = 0$ for $i \leq 3$ and $\varepsilon_i = 0$ for $i \notin \{6, 7, 8\}$, we write symbolically $\mathcal{C} = \sum_{n \geq 1} \alpha_n A_n + \delta_n D_n + \varepsilon_n E_n$ if for any $n \geq 1$, $\mathcal{C}$ contains α_n (resp. δ_n, ε_n) configurations of type A_n (resp. D_n, E_n).

Let $\mathcal{C}$ be an ADE configuration. We are looking for obstructions or criteria for some sub-configurations of $\mathcal{C}$ to be part of an even set or a 3-divisible set. In Remark 6 below, when we speak of a configuration A_n or D_n, we implicitly assume it is maximal in $\mathcal{C}$; i.e., it is not contained in an A_m or D_m contained in $\mathcal{C}$ for some $m > n$.

Remark 6.

(A) An irreducible component C of a configuration A_n in $\mathcal{C}$ can be part of an even set E if and only if n is odd, the $\frac{n+1}{2}$ disjoint curves in that A_n configuration are in E, and C is among these curves, since otherwise there always exists a curve C' supported on A_n such that $C'E = 1$, and therefore E cannot be even.

(B) The discriminant group of D_n is $(\mathbb{Z}_2)^2$ if $n \geq 4$ is even and is $\mathbb{Z}_4$ if n is odd (see [9, Theorem 2.3.5]). Accordingly, k disjoint curves and the 2 extremal disjoint closest curves on a D_{2k} can possibly be part of an even set; the two closest extremal disjoint curves on a D_{2k+1} can possibly be part of an even set.

(C) The discriminant group of A_n is $\mathbb{Z}_{n+1}$. Since $\mathbb{Z}_4$ does not contain the group $\mathbb{Z}_3$, a sub-configuration A_2 of a configuration A_3 in $\mathcal{C}$ cannot be part of a 3-divisible set. There is no such a obstruction for the two disjoint A_2 in a configuration A_5.

2.3. **Double, bi-double, and triple covers, lifts of automorphisms.** To an even set E (resp. a 3-divisible divisor $E = \sum_{i=1}^{n} C_i^1 + 2C_i^2$) on a K3 surface X, one can associate a double (resp. triple) cyclic cover of X branched on the support of E. The minimal desingularisation Y of that cyclic cover has an involution (resp. an automorphism of order 3) τ such that Y/τ is (isomorphic to) $\mathcal{X}$, the surface obtained by contracting the curves on the support of E. We call Y the *surface associated* to E.

Lemma 7. *Let E be an even set on a K3 surface X and let Y be the surface associated to E.*

(A) *An automorphism σ of order n of the K3 surface X lifts to Y if and only if $E = \sigma^* E$.*

(B) *Suppose that σ lifts to an automorphism $\sigma' \in \mathrm{Aut}(Y)$. Let τ be an element of the transformation group of the cover Y (thus Y/τ is bi-rational to X). There is an exact sequence*

$$0 \to \langle \tau \rangle \to \langle \tau, \sigma' \rangle \to \langle \sigma \rangle \to 1.$$

Proof. A K3 surface satisfies $\mathrm{NS}(X) = \mathrm{Pic}(X)$. Then part (A) is [20, Proposition 4.2]. Part (B) follows from the fact that $\sigma' \tau \sigma'^{-1}$ is a lift of the identity, thus a power of τ, and $\langle\tau\rangle$ is normal in $\langle\tau,\sigma'\rangle$. The map $\langle\tau,\sigma'\rangle \to \langle\sigma\rangle$ maps a lift μ' of $\mu \in \mathrm{Aut}(X)$ to μ. □

A bi-double cover $Y \to X$ of a surface X is a Galois cover of group $(\mathbb{Z}_2)^2$. It is determined by divisors D_1, D_2, D_3 and invertible sheaves L_1, L_2, L_3 such that for $\{i,j,k\} = \{1,2,3\}$, one has

$$2L_i \equiv D_j + D_k \tag{2.1}$$

(see [3]). The surface Y is embedded in the total space of the vector bundle $\mathbb{L} = L_1 \oplus L_2 \oplus L_3$ as the variety with equation

$$\mathrm{rk}\begin{pmatrix} x_1 & w_3 & w_2 \\ w_3 & x_2 & w_1 \\ w_2 & w_1 & x_3 \end{pmatrix} = 1,$$

where $D_i = \mathrm{div}(x_i)$ and w_1, w_2, w_3 are coordinates of the L_i.

Example 8. Let $E = \sum_{i=1}^{12} C_i$ be a $12A_1$ configuration on a K3 surface such that E has 2 linearly independent even sets ℓ_1, ℓ_2. Up to reordering, one can suppose that

$$\ell_1 = \sum_{i=5}^{12} C_i, \qquad \ell_2 = \sum_{i=1}^{4} C_i + \sum_{i=9}^{12} C_i.$$

Then $\ell_3 = \sum_{i=1}^{8} C_i$ is also even. Let $L_i := \frac{1}{2}\ell_i$ and let $D_j = E - \ell_j$. The data D_i, L_j satisfy the relations (2.1) and determine a bi-double cover $Y \to X$; Y is a smooth K3.

Let σ be an automorphism of a smooth surface X admitting a bi-double cover determined by divisors D_i, $i \in \{1,2,3\}$, and invertible sheaves L_i, $i \in \{1,2,3\}$, as above. Suppose that there is an action of σ on $\{1,2,3\}$ such that $\sigma^* L_i = L_{\sigma i}$ and $\sigma^* D_i = D_{\sigma i}$. Then we have the following.

Lemma 9. *The automorphism σ lifts to an automorphism of Y.*

Proof. One can choose coordinates w_i so that $\sigma^* w_i = w_{\sigma i}$ and equations x_i of D_i such that $\sigma^* x_i = x_{\sigma i}$. Then the automorphism σ lifts to an automorphism of $\mathbb{L}$, and the equations of Y are preserved; thus it restricts to an automorphism of Y. □

2.4. **Roots of a lattice and (-2)-curves.** Let X be an algebraic K3 surface. Let h be a pseudoample divisor on X (i.e., $h^2 > 0$ and $hD \geq 0$ for all effective divisors D) and let

$$L = h^{\perp} := \{l \in \mathrm{NS}(X) \text{ such that } lh = 0\}$$

be the orthogonal of h in $\mathrm{NS}(X)$. We will use the following result proved by Garbagnati [7, Proposition 3.2] (see also [2, Lemma 3.1] of Bertin).

Proposition 10. *Let us assume that there exists a root lattice R such that:*

(1) *L is an overlattice of finite index of R,*
(2) *the roots of R and of L coincide.*

Then there exists a basis of R which is supported on smooth irreducible rational curves.

Remark 11. According to a recent preprint of Schütt [22], hypothesis (2) is always satisfied.

Let X be a K3 surface and let $F \subset \mathrm{NS}(X)$ be a sub-lattice. A minimal primitive sub-lattice of $H^2(X,\mathbb{Z})$ containing F is a lattice K_F containing F such that K_F/F is finite and $H^2(X,\mathbb{Z})/K_F$ is free. That lattice K_F is unique and is equal to the lattice $\mathrm{NS}(X) \cap F \otimes \mathbb{Q}$.

Let X be a non-algebraic K3 surface. By Grauert's ampleness criterion for complex surfaces, since X is not algebraic any divisor on X has self-intersection ≤ 0. Therefore the irreducible curves are -2-curves or of arithmetic genus 1. By Riemann-Roch, if there is a curve of arithmetic genus 1, then it is a fiber of a fibration $X \to \mathbb{P}^1$. That fibration is then unique and contracts every (-2)-curve on X (this is still because the divisors have self-intersection ≤ 0). The class of a fiber generates the kernel of the natural map $\mathrm{NS}(X) \to \mathrm{Num}(X)$; in particular $\mathrm{NS}(X)$ is degenerate. Therefore since the signature of the intersection form on $H^{1,1}$ is $(1, 19)$, if $\mathrm{NS}(X)$ contains a negative definite sub-group of rank 19, then there are no curves of arithmetic genus 1 on the non-algebraic K3 surface X.

Remark 12. Let X be a non-algebraic K3 surface containing no curves of arithmetic genus 1. The negative definite lattice $\mathrm{NS}(X)$ has rank $\rho \leq 19$, and ρ is equal to the number of (-2)-curves on X. In particular the minimal primitive sub-lattice containing the (-2)-curves on X is $\mathrm{NS}(X)$ itself.

Let δ be a (-2)-class on X, i.e., an element of $\mathrm{NS}(X)$ such that $\delta \in NS(X)$ satisfies $\delta^2 = -2$. By Riemann-Roch $-\delta$ or δ is effective; say $\delta \geq 0$: there exist (-2)-curves C_i such that $\delta = \sum m_i C_i$, with $m_i \geq 1$. Therefore the (-2)-classes on X form a root system of the lattice F generated by the (-2)-curves on X, and the (-2)-curves form a simple base B of F. One can then apply [2, Lemma 3.2] (see also [7, Remark 4.5]) and conclude that if one has a direct sum decomposition as root lattice $F = \sum_{n\geq 1} A_n^{\oplus \alpha_n}$, then for each of the factors A_n there is a simple base constituted of (-2)-curves.

2.5. **Orbifold settings.** Let $\mathcal{C} = \sum_{n\geq 1} \alpha_n A_n + \delta_n D_n + \varepsilon_n E_n$ be an ADE configuration of curves on a smooth surface X. Let us define the quantity

$$m(\mathcal{C}) := \sum_{n\geq 1} (\alpha_n + \delta_n + \varepsilon_n)(n+1) - \sum_{n\geq 1} \frac{\alpha_n}{n+1} - \sum_{n\geq 4} \frac{\delta_n}{4(n-2)} - \frac{\varepsilon_6}{24} - \frac{\varepsilon_7}{48} - \frac{\varepsilon_8}{120}.$$

Let $X \to \mathcal{X}$ be the contraction map of the curves contained in $\mathcal{C}$. Since $\mathcal{X}$ contains only ADE singularities, the orbifold Chern numbers of $\mathcal{X}$ are

$$c_1^2(\mathcal{X}) = K_X^2 \quad \text{and} \quad c_2(\mathcal{X}) = c_2(X) - m(\mathcal{C})$$

(see e.g. [21]). Let X be a K3 surface. The orbifold Miyaoka-Yau inequality (1.1) tells us that

$$m(\mathcal{C}) \leq 24.$$

Moreover, since each configuration A_n, D_n, or E_n contributes for an n-dimensional sub-space in the negative definite part (of rank at most 19) of the Néron-Severi group, one has the restriction

$$\sum_{n\geq 1} n(\alpha_n + \delta_n + \varepsilon_n) \leq 19.$$

Suppose that the K3 orbifold $\mathcal{X}$ has a finite uniformization $Y \to \mathcal{X}$; i.e., Y is a smooth surface with an action by a finite group G of order n such that $\mathcal{X} = Y/G$ and $Y \to Y/G$ is ramified in codimension 2. Then we have the following.

Lemma 13. *The Chern numbers of* Y *are* $K_Y^2 = nc_1^2(\mathcal{X})$ *and* $c_2(Y) = nc_2(\mathcal{X})$. *The surface* Y *is Abelian or a K3.*

Proof. The first part follows by the definition of the orbifold Chern numbers; see e.g. [13, 21]. Since $Y \to \mathcal{X}$ is ramified in codimension 2, the canonical divisor K_Y is the pull-back of $K_{\mathcal{X}}$, which is trivial. Thus K_Y is trivial, and Y is a K3 or is an Abelian surface. □

The double cover of an even set of 8 (resp. 16) A_1 is a K3 (resp. a torus). The triple cover of a 3-divisible set of 6 (resp. 9) A_2 is a K3 (resp. a torus).

3. Classification of symplectic groups and generalized Kummer surfaces

3.1. **Fujiki's constructions of Abelian tori with symplectic action of a group.** In [6], Fujiki constructs and classifies pairs (T, G) of complex tori T with a faithful action by a group G containing no translations. Let us describe his results when G acts symplectically and is not cyclic.

Let $\mathbb{H} = \mathbb{R}[1, i, j, k]$ be the quaternion field, so that

$$i^2 = j^2 = k^2 = -1, \qquad ij = -ji = k.$$

Let

$$\mathfrak{a} = \mathbb{Z}[1, i, j, t]$$

be the ring of Hurwitz quaternions, where $t = \frac{1}{2}(1 + i + j + k)$. This is a maximal order of $F = \mathbb{Q}[1, i, j, k]$, and its group of invertible elements is

$$\mathfrak{a}^\times = \{1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k)\},$$

which is the binary tetrahedral group T_{24}. Let

$$\mathfrak{a}_0 = \mathbb{Z}[1, i, j, k];$$

the sub-group $\mathfrak{a}_0^\times = \{1, \pm i, \pm j, \pm k\}$ is the quaternion group Q_8. Let

$$F' = \mathbb{Q}[1, i, \sqrt{3}j, \sqrt{3}k],$$

and let $\mathfrak{b} = \mathbb{Z}[1, i, h, l]$, where

$$h = \frac{1}{2}(i + \sqrt{3}j), \qquad l = \frac{1}{2}(1 + \sqrt{3}k).$$

The sub-group

$$\mathfrak{b}^\times = \{\pm 1, \pm i, \pm h, \pm l, \pm ih, \pm il\}$$

is the binary dihedral group D_{12} of order 12.

Let us define the following lattices in $\mathbb{H}$:

$$\Lambda_{Q_8} = \mathfrak{a}_0, \qquad \Lambda_{D_{12}} = \mathfrak{b}, \qquad \Lambda_{T_{24}} = \mathfrak{a}.$$

The set $\mathcal{X}$ of pure quaternions,

$$\mathcal{X} = \{q \in \mathbb{H} \,|\, q^2 = -1\} = \{ai + bj + ck \,|\, a^2 + b^2 + c^2 = 1\},$$

is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$. For $q \in \mathcal{X}$, one can identify $\mathbb{R} + q\mathbb{R}$ with $\mathbb{C}$ by sending q to $\sqrt{-1}$. By multiplication on the right, $\mathcal{X}$ parametrizes complex structures of $\mathbb{H} = F \otimes \mathbb{R} = \mathbb{R}^4$.

For $G = Q_8$, D_{12}, or T_{24}, such a complex structure induces a complex structure on the real torus $T_q := \mathbb{H}/\Lambda_G$. The left multiplication on $\mathbb{H}$ induces a left action of

$G = \Lambda_G^\times$ on $T_q = \mathbb{H}/\Lambda_G$, which is compatible with the complex structure induced by q; in other words, that action is holomorphic. In that way we get a holomorphic family of pairs

$$(T_q, G)_{q\in\mathcal{X}}$$

of a complex tori T_q with an action of the group automorphism G (preserving $0 \in T_q$), parametrized by $q \in \mathcal{X}_G = \mathcal{X} \simeq \mathbb{P}^1$.

We say that a group acts *symplectically* on a torus (or is *symplectic*) if its analytic representation is in $\mathrm{SL}_2(\mathbb{C}) \subset \mathrm{GL}_2(\mathbb{C})$. According to [6], the three groups $G = Q_8, D_{12}$, and T_{24} act symplectically on the torus $T_q = \mathbb{H}/\Lambda_G$.

Definition. We say that two pairs (T_1, G_1) and (T_2, G_2) of tori T_1, T_2 with action by groups G_1, G_2 are *isomorphic* if there is an isomorphism of T_1 with T_2 such that the action of G_1 on T_2 (induced by transport of structure) is G_2 (in particular $G_1 \simeq G_2$). We say that a symplectic group G acting on a torus T is *reduced* if it contains no translations.

Let G be a symplectic group of automorphisms of a torus T and let G_0 be its sub-group of translations.

Lemma 14. *The group G_0 is normal in G and G/G_0 is a reduced symplectic group of automorphisms of the torus T/G_0.*

Proof. It is easy to check that G_0 is normal (the translation sub-group of a torus is normal). The quotient T/G_0 is of course a torus; the group G/G_0 acts on T/G_0 symplectically since the analytic representations of an element in G or its image in G/G_0 are the same. □

We say that a finite reduced group G is *maximal* if G is not a strict sub-group of another reduced finite symplectic group. Let G be a non-cyclic group of symplectic automorphisms of a torus T, fixing one point globally (which we can suppose to be the origin; that hypothesis implies that G is reduced). We have the following theorem.

Theorem 15 (Fujiki [6, Proposition 3.5 and Theorem 3.11])**.** *The group G is isomorphic to one of the groups Q_8, D_{12}, or T_{24}.*

If G is maximal, then there exists $q \in \mathcal{X}$ such that (T, G) is isomorphic to (T_q, G), where $T_q = \mathbb{H}/\Lambda_G$ with complex structure given by q.

If G is not maximal, then $G = Q_8$ and there exists $q \in \mathcal{X}$ such that (T, G) is isomorphic to (T_q, Q_8), where $T_q = \mathbb{H}/\Lambda_{T_{24}}$ and $Q_8 \subset T_{24}$ is the unique quaternion group of order 8 contained in T_{24}.

For $q \in \mathcal{X}$ and $T_q = \mathbb{H}/\Lambda_{T_{24}}$, let us now denote by $\mathbb{A}(T_q)$ and $\mathbb{A}_0(T_q)$ respectively the group of real affine automorphisms and the group of translations of T_q. Then $\mathbb{A}(T_q)$ is naturally a semi-direct product $\mathbb{A}(T_q) = \mathrm{Aut}_{\mathbb{Z}}\,\Lambda_{T_{24}} \ltimes \mathbb{A}_0(T_q)$. Let $\lambda \in \Lambda_{T_{24}}^\times$ (acting by left multiplication) and $r \in T_q$. Then the action $(\lambda; r)\,x \to \lambda x + r$ is biholomorphic on T_q so that we have the natural embedding $\Lambda_{T_{24}}^\times \ltimes \mathbb{A}_0(T_q) \subset \mathbb{A}(T_q)$. Let us define the sub-groups $\hat{Q}_8$ and $\hat{T}_{24}$ of $\mathbb{A}(T_q)$ as follows:

$$\hat{Q}_8 = \{1, \pm i, \pm j', \pm k'\}$$

for $j' = (j; \alpha)$, $k' = (k; \alpha)$ where $\alpha = \frac{1}{2}(1 + i)$, and

$$\hat{T}_{24} = \langle \hat{Q}_8,\, (t; \frac{1}{2}s)\rangle, \text{ for } s = \frac{1}{2}(1 + i - j + k)$$

(we recall that $t = \frac{1}{2}(1+i+j+k)$). Thus by definition $\hat{Q}_8 \subset \hat{T}_{24}$. For $q \in \mathcal{X}$, the group $\hat{T}_{24}$ acts symplectically on the torus $T_q = \mathbb{H}/\Lambda_{\hat{T}_{24}}$. That action is without global fixed points, and so is the action of the sub-group $\hat{Q}_8 \subset \hat{T}_{24}$. One has $\hat{Q}_8 \simeq Q_8$ and $\hat{T}_{24} \simeq T_{24}$ as abstract groups.

Theorem 16 (Fujiki [6, Theorem 3.17])**.** *Let G be a reduced finite group acting symplectically on a torus T such that there is no global fixed point.*

The group G is isomorphic to Q_8 or T_{24}. If $G \simeq Q_8$ (resp. T_{24}), then there exists $q \in \mathcal{X}$ such that (T, G) is isomorphic to $(T_q, \hat{Q}_8)$ (resp. $(T_q, \hat{T}_{24})$), where in both cases $T_q = \mathbb{H}/\Lambda_{24}$.

Remark 17.

(A) Any action of Q_8 on the torus $\mathbb{H}/\Lambda_{Q_8}$ has a global fixed point.

(B) By [6, Proposition 5.7, p. 62], the complex torus T_q is algebraic if and only if $\exists \mu \in \mathbb{R}$, $\mu q \in \Lambda_G$. There are an infinite number of such $q \in X$. Moreover if T_q is algebraic, it has maximal Picard number.

In the following table we summarize the 10 ADE configurations on generalized Kummer surfaces:

Configuration	Groups	References for K_G	ρ
$16A_1$	$\mathbb{Z}/2\mathbb{Z}$	[15, 17]	16
$9A_2$	$\mathbb{Z}/3\mathbb{Z}$	[2]	18
$6A_1 + 4A_3$	$\mathbb{Z}/4\mathbb{Z}$	[2]	18
$5A_1 + 4A_2 + A_5$	$\mathbb{Z}/6\mathbb{Z}$	[2]	18
$2A_1 + 3A_3 + 2D_4$	Q_8	[7, §4.2.2], [23, Prop. 2.1]	19
$3A_1 + 4D_4$	$Q_8 \subset T_{24}$	[7, §4.2.3]	19
$A_1 + 6A_3$	$\hat{Q}_8$		19
$A_1 + 2A_2 + 3A_3 + D_5$	Q_{12}	[7, §4.2.5]	19
$A_1 + 4A_2 + D_4 + E_6$	T_{24}	[7, §4.2.4], [23, Prop. 2.1]	19
$4A_2 + 2A_3 + A_5$	$\hat{T}_{24}$		19

The column ρ gives the contribution of the given configuration of (-2)-curves to the Picard number of the K3 surface.

About generalized Kummer surfaces, one must cite the work of Enriques and Severi [5], who were the first to study generalized Kummer surfaces obtained as quotients of Jacobians of curves more than one century ago. They saw the 10 cases of the above table. They also described the resulting singularities (with errors for some non-cyclic groups).

In [4] Çinkir and Önsiper study generalized Kummer surfaces and describe the quotient singularities (but some cases are missing). In [19] Önsiper and Sertöz give a generalization of Shioda-Inose structures to these generalized Kummer surfaces. In [2], Bertin describes the primitive sub-lattices containing the configurations for cyclic groups $\mathbb{Z}_n$, $n \in \{3, 4, 6\}$, after the work of Nikulin [17] (and Morrison [15]) for $n = 2$. In [23], Wendland studies that problem for some non-cyclic groups preserving globally a point, a work which was later corrected and completed by Garbagnati in [7].

To be more exhaustive, one must also mention that Fujiki studied the possible ADE singularities in [6], and Katsura [11] worked out the possible symplectic groups in characteristic > 0, illustrating each case by examples.

3.2. **The configuration $\hat{Q}_8$: A_1+6A_3.** Let X be the K3 surface obtained as the desingularization of the quotient $T_q/\hat{Q}_8$ of a torus $T_q = \mathbb{H}/\Lambda_{T_{24}}$ by the action of $\hat{Q}_8 \subset \hat{T}_{24}$ described in section 3.1.

Lemma 18. *The singularities of the quotient surface $T_q/\hat{Q}_8$ are A_1+6A_3.*

Proof. Since the square of any order 4 elements in Q_8 is the multiplication by -1 map $[-1]_T$, the fixed point sets of these elements are included in the fixed point set of $[-1]_T$, i.e., the set of 2-torsion points of T_q. For $a,b,c,d \in \{0,1\}$, let us denote by $abcd$ the 2-torsion point $\frac{a}{2}+\frac{b}{2}i+\frac{c}{2}j+\frac{d}{2}t \in T_q$. One has $i(0011)=0101$, $i(1001)=1111$, $i(0001)=1011$, $i(0111)=1101$, $j'(0000)=1100$, $j'(1010)=0110$, etc., and we obtain that the fixed point sets of the order 4 elements i, $j'=(j;\alpha)$, $k'=(k;\alpha)$ (where $\alpha=\frac{1}{2}(1+i)$) of $\hat{Q}_8$ are

$$\begin{aligned}\mathrm{Fix}(i) &= \{0000,\ 1100,\ 1010,\ 0110\},\\ \mathrm{Fix}(j') &= \{0011,\ 0101,\ 1001,\ 1111\},\\ \mathrm{Fix}(k') &= \{0001,\ 1011,\ 0111,\ 1101\}.\end{aligned}$$

Using that $k'=ij'$, $j'=-ik'$, etc., we compute that on the quotient surface there are $2A_3$ which are the images of $\mathrm{Fix}(i)$, $2A_3$ images of $\mathrm{Fix}(j')$, and $2A_3$ images of $\mathrm{Fix}(k')$. The image of the 4 remaining 2-torsion points in T_q (the orbit of 1000) is an A_1. □

Now let X be any K3 surface containing a configuration A_1+6A_3. For $1 \le r \le 6$, we denote by

$$C_r^s, \qquad 1 \le s \le 3,$$

the resolution of the $6A_3$, where $C_r^1C_r^2 = C_r^2C_r^3 = 1$ and the other intersection numbers among the curves C_r^s are 0 or -2. Let C_0 be the resolution of the A_1. The discriminant group of the lattice $F_{\hat{Q}_8}$ generated by the curves C_r^s, $1 \le r \le 6$, $s \in \{1,2,3\}$, and C_0 is $\mathbb{Z}_2 \times (\mathbb{Z}_4)^6$; it is generated by $t_0 = \frac{1}{2}C_0$ and

$$t_r = \frac{1}{4}(C_r^1+2C_r^2+3C_r^3), \qquad r \in \{1,\dots,6\}.$$

Let $K_{\hat{Q}_8}$ be the Kummer lattice of $\hat{Q}_8$, the minimal primitive sub-lattice of $\mathrm{NS}(X)$ containing the lattice $F_{\hat{Q}_8}$.

Proposition 19. *The lattice $K_{\hat{Q}_8}$ is generated by $F_{\hat{Q}_8}$ and by the divisors*

$$\delta_1 = (1,1,1,1,2,0), \qquad \delta_2 = (1,3,2,0,1,3)$$

in the base $t_1,\dots,t_6$ (up to reorder of the t_r and C_r^i).

The lattice $K_{\hat{Q}_8}$ has discriminant group $\mathbb{Z}_2 \times (\mathbb{Z}_4)^2$; the index of $F_{\hat{Q}_8}$ in $K_{\hat{Q}_8}$ equals 16.

Proof. The curves C_r^1, C_r^3, $r \in \{1,\dots,6\}$, and C_0 form a configuration of 13 disjoint A_1. Therefore there exist two linearly independent even sets supported on 12 of these curves. The curve C_0 cannot be part of such an even set (see Remark 6). Therefore, up to permuting the indices, the three even sets are

$$\begin{aligned}v_1 &= C_1^1+C_1^3+C_2^1+C_2^3+C_3^1+C_3^3+C_4^1+C_4^3,\\ v_2 &= C_3^1+C_3^3+C_4^1+C_4^3+C_5^1+C_5^3+C_6^1+C_6^3,\\ v_3 &= C_1^1+C_1^3+C_2^1+C_2^3+C_5^1+C_5^3+C_6^1+C_6^3,\end{aligned}$$

and $\frac{1}{2}v_1, \frac{1}{2}v_2, \frac{1}{2}v_3$ are in fact elements of $\mathrm{NS}(X)$. In the discriminant group of $F_{\hat{Q}_8}$, one has $\frac{1}{2}v_1 + \frac{1}{2}v_2 = \frac{1}{2}v_3$. Let us denote by L_8 the lattice spanned by $F_{\hat{Q}_8}$ and $\frac{1}{2}v_1, \frac{1}{2}v_2, \frac{1}{2}v_3$. The discriminant group of L_8 is $(\mathbb{Z}/2\mathbb{Z})^5 \times (\mathbb{Z}/4\mathbb{Z})^2$, of length $7 > \mathrm{rk}(L_8^{\perp}) = 3$; thus there exist other divisibilities.

Since a set of 12 disjoint A_1 contains at most two linearly independent even sets, these are divisibilities by 4. Comparing the length, one obtains that there are two linearly independent 4-divisible classes. In $A_1 + 6A_3$ there is no sub-configuration $4A_3{+}6A_1$ that could come from the quotient of a torus by an order 4 automorphism. A quotient of a K3 by an order 4 automorphism has singularities $4A_3 + 2A_1$, and the 4×2 disjoint configurations A_1 supported on the sub-configuration $4A_3$ of $4A_3 + 2A_1$ must be divisible by 2. Thus in our configuration $6A_3 + A_1$, these $4A_3$ are (supported 4 times on the following elements)

$$t_1, t_2, t_3, t_4 \quad \text{or} \quad t_1, t_2, t_5, t_6 \quad \text{or} \quad t_3, t_4, t_5, t_6.$$

Once the $4A_3$ are chosen, there are two choices for the $2A_1$ such that $4A_3 + 2A_1$ becomes 4-divisible: one can take two disjoint curves in the resolution of the 5th or of the 6th A_3's. Up to permuting the t_j, and also since one has some freeness to permute C_r^1 with C_r^3, one can suppose that

$$\delta_1 = (1, 1, 1, 1, 2, 0)$$

(written in the canonical base of the sub-group $\mathbb{Z}^6 \subset F_{\hat{Q}_8}^{\vee}$ generated by the t_i, $i \in \{1, \dots, 6\}$), is integral. The relations $\delta_1\delta_2 \in \mathbb{Z}$, $\delta_2^2 \in 2\mathbb{Z}$ forces the other generator δ_2 supported on t_1, t_2, t_5, t_6 to be $\delta_2 = (1, 3, 2, 0, 1, 3)$ (or $(3, 1, 2, 0, 3, 1)$, but both generate the same group in the discriminant group). Then $\delta_3 = (2, 0, 3, 1, 3, 3)$ is supported on t_3, t_4, t_5, t_6 and equals $\delta_1 + \delta_2$ in the discriminant group.

The lattice generated by $F_{\hat{Q}_8}$ and the δ_i, $i \in \{1, 2, 3\}$, has discriminant group $\mathbb{Z}_2 \times (\mathbb{Z}_4)^2$ (of length 3). Another divisibility by 2 is not possible because a set of 13 disjoint A_1 supports at most two linearly independent even sets. If there were another independent 4-divisible set, it would create other even sets. Therefore that lattice is primitive and equals $K_{\hat{Q}_8}$. ◻

Remark 20. Let Y be the K3 associated to the $\mathbb{Z}_4$-cover defined by δ_1. There exists on Y a configuration $4A_3 + 6A_1$; therefore $Y = \mathrm{Km}(T', \mathbb{Z}_4)$ for some torus T'. The order 4 automorphism τ such that Y/τ is birational to X lifts to an automorphism of T'. The group generated by the lifts and the automorphism $\sigma \in \mathrm{Aut}(T')$ such that Y is birational to T'/σ has order 16. Thus by the classification of Fujiki, it contains a translation.

Let us now prove the following result.

Proposition 21. *Let X be a $K3$ surface containing a configuration $A_1 + 6A_3$. Then there exists $q \in \mathcal{X}$ such that $X = \mathrm{Km}(T_q, \hat{Q}_8)$ where $T_q = \mathbb{H}/\mathfrak{a}$.*

Proof. By the proof of Proposition 19, there exist two linearly independent even sets which are supported on the 12 disjoint rational curves of the sub-configuration $6A_3$ in $A_1 + 6A_3$.

Taking the associated bi-double cover and its minimal model, the pull-back of the A_1 and central curves in the six A_3 comprise a set of 16 disjoint A_1 curves C_i; this is therefore a Kummer surface $\mathrm{Km}(T)$. Since the automorphisms in the group $(\mathbb{Z}/2\mathbb{Z})^2$ preserve the branch locus $\sum_1^{16} C_i$, these automorphisms lift to automorphisms of

T. By the classification of Fujiki that group must be isomorphic to Q_8, and the result follows from Fujiki's classification, Theorem 16. □

Now let X be a K3 surface such that there exists a primitive embedding of $K_{\hat{Q}_8}$ into $\mathrm{NS}(X)$.

Theorem 22. *There exists a complex torus T and a group of automorphism $G \simeq Q_8$ such that $X = \mathrm{Km}(T, G)$.*

Proof. Using Magma, one computes that the number of roots in $K_{\hat{Q}_8}$ (37 of such) equals the number of roots of $F_{\hat{Q}_8}$. Thus by Proposition 10, there exists a configuration $A_1 + 6A_3$ of smooth irreducible rational curves. We then apply Proposition 21 and Remark 12. □

3.3. **The configuration $\hat{T}_{24}$: $4A_2 + 2A_3 + A_5$.** Let $X = \mathrm{Km}(T, \hat{T}_{24})$ be a K3 surface obtained as the desingularization of the quotient of a complex torus $T = T_q$ by the action of $\hat{T}_{24}$.

One computes that the order 3 automorphism $w = (t; \frac{1}{2}s)^2$ fixes a unique 2-torsion point on the torus $T_q = \mathbb{H}/\Lambda_{\hat{T}_{24}}$; that point is not in the fixed point sets of the automorphisms i, $j' = (j, \alpha)$, $k' = (k, \alpha)$. The K3 surface $T_q/\hat{T}_{24}$ is a quotient of $T_q/\hat{Q}_8$ (where $\hat{Q}_8 \subset \hat{T}_{24}$ is the unique normal sub-group of order 8) by the order 3 automorphism w' induced by w.

An order 3 automorphism on a smooth K3 has 6 isolated fixed points. In our situation, two of these fixed points are on the isolated A_1 in $6A_3 + A_1$; thus taking the resolution one gets an A_5. The configurations $6A_3$ on $T_q/\hat{Q}_8$ are permuted by 3, creating $2A_3$ on the quotient surface. There are moreover $4A_2$ coming from the 4 other fixed points of w'. We thus obtain the following lemma.

Lemma 23. The K3 surface $X = \mathrm{Km}(T, \hat{T}_{24})$ contains a configuration $4A_2 + 2A_3 + A_5$.

Now let X be any K3 surface containing a configuration $4A_2 + 2A_3 + A_5$.

Proposition 24. *There exists a torus T with an action of the group $\hat{T}_{24}$ such that $X = \mathrm{Km}(T, \hat{T}_{24})$.*

Proof. The configuration $4A_2 + 2A_3 + A_5$ contains 8 disjoint A_2 sub-configurations. The discriminant group of $8A_2$ is $(\mathbb{Z}_3)^8$. It has length $8 > \min(16, 22 - 16) = 6$; therefore there exists a non-trivial 3-divisible class D with support on 6 of the $8A_2$. By Remark 6, the support of D is the sub-configuration $6A_2$ contained in $4A_2 + A_5$.

The surface associated to the triple cover branched on the support of D is a K3 surface Y with a configuration $6A_3 + A_1$ and having an order 3 automorphism σ. We proved in Proposition 21 that the surface Y is of type $Y = \mathrm{Km}(T, \hat{Q}_8)$. The automorphism σ must preserve the 2 linearly independent even sets on Y supported on the $6A_3$; otherwise there would be other divisibility relations. Therefore by Lemma 9, the automorphism σ lifts to the $(\mathbb{Z}_2)^2$-cover of Y, which contains a $16A_1$ configuration. These $16A_1$ are pull-backs of curves in X. Thus σ lifts to an automorphism $\tilde{\sigma}$ of T, and X is the Kummer surface associated to the group generated by $\hat{Q}_8$ and $\tilde{\sigma}$, which has order divisible by 3. By Theorem 16, that group is $\hat{T}_{24}$. □

Again let X be a K3 surface containing a configuration $4A_2+2A_3+A_5$. The discriminant group of the lattice $F_{\hat{T}_{24}}$ generated by the curves in $4A_2+2A_3+A_5$ is

$$(\mathbb{Z}_3)^4\times(\mathbb{Z}_4)^2\times\mathbb{Z}_6.$$

It has length 5. There exists an integral class $\gamma=\frac{1}{3}D$, where D is supported on the 6 disjoint A_2 on the sub-configuration $4A_2+A_5$ (see proof of Proposition 24). The discriminant group of the lattice generated by γ and $F_{\hat{T}_{24}}$ is

$$(\mathbb{Z}_3)^2\times(\mathbb{Z}_4)^2\times\mathbb{Z}_6\simeq(\mathbb{Z}_{12})^2\times\mathbb{Z}_6,$$

which has length $3=22-19$. By Remark 6, there are no other sets of 6 disjoint A_2 which are 3-divisible, nor are there even sets. Therefore we get the following result.

Proposition 25. *The lattice generated by $F_{\hat{T}_{24}}$ and δ is the minimal primitive sub-lattice $K_{\hat{T}_{24}}\subset\mathrm{NS}(X)$ containing $F_{\hat{T}_{24}}$. The discriminant group of $K_{\hat{T}_{24}}$ is $(\mathbb{Z}_{12})^2\times\mathbb{Z}_6$.*

Thus if $X=\mathrm{Km}(T,\hat{T}_{24})$, then there is a primitive embedding of $K_{\hat{T}_{24}}$ into $\mathrm{NS}(X)$. Conversely, let X be any K3 surface.

Theorem 26. *Suppose that there is a primitive embedding of $K_{\hat{T}_{24}}$ into $\mathrm{NS}(X)$. Then $X=\mathrm{Km}(T,\hat{T}_{24})$.*

Proof. Using MAGMA, it turns out that $K_{\hat{T}_{24}}$ has the same roots as $F_{\hat{T}_{24}}$. We then apply Proposition 10 and Remark 12. □

4. The case of equality in the orbifold Bogomolov-Miyaoka-Yau inequality

4.1. **K3 surfaces.** For an orbifold $\mathcal{X}$ with only ADE singularities such that X has Kodaira dimension 0 or 1, one has $c_1^2(\mathcal{X})=0$, and the second orbifold Chern number is defined by $c_2(\mathcal{X})=c_2(X)-m(\mathcal{C})$, where the rational number $m(\mathcal{C})\geq 0$ depends only on the type and number of the singularities of $\mathcal{X}$ (see section 2). For a K3 surface, part (A) of Theorem 2 is thus equivalent to $m(\mathcal{C})\leq c_2(X)=24$. Our aim is to characterize configurations $\mathcal{C}$ for which equality

$$c_2(\mathcal{X})=0$$

holds, i.e., when $m(\mathcal{C})=c_2(X)$. For any configuration $\mathcal{C}$ among the 10 configurations

$$16A_1,\ 9A_2,\ 6A_1+4A_3,\ 5A_1+4A_2+A_5,$$
$$3A_1+4D_4,\ 2A_1+3A_3+2D_4,\ A_1+2A_2+3A_3+D_5,$$
$$A_1+4A_2+D_4+E_6,\ 4A_2+2A_3+A_5,\ A_1+6A_3,$$

one has $m(\mathcal{C})=24$. Moreover:

Theorem 27. *Suppose that a K3 X contains the configuration $\mathcal{C}$ and let $X\to\mathcal{X}$ be the contraction of the curves in $\mathcal{C}$. There exists a finite group of automorphisms G acting on a torus T such that $X=\mathrm{Km}(T,G)$ and $\mathcal{X}=T/G$.*

This is a result of Nikulin [17] for $16A_1$, Bertin [2] for the cases $9A_2$, $6A_1+4A_3$, $5A_1+4A_2+A_5$ of Propositions 21 and 24 for the two last cases, and Garbagnati [7] for the remaining cases. A direct consequence follows.

Theorem 28. *For each of the* 10 *above cases, there exists a lattice* Γ *in the affine automorphism group of* $\mathbb{C}^2$ *such that* $\mathcal{X} = \mathbb{C}^2/\Gamma$.

In other words, each of the orbifold surfaces $\mathcal{X}$ is uniformizable by $\mathbb{C}^2$.

It is easy to compute that there are 8 other possible configurations $\mathcal{C}$ with Milnor number $\rho \leq 19$ (since $H^2(X,\mathbb{Z})$ has signature $(3,19)$) and $m(\mathcal{C}) = 24$. These configurations are

$$\begin{array}{llll} \mathcal{C}_1 = & 11A_1 + 2A_3, & \mathcal{C}_5 = & 5A_1 + A_2 + D_4 + D_8, \\ \mathcal{C}_2 = & 7A_1 + A_3 + 2D_4, & \mathcal{C}_6 = & 5A_1 + A_3 + A_4 + D_7, \\ \mathcal{C}_3 = & 5A_1 + A_3 + A_7 + D_4, & \mathcal{C}_7 = & 2A_1 + 2A_2 + 2D_4 + D_5, \\ \mathcal{C}_4 = & 6A_1 + 2A_2 + A_3 + D_5, & \mathcal{C}_8 = & A_1 + 4A_2 + 2D_5. \end{array}$$

The aim of this section is to prove the following result, which with Theorem 28 implies Theorem 3.

Proposition 29. *For any* $i \in \{1, \dots, 8\}$ *there is no complex K3 surface containing a configuration* $\mathcal{C}_i$.

Remark 30. Some of these configurations $\mathcal{C}_i$ may exist in characteristic $p > 0$. Indeed by [11, Corollary 3.17 and Remark 7.3], the cyclic groups $\mathbb{Z}_5, \mathbb{Z}_8, \mathbb{Z}_{10}$, $\mathbb{Z}_{12}$, the binary dihedral groups $\mathbb{D}_{n-2}$ (of order $4n-8$, creating singularity D_n) with $n \in \{4, \dots, 8\}$, and the binary octahedral and icosahedral groups act symplectically on some Abelian surfaces in characteristic $p > 0$.

4.1.1. *Configuration* $\mathcal{C}_1 = 11A_1 + 2A_3$. The $11A_1$ plus one curve from each A_3 form a set of 13 disjoint curves. By Remark 6 there are two linearly independent even sets supported on 12 curves. This is impossible since a unique curve on an A_3 cannot be part of an even set. Such a configuration $\mathcal{C}_1 = 11A_1 + 2A_3$ does not exist on a complex K3.

4.1.2. *Configuration* $\mathcal{C}_2 = 7A_1 + A_3 + 2D_4$. There are 14 disjoint rational curves: $7A_1$, one curve in A_3, plus 3 curves for each D_4. If there are 14 disjoint rational curves on a K3 surface, then there are three independent even sets, supported on all the curves. But one curve in A_3 cannot be in the support of an even set. Therefore that configuration does not exist on a complex K3 surface.

4.1.3. *Configuration* $\mathcal{C}_3 = 5A_1 + A_3 + A_7 + D_4$. Let us consider the following set of 12 disjoint rational curves supported on $\mathcal{C}_3$: $5A_1$, plus the two disjoint curves in A_3, plus two disjoint curves in A_7 (at the extrema), and three disjoint curves in D_4. It contains an even set of curves E. The two curves on the A_7 cannot be on the support of E. One must take 0 or 2 curves in the D_4; thus the even set is made of $4A_1$ plus the two disjoint curves on the A_3 and two disjoint curves on the D_4. The K3 double cover will have a configuration

$$2A_1 + A_1 + 2A_7 + A_3,$$

but it would have Picard number > 20, a contradiction.

4.1.4. *Configuration* $\mathcal{C}_4 = 6A_1 + 2A_2 + A_3 + D_5$. There is a set of 13 disjoint rational curves on $\mathcal{C}$. There must be two linearly independent even sets supported on 12 of these curves. But an even set cannot contain the curves in an A_2.

4.1.5. *Configuration* $\mathcal{C}_5 = 5A_1 + A_2 + D_4 + D_8$. Let us consider the following 14 curves: $5A_1$, plus one curve in A_2, plus the 3 disjoint curves in D_4 and the 5 disjoint curves in D_8. As for configuration $\mathcal{C}_2$, there are three independent even sets, supported on all the curves. But the curve in A_2 cannot be in the support of a 2-divisible even set.

4.1.6. *Configuration* $\mathcal{C}_6 = 5A_1 + A_3 + A_4 + D_7$. The sub-configuration $4A_1 + A_3 + A_4 + D_7$ contains 13 disjoint rational curves (the $5A_1$ plus 2 disjoint curves in A_3, 2 in A_4, and 4 curves in D_7). Thus there exist two independent even sets supported on 12 curves. However the 2 disjoint curves in A_4 cannot be part of an even set.

4.1.7. *Configuration* $\mathcal{C}_7 = 2A_1 + 2A_2 + 2D_4 + D_5$. There are 13 disjoint rational curves on $\mathcal{C}_7$. Thus there exist two linearly independent even sets, and we obtain a contradiction as before by looking at the possible supports for these two even sets.

4.1.8. *Configuration* $\mathcal{C}_8 = A_1 + 4A_2 + 2D_5$. The discriminant group of $\mathcal{C}_8$ is

$$\mathbb{Z}_2 \times (\mathbb{Z}_3)^2 \times (\mathbb{Z}_{12})^2.$$

It has length 5, but the lattice has rank 19, and a minimal primitive sub-lattice of rank 19 has a discriminant with length at most 3. Thus there exist some divisibilities by 2 or 3. But it is easy to check using Remark 6 that no such even set can exist, nor does there exist a 3-divisible set of $6A_2$.

4.2. **Enriques surfaces.** An Enriques surface Z has invariants $K_Z^2 = 0$, $c_2 = 12$ with $2K_Z = 0$. It is the quotient of a K3 by a fix-point free involution. Let $\mathcal{C}$ be a configuration of ADE curves on an Enriques surface Z such that the associated orbifold $\mathcal{Z}$ has Chern numbers $c_1^2(\mathcal{Z}) = 3c_2(\mathcal{Z})$.

Proposition 31. *The configuration $\mathcal{C}$ is $\mathcal{C} = 8A_1$. There exist an Abelian surface A isogeneous to the product of two elliptic curves, a group of automorphisms $G \simeq (\mathbb{Z}_2)^2$ of the surface A generated by the involution $[-1]$, and a fix-point free involution such that Z is the minimal resolution of A/G.*

Proof. For an Enriques surface the condition $c_1^2(\mathcal{Z}) = 3c_2(\mathcal{Z})$ is equivalent to $c_2(\mathcal{Z}) = 0$, i.e., $m(\mathcal{C}) = 12$.

Let $X \to Z$ be the étale double cover of Z. The K3 surface X contains the configuration $2\mathcal{C}$, which verifies $m(2\mathcal{C}) = 24$; thus the only possibilities are $\mathcal{C} = 8A_1$ and $\mathcal{C} = 3A_1 + 2A_2$.

Let σ be the Enriques involution on X so that $Z = X/\sigma$. The involution σ preserves the $16A_1$ (resp. $6A_1 + 4A_2$) on X; thus it lifts to an automorphism σ' on the Abelian surface A such that $X = \mathrm{Km}(A)$ (resp. $X = \mathrm{Km}(A, \mathbb{Z}_4)$). Since σ has no fixed points on X, σ' has no fixed points on A either.

Let us study the case $\mathcal{C} = 8A_1$. Suppose that a lift σ' of σ has order 4; then σ'^2 is the transformation of the double cover $A \to X$, i.e., $\sigma'^2 = [-1]$. Since $H^0(Z, K_Z) = 0$, σ' must not preserve the space $H^0(A, K_A)$. Thus (up to replacing σ' by σ'^3) the eigenvalues of the analytic representation of σ' are i, i, and A is the surface $(\mathbb{C}/\mathbb{Z}[i])^2$; σ' is the multiplication by an i map composed by some translation. But such an morphism always has fixed points.

Therefore σ' has order 2, commutes with $[-1]$, and the eigenvalues of its analytic representation are $(1, -1)$. Then there exist coordinates of $T_A \simeq \mathbb{C}^2$ such that $\sigma' : A \to A$ is given by

$$\sigma'(z_1, z_2) = (-z_1, z_2) + v$$

where $v \in A$. Thus there exist a product $E_1 \times E_2$ of elliptic curves and an isogeny $E_1 \times E_2 \to A$. Moreover since σ' commutes with $[-1]$, one must have $v = -v$; i.e., v is a 2-torsion point. Since σ' has no fixed points v is non-trivial.

Let us study the case $3A_1 + 2A_3$. Suppose there exist an Abelian surface A and a group G of order 8 such that A/G is an Enriques surface with a configuration $2A_3 + 3A_1$. For an automorphism τ let τ_0 be the linear part of τ, and let G_0 be the group $\{\tau_0 \,|\, \tau \in G\}$. An element τ in the kernel K of $G \to G_0$ is a translation, but then A/G is the surface A'/G_0 where $A' = A/K$ and G_0 has order 4, which leads to a contradiction. The group G is therefore isomorphic to G_0, and since it contains an order 4 element, it is among the following groups:

$$\mathbb{Z}_8, \qquad \mathbb{Z}_4 \times \mathbb{Z}_2, \qquad \mathbb{D}_4, \quad \text{or} \quad Q_8.$$

There are no order 8 automorphisms acting on an Abelian surface [6]; thus $G \neq \mathbb{Z}_8$. The group G is generated by a fix-point free involution σ and an automorphism μ of order 4 such that A/μ is a K3 with $4A_3 + 6A_1$ (in particular $\mu^2 = [-1]$). Moreover the involution σ induces a fix-point free non-symplectic involution on A/μ. The group G is not Q_8 since that group has a unique involution.

Suppose that this is $\mathbb{Z}_4 \times \mathbb{Z}_2 = \langle \mu \rangle \times \langle \sigma \rangle$. Then σ_0 is $\sigma_0(z_1, z_2) = (-z_1, z_2)$, and

$$\sigma(z_1, z_2) = (-z_1, z_2) + v,$$

where v is a non-trivial 2-torsion point. Moreover since $\mu_0\sigma_0 = \sigma_0\mu_0$, the element μ must act diagonally; thus

$$\mu(z_1, z_2) = (iz_1, -iz_2).$$

Therefore $A = C \times C$, where C is the elliptic curve $\mathbb{C}/\mathbb{Z}[i]$. One has

$$\sigma\mu(z_1, z_2) = (-iz_1, -iz_2) + v,$$

which always has some fixed points, creating $\frac{1}{4}(1, 1)$ singularities, but there are no such singularities on Enriques surfaces.

The dihedral group $\mathbb{D}_4$ of order 8 remains. There is only one faithful 2-dimensional representation of $\mathbb{D}_4$, which is generated by

$$\sigma_0(z_1, z_2) = (-z_1, z_2), \qquad \mu(z_1, z_2) = (-z_2, z_1).$$

Taking $\sigma(z_1, z_2) = (-z_1, z_2) + v$ where v is a 2-torsion point, the involution $\sigma\mu$ has a one-dimensional fixed point set; thus the quotient of A by $\mathbb{D}_4$ is a rational surface (see e.g. [11]). We have thus proved that there is no Enriques surface containing a configuration $3A_1 + 2A_3$. $\square$

Example 32 (Lieberman, [12])**.** Let A be the product of two elliptic curves $A = E_1 \times E_2$ and let (e_1, e_2) be a 2-torsion point on A, with $e_1 \neq 0$, $e_2 \neq 0$. Then the endomorphism $\tau : A \to A$ given by

$$\tau(z_1, z_2) = (-z_1 + e_1, z_2 + e_2)$$

induces a fix-point free involution on the Kummer surface $\mathrm{Km}(A)$. The associated Enriques surface contains an $8A_1$ configuration.

Let T_0 be a sub-group of torsion points on $A = E_1 \times E_2$ as above such that $\tau(T_0) = T_0$ and $(e_1, 0)$, $(0, e_2)$ are not elements of T_0. Then τ induces a fix-point free involution τ' on the quotient A/T_0, and $A/\langle T_0, [-1], \tau \rangle$ is an Enriques surface containing $8A_1$. Reciprocally, from the proof of Proposition 31, every Enriques surface containing $8A_1$ is obtained by that construction.

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Aix-Marseille Université, CNRS, Centrale Marseille, I2M UMR 7373, 13453 Marseille, France
Email address: Xavier.Roulleau@univ-amu.fr