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The Bolza Curve and Some Orbifold Ball Quotient Surfaces

Vincent Koziarz, Carlos Rito, & Xavier Roulleau

Abstract. We study Deraux's nonarithmetic orbifold ball quotient surfaces obtained as birational transformations of a quotient X of a particular Abelian surface A. Using the fact that A is the Jacobian of the Bolza genus 2 curve, we identify X as the weighted projective plane $\mathbb{P}(1, 3, 8)$. We compute the equation of the mirror M of the orbifold ball quotient (X, M), and by taking the quotient by an involution we obtain an orbifold ball quotient surface with mirror birational to an interesting configuration of plane curves of degrees 1, 2, and 3. We also exhibit an arrangement of four conics in the plane that provides the above-mentioned ball quotient orbifold surfaces.

1. Introduction

Chern numbers of smooth complex surfaces of general type X satisfy the Bogomolov–Miyaoka–Yau inequality $c_1^2(X) \leq 3c_2(X)$. Surfaces for which the equality is reached are ball quotient surfaces: there exists a cocompact torsion-free lattice Γ in the automorphism group $\mathrm{PU}(2, 1)$ of the ball B_2 such that $X = B_2/\Gamma$. This description of ball quotient surfaces by uniformization is of transcendental nature, and in fact among ball-quotient surfaces, very few are constructed geometrically (e.g. by taking cyclic covers of known surfaces or by explicit equations of an embedding in a projective space).

Among lattices in $\mathrm{PU}(2, 1)$, only 22 commensurability classes are known to be nonarithmetic. The first examples of such lattices were given by Mostow [22] and Deligne and Mostow [10], and recently Deraux, Parker, and Paupert [14; 15] constructed some more, sometimes related to an earlier work of Couwenberg, Heckman, and Looijenga [9].

Being rare and difficult to produce, these examples are particularly interesting, and we would like to have a geometric description of them. To this end, Deraux [12] studied the quotient of the Abelian surface $A = E \times E$, where E is the elliptic curve $E = \mathbb{C}/\mathbb{Z}[i\sqrt{2}]$, by an order 48 automorphism group isomorphic to $\mathrm{GL}_2(\mathbb{F}_3)$, which we denote by G_{48}. The ramification locus of the quotient map $A \to A/G_{48}$ is the union of 12 elliptic curves and two orbits of isolated fixed points. The images of these two orbits are singularities of type A_2 and $\frac{1}{8}(1, 3)$, respectively.

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Then Deraux proves that (on some birational transforms) the one-dimensional branch locus M_{48} of the quotient map $A \to A/G_{48}$ and the two singularities are the support of four ball-quotient orbifold structures, three of these corresponding to nonarithmetic lattices in PU(2, 1). Knowing the branch locus M_{48} is therefore important for these ball-quotient orbifolds, since it gives an explicit geometric description of the uniformization maps from the ball to the surface.

Deraux [12] also remarks that the invariants of A/G_{48} and its singularities are the same as for the weighted projective plane $\mathbb{P}(1, 3, 8)$, and, in analogy with cases in [11] and [13], where weighted projective planes appear in the context of ball-quotient surfaces, he asks whether the two surfaces are isomorphic.

In fact, the quotient A/G_{48} can also be seen as a quotient $\mathbb{C}^2/G$ where G is an affine crystallographic complex reflection group. A conjecture states that if G' is any affine crystallographic complex reflection group acting on a complex affine space V, then the quotient V/G' is a weighted projective space. Using theta functions, Bernstein and Schwarzman [2] observed that for many examples the conjecture is true. Kaneko, Tokunaga, and Yoshida [20] worked out some other cases, and it is believed that this analog of the Chevalley theorem always happens (see [2], [16, p. 17]), although no general method is known (see also the presentation of the problem given by Deraux [12], where more details can be found).

In this paper, we prove that indeed:

THEOREM A. *The surface A/G_{48} is isomorphic to* $\mathbb{P}(1, 3, 8)$.

We obtain this result by exploiting the fact that A is the Jacobian of a smooth genus 2 curve θ, a curve that was first studied by Bolza [5]. The automorphism group of the curve θ induces the action of G_{48} on the Jacobian A. The main idea to obtain Theorem A is to understand the image of the curve θ in A by the quotient map $A \to A/G_{48}$ and to prove that its strict transform in the minimal resolution is a (-1)-curve.

We then construct birational transformations of $\mathbb{P}(1, 3, 8)$ to $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ and obtain the equations of the images $M_{\mathbb{P}^1\times\mathbb{P}^1}$ and $M_{\mathbb{P}^2}$ of the branch curve M_{48} in these surfaces (and also $M_{48} \subset \mathbb{P}(1, 3, 8)$). In particular:

THEOREM B. *In the projective plane the mirror $M_{\mathbb{P}^2}$ is the quartic curve*

$$(x^2 + xy + y^2 - xz - yz)^2 - 8xy(x + y - z)^2 = 0.$$

This curve has two smooth flex points and a singular set $\mathfrak{a}_1 + 2\mathfrak{a}_2$ (where an $\mathfrak{a}_k$ singularity has the local equation $y^2 - x^{k+1} = 0$). The line L_0 through the two residual points of the flex lines F_1 and F_2 contains the node (by flex line we mean the tangent line to a flex point).

The curve $M_{\mathbb{P}^2}$ with the two flex lines F_1 and F_2 gives rise to the four orbifold ball-quotient surfaces (previously described by Deraux [12]) on suitable birational transformations of the plane. We prove that the configuration of curves described in Theorem B is unique up to projective equivalence.

Hirzebruch [18] constructed ball-quotient surfaces using arrangements of lines and performing Kummer coverings. It is a well-known question whether we can construct other ball-quotient surfaces using higher-degree curves, the next case being arrangements of conics.

Let φ be the Cremona transformation of the plane centered at the three singularities of $M_{\mathbb{P}^2}$. The image by φ of the curves $M_{\mathbb{P}^2}$, F_1, F_2, L_0 described in Theorem B is a special arrangement of four plane conics. We remark that by performing birational transforms of $\mathbb{P}^2$ and by taking the images of the four conics we can obtain the orbifold ball-quotients of [12]. To our knowledge, that gives the first example of orbifold ball-quotients obtained from a configuration of conics (ball-quotient orbifolds obtained from a configuration of a conic and three tangent lines are studied in [19] and [28]). However, we do not know whether we can obtain ball-quotient surfaces by performing Kummer coverings branched at these conics.

When preparing this paper, we observed that the mirror $M_{\mathbb{P}^1\times\mathbb{P}^1}$ and one related orbifold ball-quotient surface among the four might be invariant by an order 2 automorphism. Using the equation we have obtained for $M_{\mathbb{P}^2}$, we prove that this is actually the case: there is an involution σ on $\mathbb{P}^1 \times \mathbb{P}^1$ with fixed point set a $(1, 1)$-curve D_i such that the quotient surface is $\mathbb{P}^2$; moreover, the image of D_i is a conic C_o, and the image of $M_{\mathbb{P}^2}$ is the unique cuspidal cubic curve C_u. In the last section we obtain and describe the following result.

THEOREM C. *There is an orbifold ball-quotient structure on a surface W birational to $\mathbb{P}^2$ such that the strict transforms on W of C_o and C_u have weights* 2 *and* ∞, *respectively.*

The paper is structured as follows. In Section 2, we recall some results of Deraux on the quotient surface A/G_{48} and introduce some notation. In Section 3, we study properties of the surface $\mathbb{P}(1, 3, 8)$. In Section 4, we introduce the Bolza curve θ and prove that A/G_{48} is isomorphic to $\mathbb{P}(1, 3, 8)$. Section 5 is devoted to the equation of the mirror $M_{\mathbb{P}^2}$. Moreover, we describe the four conics configuration. Section 6 deals with Theorem C.

Some of the proofs in Sections 5 and 6 use the computational algebra system Magma, version V2.24-5. A text file containing only the Magma code that appears further is available as an auxiliary file on arXiv and in [25].

Along this paper, we use intersection theory on normal surfaces as defined by Mumford [23, Section 2].

2. Quotient of A by G_{48} and Image of the Mirrors

2.1. Properties of A/G_{48} and Image of the Mirrors

In this section, we collect some facts from [12] about the action of the automorphism subgroup G_{48} on the Abelian surface

$$A := \mathbb{C}^2/\big(\mathbb{Z}[i\sqrt{2}]\big)^2.$$

There exists a group G_{48} of order 48 acting on A that is isomorphic to $\mathrm{GL}_2(\mathbb{F}_3)$ (see [12, Section 3.1] for generators). The action of G_{48} on A has no global fixed points (in particular, some elements have a nontrivial translation part).

The group G_{48} contains 12 order 2 reflections, that is, their linear parts acting on the tangent space $T_A \simeq \mathbb{C}^2$ are complex order 2 reflections. The fixed point set of a reflection being usually called a mirror, we similarly call the fixed point set of a reflection τ of G_{48} a *mirror*. The mirror of such τ is an elliptic curve on A. The group G_{48} acts transitively on the set of the 12 mirrors, whose list can be found in [12, Table 1].

We denote by M the union of the mirrors in A and by M_{48} the image of M in the quotient surface A/G_{48}. The curve M_{48} is also called the mirror of A/G_{48}.

Except the points on M, there are two orbits of points in A with nontrivial isotropy, one with isotropy group of order 3 at each point and the other with isotropy group of order 8; see [12, Prop. 4.4]. Correspondingly, the quotient A/G_{48} has two singular points, which are the images of two special orbits.

PROPOSITION 1. *The surface A/G_{48} is rational, and its singularities are of type $A_2 + \frac{1}{8}(1,3)$.*

The minimal resolution $p : X_{48} \to A/G_{48}$ of the surface A/G_{48} has invariants $K^2_{X_{48}} = 5$ and $c_2(X_{48}) = 7$.

Proof. Let us compute the invariants of X_{48}. Let $\pi : A \to A/G_{48}$ be the quotient map. We have

$$\mathcal{O}_A = K_A = \pi^* K_{A/G_{48}} + M. \tag{2.1}$$

Moreover, according to [12, §4], each mirror M_i, $i = 1, \dots, 12$, satisfies $M_i M = 24$; therefore $M^2 = 288$, and

$$(K_{A/G_{48}})^2 = \frac{1}{48} M^2 = 6.$$

We observe that $M = \pi^*(\frac{1}{2} M_{48})$, and thus by (2.1) we get $M_{48} = -2K_{A/G_{48}}$.

The singularities of the quotient surface A/G_{48} are computed in [12, Table 2]. Let C_1, C_2 be the two (-3)-curves above the singularity $\frac{1}{8}(1,3)$; they are such that $C_1 C_2 = 1$. Since the singularity of type A_2 is an *ADE* singularity, we obtain

$$K_{X_{48}} = p^* K_{A/G_{48}} - \frac{1}{2}(C_1 + C_2)$$

and $(K_{X_{48}})^2 = 5$.

Let τ be a reflection in G_{48}, and let G be the Klein group of order 4 generated by τ and the involution $[-1]_A \in G_{48}$. We can check that the quotient surface A/G is rational. Being dominated by the rational surface A/G, the surface A/G_{48} is also rational. Thus the second Chern number is $c_2(X_{48}) = 7$ by Noether's formula. □

The mirror M_{48} (the image of M by the quotient map) contains no singularities of A/G_{48}. Moreover:

LEMMA 2. *The pull-back $\tilde{M}_{48}$ of the mirror M_{48} by the resolution map $p : X_{48} \to A/G_{48}$ has self-intersection* 24. *Its singular set is*

$$2\mathfrak{a}_2 + \mathfrak{a}_3 + \mathfrak{a}_5,$$

where $\mathfrak{a}_k$ denotes a singularity with local equation $y^2 - x^{k+1} = 0$.

Proof. The singularities of $\tilde{M}_{48} = p^* M_{48}$ are the same as the singularities of M_{48} since M_{48} is in the smooth locus of A/G_{48}. For the computation of the singularities of M_{48}, we refer to [12, Table 3], and for the self-intersection of $\tilde{M}_{48}$ (which is the same as the one of M_{48}), to [12, §6.2]. □

3. The Weighted Projective Space $\mathbb{P}(1, 3, 8)$

Since we aim to prove that the quotient surface A/G_{48} is isomorphic to $\mathbb{P}(1, 3, 8)$, we first have to study that weighted projective space; this is the goal of this (technical) section. The reader might at first browse through the main results and notation and proceed to the next section.

3.1. *The Surface* $\mathbb{P}(1, 3, 8)$ *and Its Minimal Resolution*

The weighted projective space $\mathbb{P}(1, 3, 8)$ is the quotient of $\mathbb{P}^2$ by the group $\mathbb{Z}_3 \times \mathbb{Z}_8$ generated by

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & \zeta \end{pmatrix} \in \mathrm{PGL}_3(\mathbb{C}),$$

where $j^2 + j + 1 = 0$, and ζ is a primitive 8th root of unity. The fixed point set of the order 24 element σ is

$$p_1 = [1 : 0 : 0], \qquad p_2 = [0 : 1 : 0], \qquad p_3 = [0 : 0 : 1].$$

For $i, j \in \{1, 2, 3\}$ with $i \neq j$, let L'_{ij} be the line through p_i and p_j. The fixed point set of an order 3 element (e.g., σ^8) is p_2 and the line L'_{13}. The fixed point set of an order 8 element (e.g., σ^3) and its nontrivial powers is p_3 and the line L'_{12}. Let $\pi : \mathbb{P}^2 \to \mathbb{P}(1, 3, 8)$ be the quotient map: π is ramified with order 3 over L'_{13} and with order 8 over L'_{12}. The surface $\mathbb{P}(1, 3, 8)$ has two singularities, images of p_2 and p_3, which are respectively a cusp A_2 and a singularity of type $\frac{1}{8}(1, 3)$. We denote by $p : Z \to \mathbb{P}(1, 3, 8)$ the minimal desingularization map. The singularity of type $\frac{1}{8}(1, 3)$ is resolved by two rational curves C_1 and C_2 with $C_1 C_2 = 1$ and $C_1^2 = C_2^2 = -3$, and the singularity A_2 is resolved by two rational curves C_3 and C_4 with $C_3 C_4 = 1$ and $C_3^2 = C_4^2 = -2$ (see, e.g., [1, Chap. III]).

LEMMA 3. *The invariants of the resolution Z are*

$$K_Z^2 = 5, \qquad c_2(Z) = 7, \qquad p_q = q = 0.$$

Proof. We have:

$$K_{\mathbb{P}^2} \equiv \pi^* K_{\mathbb{P}(1,3,8)} + 2L'_{13} + 7L'_{12}.$$

Therefore, since $K_{\mathbb{P}^2} \equiv -3L$, we obtain $\pi^* K_{\mathbb{P}(1,3,8)} \equiv -12L$ and

$$(K_{\mathbb{P}(1,3,8)})^2 = \frac{(-12L)^2}{24} = 6.$$

We have

$$K_Z \equiv p^* K_{\mathbb{P}(1,3,8)} - \sum_{i=1}^{4} a_i C_i,$$

where a_i are rational numbers. The divisor K_Z must satisfy the adjunction formula, that is, we must have $C_i K_Z = -2 - C_i^2$ for $i \in \{1,2,3,4\}$, which gives

$$K_Z = p^* K_{\mathbb{P}(1,3,8)} - \frac{1}{2}(C_1 + C_2),$$

and therefore $K_Z^2 = 5$. For the Euler number, we may use the formula in [26, Lemma 3]:

$$e(\mathbb{P}(1,3,8)) = \frac{1}{24}(3 + 2(2-2) + 7(2-2) + 23 \cdot 3) = 3.$$

Thus $e(Z) = e(\mathbb{P}(1,3,8)) - 2 + 3 + 3 = 7$. Since $\mathbb{P}(1,3,8)$ is dominated by $\mathbb{P}^2$, the surface Z is rational, so that $q = p_g = 0$. □

3.2. *The Branch Curves in* $\mathbb{P}(1,3,8)$ *and Their Pullback in the Resolution*

Let L_{ij} be the image of the line L'_{ij} on $\mathbb{P}(1,3,8)$, and let $\bar{L}_{ij}$ be the strict transform of L_{ij} in Z.

PROPOSITION 4. *We have*:

$$\bar{L}_{23}^2 = -1, \quad \bar{L}_{23}C_1 = \bar{L}_{23}C_3 = 1, \quad \bar{L}_{23}C_2 = \bar{L}_{23}C_4 = 0,$$
$$\bar{L}_{13}^2 = 0, \quad \bar{L}_{13}C_2 = 1, \quad \bar{L}_{13}C_1 = \bar{L}_{13}C_3 = \bar{L}_{13}C_4 = 0,$$
$$\bar{L}_{12}^2 = 2, \quad \bar{L}_{12}C_4 = 1, \quad \bar{L}_{12}C_1 = \bar{L}_{12}C_2 = \bar{L}_{12}C_3 = 0.$$

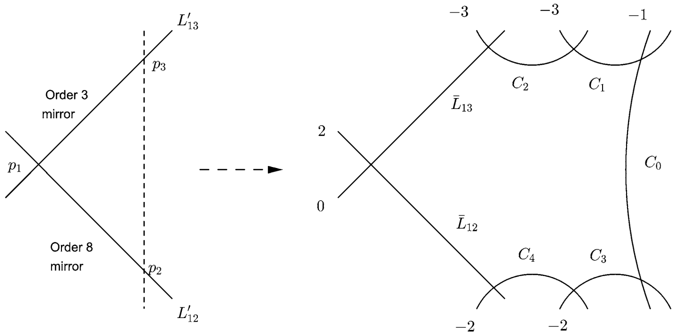

Figure 1 Image of the lines L'_{ij} in the desingularization of $\mathbb{P}(1,3,8)$.

Proof. On $\mathbb{P}(1,3,8)$, we have $L_{23}^2 = \frac{1}{24} L_{23}'^2 = \frac{1}{24}$. Recall that the resolution map is $p : Z \to \mathbb{P}(1,3,8)$. Let $a_1, \ldots, a_4 \in \mathbb{Q}$ be such that

$$\bar{L}_{23} = p^* L_{23} - \sum_{i=1}^{4} a_i C_i.$$

Then $C_i p^* L_{23} = 0$ for $i \in \{1,2,3,4\}$. Let $u_i \in \mathbb{N}$ be such that $C_i \bar{L}_{23} = u_i$. We get that

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad \begin{pmatrix} a_3 \\ a_4 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} u_3 \\ u_4 \end{pmatrix}.$$

We have $\pi^* K_{\mathbb{P}(1,3,8)} = -12 L'_{23}$, and thus

$$K_{\mathbb{P}(1,3,8)} L_{23} = \frac{1}{24}(-12 L'_{23} \cdot L'_{23}) = -\frac{1}{2}.$$

Since $K_Z = p^* K_{\mathbb{P}(1,3,8)} - \frac{1}{2}(C_1 + C_2)$, we get

$$\begin{aligned} K_Z \bar{L}_{23} &= \left(p^* K_{\mathbb{P}(1,3,8)} - \frac{1}{2}(C_1 + C_2) \right) \left(p^* L - \sum_{i=1}^{4} a_i C_i \right) \\ &= -\frac{1}{2} - a_1 - a_2 = -\frac{1}{2}(1 + u_1 + u_2), \end{aligned}$$

which is in $\mathbb{Z}$, with $u_1, u_2 \in \mathbb{N}$. We compute that

$$\bar{L}_{23}^2 = \frac{1}{24} - \frac{1}{8}(3u_1^2 + 3u_2^2 + 2u_1 u_2) - \frac{2}{3}(u_3^2 + u_3 u_4 + u_4^2) \in \mathbb{Z}_{\leq 0}.$$

Since $K_Z \bar{L}_{23} + \bar{L}_{23}^2 = -2$, the only possibility is

$$\{u_1, u_2\} = \{0, 1\}, \qquad \{u_3, u_4\} = \{0, 1\},$$

which gives the intersection numbers with $\bar{L}_{23}$.

For the curve L_{13}, we have $L_{13} K_{\mathbb{P}(1,3,8)} = -\frac{3}{2}$ and $L_{13}^2 = \frac{3}{8}$. Let $u := \bar{L}_{13} C_1 \in \mathbb{N}$, $v := \bar{L}_{13} C_2 \in \mathbb{N}$. Then we similarly compute that

$$\bar{L}_{13} K_Z = -\frac{1}{2}(3 + u + v) \leq -\frac{3}{2}$$

and

$$\bar{L}_{13}^2 = \frac{1}{8}(3 - 3u^2 - 3v^2 - 2uv) \leq \frac{3}{8}.$$

Therefore $\bar{L}_{13}^2 + K_Z \bar{L}_{13} \leq -\frac{9}{8}$, and since $\bar{L}_{13}^2 + K_Z \bar{L}_{13} \geq -2$, the only solution is $\{u, v\} = \{0, 1\}$, and thus $\bar{L}_{13}^2 = 0$ and $\bar{L}_{13} K_Z = -2$.

For the curve L_{12}, which does not go through the $\frac{1}{8}(1,3)$ singularity, we have

$$\bar{L}_{12} K_Z = L_{12} K_{\mathbb{P}(1,3,8)} = -4$$

and $L_{12}^2 = \frac{8}{3}$. Let $w := \bar{L}_{12} C_3$, $t := \bar{L}_{12} C_4$. Then

$$\bar{L}_{12}^2 = \frac{1}{3}(8 - 2w^2 - 2t^2 - 2wt) \leq \frac{8}{3}.$$

Therefore $\bar{L}_{12}^2 + K_Z\bar{L}_{12} \leq -\frac{4}{3}$, and the only solution is $\{w, t\} = \{0, 1\}$; thus $\bar{L}_{12}^2 = 2$. □

3.3. *From* $\mathbb{P}(1, 3, 8)$ *to the Hirzebruch Surface* $\mathbb{F}_3$ *and Back*

By contracting the (-1)-curve $C_0 := \bar{L}_{23}$ and then the other (-1)-curves appearing from the configuration $C_1, \dots, C_4, \bar{L}$, we get a rational surface with

$$K^2 = 2c_2 = 8,$$

containing (depending on the choice of the (-1)-curves we contract) a curve, which is either a (-2)-curve or a (-3)-curve. Thus that surface is one of the Hirzebruch surfaces $\mathbb{F}_2$ or $\mathbb{F}_3$. Conversely, we can reverse the process and obtain the surface $\mathbb{P}(1, 3, 8)$ by performing a sequence of blow-ups and blow-downs. This process is unique: this follows from the fact that the automorphism group of a Hirzebruch surface $\mathbb{F}_n$, $n \geq 1$, has two orbits, which are the unique $(-n)$-curve and its open complement (see, e.g., [4]). We further will use only the connection between $\mathbb{P}(1, 3, 8)$ and $\mathbb{F}_3$.

4. The Bolza Genus 2 Curve in A and Its Image by the Quotient Map

In this section, we prove that A/G_{48} is isomorphic to $\mathbb{P}(1, 3, 8)$.

Let us consider the genus 2 curve θ whose affine model is

$$y^2 = x^5 - x. \tag{4.1}$$

It was proved by Bolza [5] that the automorphism group of θ is $\mathrm{GL}_2(\mathbb{F}_3) \simeq G_{48}$ and θ is the unique genus 2 curve with such an automorphism group.

The automorphisms of θ are generated by the hyperelliptic involution λ and the lift of the automorphism group G of $\mathbb{P}^1$ that preserves the set of six branch points $0, \infty, \pm 1, \pm i$ of the canonical map $\theta \to \mathbb{P}^1$ (i.e., the set of points which are fixed by λ). Note that in fact any map of degree 2 from θ to $\mathbb{P}^1$ is the composition of this map with an automorphism of $\mathbb{P}^1$. This is a consequence of the two following facts: on the one hand, the six ramification points (by the Riemann–Hurwitz formula) of such a map are Weierstrass points, and, on the other hand, the genus 2 curve θ has exactly six Weierstrass points.

By the universal property of the Abel–Jacobi map the group $\mathrm{GL}_2(\mathbb{F}_3)$ acts naturally on the Jacobian variety $J(\theta)$ of θ, the action on θ and $J(\theta)$ being equivariant.

There is only one Abelian surface with an action of $\mathrm{GL}_2(\mathbb{F}_3)$, which is $A = E \times E$, where $E = \mathbb{C}/\mathbb{Z}[i\sqrt{2}]$ as before (see Fujiki [17] or [3]). We identify $J(\theta)$ with A. There are up to conjugation only two possible actions of $\mathrm{GL}_2(\mathbb{F}_3)$ on A (see [24]):

(a) The action of $G_{48} \simeq \mathrm{GL}_2(\mathbb{F}_3)$ described in Section 2.1; it has no global fixed points.

(b) The one obtained by forgetting the translation part of that action. That second action globally fixes the 0 point in A.

Let $\alpha : \theta \hookrightarrow J(\theta) = A$ be the embedding of θ sending the point at infinity of the affine model (4.1) to 0; we identify θ with its image.

Note that the morphism $\theta \times \theta \to A$, $(x, y) \mapsto [y] - [x] \in \mathrm{Div}_0(\theta) \simeq A$ is onto since $\theta \times \theta$ and A are both two-dimensional. In fact, this map has generic degree 2 and contracts the diagonal. Indeed, assume that $[y] - [x] = [y'] - [x']$, that is, $[y] + [x'] - [x] - [y'] = 0 \in \mathrm{Div}_0(\theta)$. If $y' = y$, then $x' = x$ (and conversely) because there is no degree 1 map from θ to $\mathbb{P}^1$. In the same way, $y = x$ iff $y' = x'$. In the remaining cases, there exists a function of degree 2 from θ to $\mathbb{P}^1$ whose zeroes are y and x' and poles are x and y'. But by the previous remark we must have $x' = \lambda(y)$ and $y' = \lambda(x)$. Conversely, by the same argument it is clear that for all x and y in θ, $[\lambda(y)] - [\lambda(x)] = [x] - [y]$.

This also implies that the points of the type $[y] - [x]$ with x and y being distinct Weierstrass points are exactly the 2-torsion points of A. Indeed, since there are six Weierstrass points on θ, we have 15 points of that type in A satisfying $[y] - [x] = [\lambda(x)] - [\lambda(y)] = [x] - [y]$, that is, they are 2-torsion points.

The induced linear action b) is given by $g([y] - [x]) = [g(y)] - [g(x)]$, for which $0 \in \mathrm{Div}_0(\theta)$ is a fixed point.

If we fix the base point $\infty \in \theta$, then for each $y \in \theta$, $\alpha(x) = [x] - [\infty]$. The induced action of $g \in \mathrm{Aut}(\theta)$ on A is then given by $g([y] - [x]) = [g(y)] - [g(x)] + [g(\infty)] - [\infty]$. This is indeed the only action of $\mathrm{Aut}(\theta)$ on A commuting with α.

LEMMA 5. *The action of* $\mathrm{GL}_2(\mathbb{F}_3)$ *on A inducing the action of* $\mathrm{Aut}(\theta)$ *on the curve* $\theta \hookrightarrow A$ *has no global fixed points.*

Proof. The fixed points on A for the action of the hyperelliptic involution λ are its points of 2-torsion (and 0). Indeed, $\lambda([y] - [x]) = [\lambda(y)] - [\lambda(x)] \in \mathrm{Div}_0(\theta)$ since $\infty \in \theta$ is fixed by λ, and, as a consequence of the previous discussion, if $[y] - [x] = [\lambda(y)] - [\lambda(x)]$, then either $y = x$ or $y = \lambda(x)$, that is, $[y] - [x] = [x] - [y]$, and we saw that this implies that x and y are Weierstrass points.

However, for any pair (x, y) of distinct Weierstrass points, it is easy to find $g \in \mathrm{Aut}(\theta)$ (lifting an automorphism of $\mathbb{P}^1$) such that $g(\infty) = \infty$ but $[g(y)] - [g(x)] \neq [y] - [x]$. □

For $t \in A$, let θ_t be the curve $\theta_t = t + \theta$. The previous result does not depend on the choice of the embedding $\theta \hookrightarrow A$: indeed, the group of automorphisms acting on A and preserving θ_t is conjugated by the translation $x \mapsto x + t$ to the group of automorphisms acting on A and preserving θ.

We denote by H_{48} the order 48 group acting on A and inducing the automorphism group of the curve $\theta \hookrightarrow A$ by restriction. As a consequence of Lemma 5, we get the following:

COROLLARY 6. *There exists an isomorphism between* H_{48} *and* G_{48}. *It is induced by an automorphism g of the surface A such that* $H_{48} = gG_{48}g^{-1}$.

By [6, Thm. (0.3)] the embedding $\alpha : \theta \hookrightarrow A$ is such that the torsion points of A contained in θ are 16 torsion points of order 6, 5 torsion points of order 2 and the origin; moreover, the x-coordinates of the 22 torsion points on θ satisfy

$$x^4 - 4ix^2 - 1 = 0, \qquad x^4 + 4ix^2 - 1 = 0,$$
$$x^5 - x = 0, \qquad x = \infty.$$

Proposition 7. (a) *These* 22 *torsion points of* θ *are not in the mirror of any of the* 12 *complex reflections of* H_{48};
(b) *Each of these* 22 *points has a nontrivial stabilizer.*

Proof. Let us prove part (a).

The hyperelliptic involution is given by $(x, y) \to (x, -y)$. By [8] the rational map

$$v : (x, y) \mapsto \left(-\frac{x+i}{ix+1}, \sqrt{2}\frac{i-1}{(ix+1)^3}y\right)$$

defines a nonhyperelliptic involution v on θ. The x-coordinates of the fixed point set of v are $x_\pm = i(1 \pm \sqrt{2})$. These coordinates $x_\pm$ are not among the x-coordinates of the 22 torsion points in θ. Let $\mathbf{v}$ be the automorphism of A induced by v. The fixed point set of $\mathbf{v}$ is a smooth genus 1 curve E_v (a mirror), and we have just proved that E_v contains no torsion points of θ. By the transitivity of the group H_{48} on its set of 12 nonhyperelliptic involutions we get that no mirror contains any of the 22 torsion points.

Let us prove part (b).

The six 2-torsion points are the Weierstrass points of the curve θ, and they are fixed by the hyperelliptic involution (whose action on A has only 16 fixed points).

The transformation

$$w : (x, y) \mapsto \left(\frac{(1+i)x - (1+i)}{(1-i)x + (1-i)}, -\frac{1}{((1-i)x + (1-i))^3}y\right)$$

defines an order 3 automorphism of θ, which acts symplectically on A, and we compute that it fixes a torsion point $p_0 = (x_0, y_0)$ on θ with x_0 such that $x_0^4 + 4ix_0^2 - 1 = 0$, that is, it is an order 6 torsion point. This torsion point is an isolated fixed point for each nontrivial element of its stabilizer (since by part (a) it is not on a mirror).

Recall that by [12, Table 2] there are exactly two orbits of points of respective orders 6 and 16 with nontrivial stabilizer under G_{48} that are isolated fixed points of the nontrivial elements of their stabilizer (by a direct computation we can check that these two orbits are 16 points of order 6 and 6 points of order 2). Since H_{48} is conjugate to G_{48}, the 15 other 6-torsion points on θ are also isolated fixed points for each nontrivial element of their stabilizer. □

Since we can change the embedding $\theta \hookrightarrow A$ by composing with the automorphism g such that $H_{48} = gG_{48}g^{-1}$, let us identify H_{48} with G_{48}.

By Section 2.1 (or [12]) the images of the 22 torsion points of θ on the quotient surface A/G_{48} give the singularities A_2 and $\frac{1}{8}(1, 3)$.

Let m be the mirror of one of the 12 complex reflections in G_{48}.

LEMMA 8. *We have $\theta \cdot m = 2$.*

Proof. The intersection number $\theta \cdot m$ is the number of fixed points of the involution ι_m with mirror m restricted to θ. Since ι_m fixes exactly one holomorphic form, the quotient of θ by ι_m is an elliptic curve, and thus by the Hurwitz formula $\theta \cdot m = 2$. □

Let θ_{48} be the image of θ in A/G_{48}. We have the following:

PROPOSITION 9. *The strict transform C_0 of θ_{48} by the resolution $X_{48} \to A/G_{48}$ is a (-1)-curve, and we have $\tilde{M}_{48}C_0 = 1$.*

Proof. We have

$$\theta_{48}^2 = \frac{1}{48}\theta^2 = \frac{1}{24}.$$

Let $\pi : A \to A/G_{48}$ be the quotient map; it is ramified with order 2 on the union M of the 12 mirrors. We have $\pi^*(K_{A/G_{48}} + \frac{1}{2}M_{48}) = K_A = 0$, and thus

$$K_{A/G_{48}}\theta_{48} = -\frac{1}{48}(M\theta) = -\frac{1}{48}12 \cdot 2 = -\frac{1}{2}.$$

The curve θ_{48} contains the singularities $\frac{1}{8}(1,3)$ and A_2 (image respectively of the 2-torsion points and the 6-torsion points of θ). We are then left with the same combinatorial situation as in the computation of $\bar{L}_{23}^2$ in Proposition 4, and thus we conclude that $C_0^2 = -1$.

The two intersection points of m and θ in Lemma 8 are permuted by the hyperelliptic involution of θ, and thus $M_{48}\theta_{48} = 1$, which implies $\tilde{M}_{48}C_0 = 1$. □

We obtain the following:

THEOREM 10. *The surface A/G_{48} is isomorphic to $\mathbb{P}(1,3,8)$.*

Proof. Let us denote the resolution map by $p : X_{48} \to A/G_{48}$. Let C_1, C_2 be the resolution curves of the singularity $\frac{1}{8}(1,3)$, and let C_3, C_4 be the resolution of A_2. Let $a \in A$ be an isolated fixed point of an automorphism τ of order 3 or 8. The tangent space $T_{\theta,a} \subset T_{A,a}$ is stable by the action of τ. Since the local setup is the same, we can reason as in Proposition 4 and obtain that the curve C_0 is such that

$$C_0C_1 = C_0C_3 = 1, \qquad C_0C_2 = C_0C_4 = 0.$$

Contracting the curves C_0, C_1, C_2, we get a rational surface with a (-3)-curve and with invariants $K^2 = 2c_2 = 8$. This is therefore the Hirzebruch surface $\mathbb{F}_3$. From Section 3 we know that reversing the contraction process, we get the weighted projective plane $\mathbb{P}(1,3,8)$ (contracting the curves C_0, C_1, C_3, we would obtain the Hirzebruch surface $\mathbb{F}_2$). □

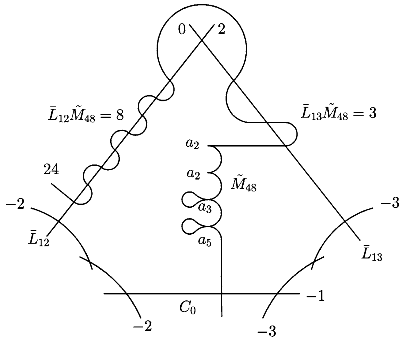

Figure 2 Configuration of curves $\tilde{M}_{48}$, $\bar{L}_{12}$, $\bar{L}_{13}$, and so on in X_{48} and their intersection numbers.

REMARK 11. Now we identify $\mathbb{P}(1,3,8)$ with A/G_{48} and use the notation in Section 3. In particular, $Z = X_{48}$ is the minimal resolution of $\mathbb{P}(1,3,8)$, the curves $C_1, \dots, C_4$ are exceptional divisors of the resolution map $Z \to \mathbb{P}(1,3,8)$, and $C_0 = \bar{L}_{23}$ is a (-1)-curve in Z.

Let us observe that the divisor $\tilde{F} = C_1 + 3C_0 + 2C_3 + C_4$ satisfies

$$\tilde{F}C_1 = \tilde{F}C_0 = \tilde{F}C_3 = \tilde{F}C_4 = 0,$$

and thus $\tilde{F}^2 = 0$; moreover, $\tilde{F}C_2 = \bar{L}_{13}\tilde{F} = 1$, $\tilde{F}\bar{L}_{13} = 0$, and $\bar{L}_{13}^2 = 0$. This implies that the curves $\tilde{F}$ and $\bar{L}_{13}$ are fibers of the same fibration onto $\mathbb{P}^1$ and C_2 is a section of that fibration.

The curves $C_0, \dots, C_4$ are exceptional divisors or strict transform of generators of the Néron–Severi group of a minimal rational surface. Thus the Néron–Severi group of the rational surface X_{48} is generated by these curves. Knowing the intersection of curves $\bar{L}_{12}$, $\bar{L}_{13}$, $\tilde{M}_{48}$ with these curves (see Propositions 4 and 9), it is easy to obtain their classes in the Néron–Severi group; in particular, we get that $\bar{L}_{12}\tilde{M}_{48} = 8$ and $\bar{L}_{13}\tilde{M}_{48} = 3$.

5. A Model of the Mirror

5.1. A Birational Map from $\mathbb{P}(1,3,8)$ *to* $\mathbb{P}^1 \times \mathbb{P}^1$*; Images of the Mirror*

5.1.1. A Rational Map $\mathbb{P}(1,3,8) \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. As before, we identify $\mathbb{P}(1,3,8)$ with A/G_{48}; we use the notation of Sections 3 and 4.

Take a point p in the Hirzebruch surface $\mathbb{F}_n$ that is not in the negative section. By blowing-up at p and then by blowing-down the strict transform of the fiber through p we get the Hirzebruch surface $\mathbb{F}_{n-1}$. This process is called an *elementary transformation.*

Recall from Sections 3 and 4 that there is a map $\psi : \mathbb{P}(1,3,8) \dashrightarrow \mathbb{F}_3$ that contracts the curves C_0, C_3, C_4 to a smooth point.

Performing any sequence of three elementary transformations as before, we get a map $\rho : \mathbb{F}_3 \dashrightarrow \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. This can be seen as a birational transform that, by

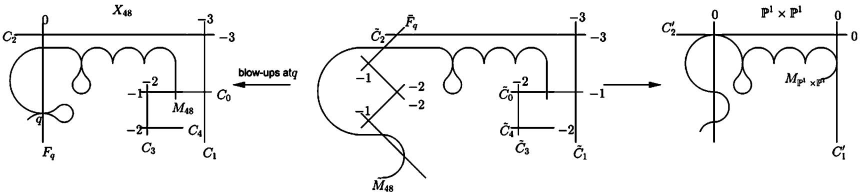

Figure 3 From X_{48} to $\mathbb{P}^1 \times \mathbb{P}^1$ and back.

blowing-up three times at a point q not contained in the negative section, takes the fiber F_q through q to a chain of curves with self-intersections (-1), (-2), (-2), (-1), then followed by the contraction of the (-1), (-2), (-2) chain (which contains the strict transform of F_q). For our purpose, we choose the three points to blow up in a specific way; see Section 5.1.2.

Consider

$$\phi := \rho \circ \psi : \mathbb{P}(1,3,8) \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1.$$

We observe that given any two points $t, t' \in \mathbb{P}^1 \times \mathbb{P}^1$ not in a common fiber, the map ϕ can be chosen such that the inverse ϕ^{-1} is not defined at t, t' and $\phi^{-1}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{P}(1,3,8)$.

5.1.2. Image of the Mirror M_{48} in $\mathbb{P}^1 \times \mathbb{P}^1$. Let us describe how to choose ϕ such that the image $M_{\mathbb{P}^1 \times \mathbb{P}^1}$ of the mirror curve M_{48} is a $(3,3)$-curve with singularities $\mathfrak{a}_3 + 2\mathfrak{a}_2$ and two special fibers tangent to it with multiplicity 3.

The map $\mathbb{P}(1,3,8) \dashrightarrow \mathbb{F}_3$ factors through a morphism $\varphi : X_{48} \to \mathbb{F}_3$. Consider the point $t_0 := \varphi(C_0)$. Since $M_{48}C_0 = 1$, $\varphi(M_{48})$ is a curve that is smooth at t_0, and its intersection number with the curve $\varphi(C_1)$ at t_0 is 3. The curve $C_1' := \rho \circ \varphi(C_1)$ is a fiber of $\mathbb{P}^1 \times \mathbb{P}^1$.

Then we choose q to be the $\mathfrak{a}_5$-singularity of M_{48}. The fiber F_q through q cuts M_{48} at q with multiplicity 2 or 3. Suppose that the multiplicity is 3. Then by taking the blowup at that point and computing the strict transform of the curves F_q and M_{48} we can check that $F_q M_{48} \geq 4$. However, $F_q M_{48} = \bar{L}_{13} M_{48} = 3$ by Remark 11. Therefore the fiber F_q through q cuts M_{48} at q with multiplicity 2 and at another point.

REMARK 12. An analogous reasoning gives that the fiber through the $\mathfrak{a}_3$-singularity has the same property: it is transverse to the tangent of the $\mathfrak{a}_3$-singularity.

The three successive blowups above q are chosen such that they resolve the singularity $\mathfrak{a}_5$. The three blowdowns we described create a multiplicity 3 tangent point between $M_{\mathbb{P}^1 \times \mathbb{P}^1}$ (the image of M_{48} in $\mathbb{P}^1 \times \mathbb{P}^1$) and the curve C_2' (the image of C_2), thus $C_2' M_{\mathbb{P}^1 \times \mathbb{P}^1} = 3$. Moreover, $C_2'^2 = 0$ and $C_1' C_2' = 1$ (see Figure 3).

The mirror M_{48} does not cut the curves C_1 and C_2. The transforms of these curves in $\mathbb{P}^1 \times \mathbb{P}^1$ are fibers C_1' and C_2' such that C_i' cuts $M_{\mathbb{P}^1 \times \mathbb{P}^1}$ at one point only

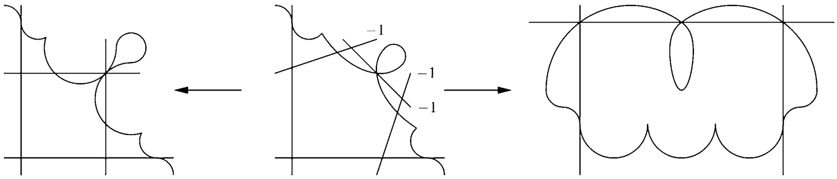

Figure 4 From $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^2$.

with multiplicity 3. In particular, the class of $M_{\mathbb{P}^1\times\mathbb{P}^1}$ in the Néron–Severi group of $\mathbb{P}^1 \times \mathbb{P}^1$ is $3C_1' + 3C_2'$. The singularities of $M_{\mathbb{P}^1\times\mathbb{P}^1}$ are $\mathfrak{a}_3 + 2\mathfrak{a}_2$.

5.1.3. From $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^2$ and Back. Let us recall that the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at a point, followed by the blowdown of the strict transform of the two fibers through that point, gives a birational map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$.

We choose to blow up the point at the $\mathfrak{a}_3$-singularity s_0, so that the strict transform of $M_{\mathbb{P}^1\times\mathbb{P}^1}$ has a node above s_0. The two fibers F_1, F_2 of $\mathbb{P}^1 \times \mathbb{P}^1$ passing through s_0 cut $M_{\mathbb{P}^1\times\mathbb{P}^1}$ in two other points respectively s_1, s_2 (see Remark 12; the result is preserved through the birational process). The fibers F_1, F_2 are contracted into points in $\mathbb{P}^2$ by the rational map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$, and the images of s_1, s_2 by that map are on the image of the exceptional divisor, which is a line L_0 through the node. This implies that the strict transform of $M_{\mathbb{P}^1\times\mathbb{P}^1}$ is a plane quartic curve $M_{\mathbb{P}^2}$. The process in illustrated in Figure 4.

The total transform of $M_{\mathbb{P}^1\times\mathbb{P}^1}$ in $\mathbb{P}^2$ is the union of $2L_0$ with $M_{\mathbb{P}^2}$. This quartic $M_{\mathbb{P}^2}$ has the following properties, which follow from its description and the choice of the transformation from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^2$:

PROPOSITION 13. *The singular set of the quartic curve $M_{\mathbb{P}^2}$ is $\mathfrak{a}_1 + 2\mathfrak{a}_2$, and the nodal point is contained in the line L_0. The curve $M_{\mathbb{P}^2}$ contains two flex points such that each corresponding tangent line meets the quartic at a second point that is contained in the line L_0.*

5.2. The Yoga Between the Mirrors $M_{\mathbb{P}^2}$ and M_{48}

Using the previous description, the reader can follow the transformations between the surfaces $\mathbb{P}(1, 3, 8)$ and the plane. The link between Deraux's ball-quotient orbifolds described in [12, Thm. 5] and the quartic $M_{\mathbb{P}^2}$ is as follows.

The singularities $\mathfrak{a}_1 + 2\mathfrak{a}_2$ of $M_{\mathbb{P}^2}$ correspond respectively to singularities $\mathfrak{a}_3 + 2\mathfrak{a}_2$ of M_{48}, so that to get the curves F, G, H in [12, Figure 1], we have to blow up and contract at these three points as it is done in [12]. To obtain the curve E in [12, Figure 1], we have to blow up the two flexes three times to separate $M_{\mathbb{P}^2}$ and the flex lines. We obtain two chains of (-1)-, (-2)-, (-2)-curves. Contracting one of the two (-2), (-2) chains, we get an A_2-singularity. The curve E is the image by the contraction map of the remaining (-1)-curve of the chain. The resolution of the singularity A_2 on $\mathbb{P}(1, 3, 8)$ corresponds to the two (-2)-curves on the other chain of (-1)-, (-2)-, (-2)-curves. After taking the blowup at the residual intersection of the quartic and the flex lines and after separating the flex lines and the mirror $M_{\mathbb{P}^2}$, we get two (-3)-curves intersecting transversally at one point. In that way the resolution of the singularity $\frac{1}{8}(1,3)$ on $\mathbb{P}(1,3,8)$ by two (-3)-curves corresponds to the two flex lines.

5.3. A Particular Quartic Curve in $\mathbb{P}^2$

The aim of this subsection is to prove the following result.

THEOREM 14. *Up to projective equivalence, there is a unique quartic curve Q in $\mathbb{P}^2$ with distinct points $p_1, \dots, p_7$ such that:*

(1) *Q has a node at p_1 and ordinary cusps at p_2, p_3;*
(2) *the points p_4 and p_5 are flex points of Q;*
(3) *the tangent lines to Q at p_4 and p_5 contain p_6 and p_7, respectively;*
(4) *the line through p_6 and p_7 contains p_1.*

We can assume that

$$p_1 = [0:0:1], \qquad p_2 = [0:1:1], \qquad p_3 = [1:0:1].$$

Then the equation of Q is

$$(x^2 + xy + y^2 - xz - yz)^2 - 8xy(x + y - z)^2 = 0,$$

and the points p_4, p_5 and p_6, p_7 are, respectively,

$$\big[\pm 2\sqrt{-2} + 8 : \mp 2\sqrt{-2} + 8 : 25\big], \qquad \big[\pm 2\sqrt{-2} : \mp 2\sqrt{-2} : 1\big].$$

COROLLARY 15. *The mirror $M_{\mathbb{P}^2}$ described in Section 5.1.3 satisfies the hypothesis of Theorem 14, and thus $M_{\mathbb{P}^2}$ is projectively equivalent to the quartic Q.*

To prove Theorem 14, let us first give a criterion for the existence of roots of multiplicity at least 3 on homogeneous quartic polynomials of two variables. We use the computational algebra system Magma; see [25] for a copy–paste ready version of the Magma code.

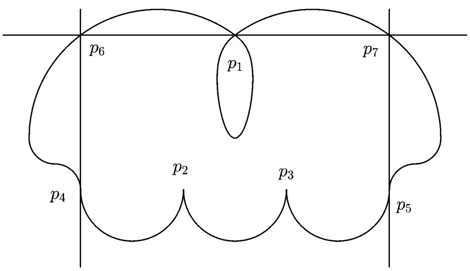

Figure 5 The quartic Q.

LEMMA 16. *The polynomial*

$$P(x,z) = ax^4 + bx^3z + cx^2z^2 + dxz^3 + ez^4$$

has a root of multiplicity at least 3 *if and only if*

$$12ae - 3bd + c^2 = 27ad^2 + 27b^2e - 27bcd + 8c^3 = 0.$$

Proof. The computation below is self-explanatory.

```
R<u,v,m,n,a,b,c,d,e>:=PolynomialRing(Rationals(),9);
P<x,z>:=PolynomialRing(R,2);
f:=(u*x+v*z)^3*(m*x+n*z);
s:=Coefficients(f);
I:=ideal<R|a-s[5],b-s[4],c-s[3],d-s[2],e-s[1]>;
EliminationIdeal(I,4);
```
□

Let us now prove Theorem 14.

Proof. We have already chosen three points p_1, p_2, p_3 in $\mathbb{P}^2$. Instead of choosing a fourth point for having a projective base, we can fix two infinitely near points over p_2 and p_3. Indeed, the projective transformations that fix points p_1, p_2, p_3 are of the form

$$\phi : [x : y : z] \longmapsto [ax : by : (a-1)x + (b-1)y + z],$$

and these transformations act transitively on the lines through p_2 and p_3. Thus up to projective equivalence, we can fix the tangent cones (which are double lines) of the curve Q at the cusps p_2 and p_3. Let us choose for these cones the lines with equations $y = z$ and $x = z$, respectively.

The linear system of quartic curves in $\mathbb{P}^2$ is 14-dimensional. The imposition of a node and two ordinary cusps (with given tangent cones) corresponds to 13 conditions, and thus we get a pencil of curves. We compute that this pencil is generated by the following quartics:

$$(x^2 + xy + y^2 - xz - yz)^2 = 0, \qquad xy(x + y - z)^2 = 0.$$

Notice that at the points p_2 and p_3 the first generator is of multiplicity 2, and the second generator is of multiplicity 3, and thus a generic element in the pencil has a cusp singularity at p_2 and p_3.

Let us compute the quartic curves Q satisfying conditions (1)–(4) of Theorem 14. The method is to define a scheme by imposing the vanishing of certain polynomials $P_i = 0$ and the nonvanishing of other ones $D_i \neq 0$, which is achieved by using an auxiliary parameter n and imposing $1 + nD_i = 0$.

```
K:=Rationals();
R<a,q1,q2,m,d1,d2,n>:=PolynomialRing(K,7);
P<x,y,z>:=ProjectiveSpace(R,2);
```

The defining polynomial of Q, depending on one parameter:

```
F:=(x^2 + x*y + y^2 - x*z - y*z)^2 + a*x*y*(x + y - z)^2;
```

The points p_6 and p_7 are in a line $y = mx$, and hence they are of the form

```
p6:=[q1,m*q1,1];
p7:=[q2,m*q2,1];
```

and we must have the vanishing of

```
P1:=Evaluate(F,[q1,m*q1,1]);
P2:=Evaluate(F,[q2,m*q2,1]);
```

The defining polynomials of lines through that points are:

```
L1:=-y+d1*x+(m*q1-d1*q1)*z;
L2:=-y+d2*x+(m*q2-d2*q2)*z;
```

We need to impose that these lines are not tangent to Q at p_6 and p_7, and thus the following matrices must be of rank 2:

```
M1:=Matrix([JacobianSequence(F),JacobianSequence(L1)]);
M1:=Evaluate(M1,[q1,m*q1,1]);
M2:=Matrix([JacobianSequence(F),JacobianSequence(L2)]);
M2:=Evaluate(M2,[q2,m*q2,1]);
```

The matrix M_i is of rank 2 if one of its minors is nonzero. Here we make a choice for these minors, but to cover all cases, the computations must be repeated for all other choices.

```
D1:=Minors(M1,2)[1];
D2:=Minors(M2,2)[1];
```

Now we intersect the quartic Q with the lines L_1, L_2:

```
R1:=Evaluate(F,y,d1*x+(m*q1-d1*q1)*z);
R2:=Evaluate(F,y,d2*x+(m*q2-d2*q2)*z);
```

and we use Lemma 16 to impose that these lines are tangent to Q at flex points of Q:

```
c:=Coefficients(R1);
P3:=c[1]*c[5]-1/4*c[2]*c[4]+1/12*c[3]^2;
P4:=c[1]*c[4]^2+c[2]^2*c[5]-c[2]*c[3]*c[4]+8/27*c[3]^3;
c:=Coefficients(R2);
P5:=c[1]*c[5]-1/4*c[2]*c[4]+1/12*c[3]^2;
P6:=c[1]*c[4]^2+c[2]^2*c[5]-c[2]*c[3]*c[4]+8/27*c[3]^3;
```

We note that the lines L_1 and L_2 cannot contain the points p_2 and p_3:

```
D3:=Evaluate(L1,[0,1,1]);
D4:=Evaluate(L1,[1,0,1]);
D5:=Evaluate(L2,[0,1,1]);
D6:=Evaluate(L2,[1,0,1]);
```

Also, the line L_i cannot contain the point p_1, $i = 1, 2$:

```
D7:=(m-d1)*(m-d2);
```

It is clear that the following must be nonzero:

```
D8:=a*q1*q2*(q1-q2);
```

Finally, we define a scheme with all these conditions:

```
A:=AffineSpace(R);
S:=Scheme(A,[P1,P2,P3,P4,P5,P6,1+n*D1*D2*D3*D4*D5*D6*D7*D8]);
```

We compute (that takes a few hours)

```
PrimeComponents(S);
```

and get the unique solution $a = -8$. □

From the equation of the quartic $Q = M_{\mathbb{P}^2}$ we can compute a 24th-degree equation for the mirror M_{48}:

```
(31072410*r+44060139)*x^24+(599304420*r-4660302600)*x^21*y
+(-106415505000*r+18054913500)*x^18*y^2+(796474485000*r
+3638808225000)*x^15*y^3+(-27123660*r-18697014)*x^16*z
+(34521715125000*r-31210968093750)*x^12*y^4+(107726220*r
+2948918400)*x^13*y*z+(-257483985484500*r-516632817969000)*x^9*y^5
+(42798843000*r-32351244300)*x^10*y^2*z+(-1747212737190000*r
+3228789525752500)*x^6*y^6+(-407331396000*r-935091495000)*x^7*y^3*z
+(-655139025450000*r+10855982580975000)*x^3*y^7+(7724970*r
-2222037)*x^8*z^2+(-3383703150000*r+9052448883750)*x^4*y^4*z
+(1544666220033750*r+11942493993804375)*y^8+(-102498120*r
-465161400)*x^5*y*z^2+(-319463676000*r+12613760073000)*x*y^5*z
+(-2705586000*r+7086771600)*x^2*y^2*z^2+(-712080*r+1186268)*z^3=0
```

where $r = \sqrt{-2}$.

5.4. A Configuration of Four Plane Conics Related to the Orbifold Ball-Quotient

In this subsection, we describe the configuration of conics, which we announced in the introduction.

Let us consider a conic tangent to two lines of a triangle in $\mathbb{P}^2$ and going through two points of the remaining line. Performing a Cremona transformation at the three vertices of the triangle, we obtain a quartic curve in $\mathbb{P}^2$ with singularities $\mathfrak{a}_1 + 2\mathfrak{a}_2$. Conversely, starting with such a quartic, its image by the Cremona transform at the three singularities is a conic with three lines having the above configuration.

Thus we consider the Cremona transform φ at the three singularities of the quartic $M_{\mathbb{P}^2}$. Let $D_1, \dots, D_4$ be respectively the images of $M_{\mathbb{P}^2}$, the line L_0 through the node and the two residual points of the flex lines, and the two flex lines. Using Magma, we see that these are four conics meeting in 10 points as follows:

	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	q_{10}
D_1	1+	1+	0	0	0	0	1	1	1	1
D_2	1	1	1	1	1	0	0	0	1	1
D_3	0	1+	1	1	1	1	1	0	0	0
D_4	1+	0	1	1	1	1	0	1	0	0

Here two + in the column of q_j mean that the two curves meet with multiplicity 3 at point q_i. The other intersections are transverse. We see that the various ball-quotient orbifolds that Deraux described in [12] may be obtained from a configuration of conics by performing birational transformations.

6. One Further Quotient by an Involution

6.1. The Quotient Morphism $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$, Image of the Mirror as the Cuspidal Cubic

Consider the plane quartic curve Q from Theorem 14. Here we show the existence of a birational map

$$\rho : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$$

and an involution σ on $\mathbb{P}^1 \times \mathbb{P}^1$ that preserves $\rho(Q)$ and fixes the diagonal D of $\mathbb{P}^1 \times \mathbb{P}^1$ pointwise. Moreover, we have $(\mathbb{P}^1 \times \mathbb{P}^1)/\sigma = \mathbb{P}^2$, and the images C_u and C_o of $\rho(Q)$ and D are curves of degrees 3 and 2, respectively. The curve C_u has a cusp singularity and intersects C_o at three points, with intersection multiplicities 4, 1, 1. The map ρ is the inverse of the birational transform $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ described in Section 5.1.3, whose indeterminacy is at the singularity $\mathfrak{a}_3$ of $M_{\mathbb{P}^1 \times \mathbb{P}^1}$.

```
K:=Rationals();
R<r>:=PolynomialRing(K);
K<r>:=ext<K|r^2+2>;
P2<x,y,z>:=ProjectiveSpace(K,2);
Q:=Curve(P2,(x^2+x*y+y^2-x*z-y*z)^2-8*x*y*(x+y-z)^2);
p6:=P2![2*r,-2*r,1];
p7:=P2![-2*r,2*r,1];
```

We compute the linear system of conics through the cuspidal points p_2 and p_3 and take the corresponding map to $\mathbb{P}^3$.

```
L:=LinearSystem(LinearSystem(P2,2),[p6,p7]);
P3<a,b,c,d>:=ProjectiveSpace(K,3);
rho:=map<P2->P3|Sections(L)>;
```

The image of $\mathbb{P}^2$ is a quadric surface Q_2 ($\cong \mathbb{P}^1 \times \mathbb{P}^1$).

```
Q2:=rho(P2);Q2;
C:=rho(Q);C;
```

There is an involution preserving both Q_2 and the curve $C := \rho(Q)$.

```
sigma:=map<P3->P3|[d,b,c,a]>;
C:=rho(Q);C;
sigma(Q2) eq Q2;
```

```
sigma(C) eq C;
```

We compute the corresponding map to the quotient. The image of C is a cubic curve, and the image of the diagonal is a conic.

```
psi:=map<P3->P2|[a+d,b,c]>;
Cu:=psi(C);
Co:=psi(Scheme(rho(P2),[a-d]));
Co:=Curve(P2,DefiningEquations(Co));
```

The curve C_u has a cusp singularity:

```
pts:=SingularPoints(Cu);
ResolutionGraph(Cu,pts[1]);
```

The intersections of C_o and C_u:

```
Degree(ReducedSubscheme(Co meet Cu)) eq 3;
pt:=Points(Co meet Cu)[1];
IntersectionNumber(Co,Cu,pt) eq 4;
```

Let C_1' and C_2' be the fibers that intersect $M_{\mathbb{P}^1\times\mathbb{P}^1}$ each at a unique point with multiplicity 3. These fibers are exchanged by the involution σ and are sent to a line F_l, which cuts the cubic curve C_u at a unique point; this is a flex line. That line F_l also cuts the conic C_o at a unique point.

Conversely, let us start from the data of a conic C_o and a cuspidal cubic C_u intersecting as before, with the flex line (at the smooth flex point) of the cubic tangent to the conic. We can take the double cover of the plane branched over C_o, which is $\mathbb{P}^1 \times \mathbb{P}^1$. The pull-back of C_u is then a curve satisfying the properties of Theorem 14, and thus the configuration (C_o, C_u) we described is unique in $\mathbb{P}^2$ up to projective automorphisms.

6.2. *An Orbifold Ball-Quotient Structure from* $(\mathbb{P}^2, (C_o, C_u))$

Let $C_u \hookrightarrow \mathbb{P}^2$ be the unique plane cuspidal curve, and let c_1 be its cuspidal point. Let F_l be the flex line through the unique smooth flex point c_2 of C_u. By the previous subsection we have the following result.

Proposition 17. *There exists a unique conic $C_o \hookrightarrow \mathbb{P}^2$ such that the following holds*:

- (i) *F_l is tangent to C_o*;
- (ii) *C_o cuts C_u at points c_3, c_4, c_5 $(\neq c_1, c_2)$ with intersection multiplicities* 4, 1, 1, *respectively.*

In this subsection, we prove that there is a natural birational transformation $W \dashrightarrow \mathbb{P}^2$ such that together with the strict transform of the curves C_o and C_u we get an orbifold ball-quotient surface. For definitions and results on orbifold theory, we use [7; 11] and [29].

Let us blow up over points c_1, c_2, c_3 and then contract some divisors as follows (for a pictural description, see Figure 6):

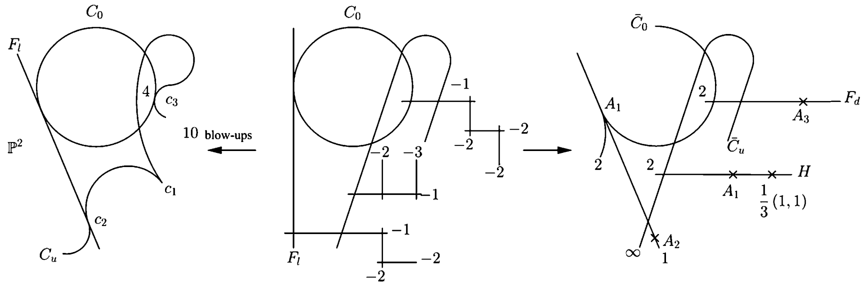

Figure 6 The plane, the surfaces Z and W.

We blow up over c_1 three times. The first blowup resolves the cusp of C_u, and the exceptional divisor intersects the strict transform of C_u tangentially, the second blowup is at that point of tangency, and the third blowup separates the strict transforms of the first exceptional divisor and the curve C_u. We obtain in that way a chain of (-3)-, (-1)-, and (-2)-curves. We then contract the (-2)- and (-3)-curves obtaining in that way the singularities A_1 and $\frac{1}{3}(1,1)$. The image of the (-1)-curve by that contraction map is denoted by H. As an orbifold, we put multiplicity 2 on H.

We blow up over c_2 (the flex point) three times in order that the strict transform of the curves F_l and C_u get separated over c_2. We obtain in that way a chain of (-1)-, (-2)-, and (-2)-curves. We then contract the two (-2)-curves and obtain an A_2-singularity. The strict transform of the line F_l is a (-2)-curve, which we also contract, obtaining in that way an A_1-singularity. The contracted curve being tangent to $\tilde{C}_0$, the image $\bar{C}_0$ has a cusp $\mathfrak{a}_2$ at the singularity A_1.

We moreover blow up over c_3 four times, in order that the strict transform of the curves C_o and C_u get separated over c_3. We obtain in that way a chain of (-1)-, (-2)-, (-2)-, (-2)-curves. We then contract the three (-2)-curves and obtain an A_3-singularity. The image of the (-1)-curve by the contraction map is a curve denoted by F_d, we give the weight 2 to that curve.

Let us denote by W the resulting surface. For a curve D on $\mathbb{P}^2$, we denote by $\tilde{D}$ its strict transform on W. Let $\mathcal{W}$ be the orbifold with the same subjacent topological space, with divisorial part:

$$\Delta = \left(1 - \frac{1}{\infty}\right)\bar{C}_u + \left(1 - \frac{1}{2}\right)(\bar{C}_o + F_d + H).$$

The singular points of W are

$$A_1 + A_1 + A_2 + A_3 + \frac{1}{3}(1,1),$$

and they have an isotropy β of order 16, 4, 3, 8, 6 respectively, for $\mathcal{W}$. The computation of the isotropy is immediate, except for the first point (that we shall denote by r_1), which is also a cusp on the curve $\bar{C}_0$ (which has weight 2). Let SD_{16} be the the semidihedral group of order 16, generated by the matrices

$$g_1 = \begin{pmatrix} 0 & -\zeta \\ -\zeta^3 & 0 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where ζ is a primitive 8th root of unity. The order 2 elements g_2 and $g_1^{-1}g_2g_1$ generate an order 8 reflection group D_4. The quotient of $\mathbb{C}^2$ by SD_{16} has an A_1-singularity, and we compute that the image of the four mirrors of D_4 is a curve with a cusp $\mathfrak{a}_2$ at the A_1-singularity of $\mathbb{C}^2/\mathrm{SD}_{16}$. The isotropy group of the point r_1 in the orbifold is therefore the semidihedral group SD_{16} of order 16. The following proposition is an application of the main result of [21].

PROPOSITION 18. *The Chern numbers of the orbifold* $\mathcal{W} = (W, \Delta)$ *satisfy*

$$c_1^2(\mathcal{W}) = 3c_2(\mathcal{W}) = \frac{9}{16};$$

in particular, $\mathcal{W}$ *is an orbifold ball-quotient.*

Proof. Let us compute the orbifold second Chern number of $\mathcal{W}$. We have (see, e.g., [27]):

$$c_2(\mathcal{W}) = e(W) - \left(\left(1 - \frac{1}{\infty}\right)e(\bar{C}_u \setminus S) + \left(1 - \frac{1}{2}\right)e(\bar{C}_o \setminus S) + \left(1 - \frac{1}{2}\right)e(F_d \setminus S) + \left(1 - \frac{1}{2}\right)e(H \setminus S)\right) - \sum_{p \in S}\left(1 - \frac{1}{\beta(p)}\right),$$

where S is the union of the singular points of W with the singular points of the round-up divisor $\lceil \Delta \rceil$, and where, moreover, $\beta(p)$ is the isotropy order of the point p, so that, for example, for p on $\bar{C}_u$, $\beta(p) = \infty$, and the unique point p in F_d and $\bar{C}_o$ has $\beta(p) = 4$. Since we have blown-up $\mathbb{P}^2$ over 10 points and we have contracted eight rational curves, we get

$$e(W) = 3 + 10 - 8 = 5.$$

We obtain

$$c_2(\mathcal{W}) = 5 - \left((2-4) + \frac{1}{2}(2-4) + \frac{1}{2}(2-3) + \frac{1}{2}(2-3)\right) - \left(10 - \frac{1}{16} - \frac{1}{4} - \frac{1}{3} - \frac{1}{8} - \frac{1}{6} - \frac{1}{4} - 4 \cdot \frac{1}{\infty}\right),$$

and thus $c_2(\mathcal{W}) = \frac{3}{16}$.

Let us compute $c_1^2(\mathcal{W})$. We have

$$c_1^2(\mathcal{W}) = (K_W + \Delta)^2,$$

so that

$$c_1^2(\mathcal{W}) = K_W^2 + 2K_W\bar{C}_u + K_W(\bar{C}_o + F_d + H) + \frac{1}{4}(\bar{C}_o^2 + F_d^2 + H^2) + \bar{C}_u^2 + \bar{C}_u(\bar{C}_o + F_d + H) + \frac{1}{2}(\bar{C}_oF_d + \bar{C}_oH + F_dH).$$

Let $p : Z \to W$ be the surface above W that resolves W and is a blowup of $\mathbb{P}^2$. Since Z is obtained by 10 blowups of $\mathbb{P}^2$, we have $K_Z^2 = 9 - 10 = -1$. Moreover, since all singularities but one are ADE, we have $K_Z = p^* K_W - \frac{1}{3} D_1$, where D_1 is the (-3)-curve on Z that is contracted to the $\frac{1}{3}(1, 1)$ singularity on W. Since $p^* K_W \cdot D_1 = 0$, we obtain

$$K_W^2 = -\frac{2}{3}.$$

The curve $\bar{C}_u$ is a smooth curve of genus 0 on the smooth locus of W. The blowup at the $\mathfrak{a}_2$-singularity of the cuspidal cubic decreases the self-intersection by 4, and the remaining blow-ups decrease the self-intersection by 1. Since we have $4 + 2 + 3 = 9$ such blowups, we get

$$\bar{C}_u^2 = 3^2 - 4 - 9 = -4,$$

and therefore $K_W \bar{C}_u = 2$. Let $\tilde{D}$ be the strict transform on Z of a curve D on W or $\mathbb{P}^2$. We have

$$\tilde{C}_o = p^* \bar{C}_o - a F_l.$$

Since $\tilde{C}_o F_l = 2$, a is equal to 1. Since, moreover, $\tilde{C}_o^{\,2} = 0$, we get $0 = (\tilde{C}_o)^2 = \bar{C}_o^2 - 2$, and thus $\bar{C}_o^2 = 2$. We have

$$K_W \bar{C}_o = (\tilde{C}_o + F_l)\left(K_W + \frac{1}{3} D_1\right) = -2.$$

Let $F_1, F_2, F_3 \subset Z$ be the chain of three (-2)-curves above the A_3-singularity in W, so that $\tilde{F}_d F_1 = 1$. We compute that

$$\tilde{F}_d = p^* F_d - \frac{1}{4}(3F_1 + 2F_2 + F_3)$$

(it is easy to check that $\tilde{F}_d F_1 = 1$ and $\tilde{F}_d F_2 = \tilde{F}_d F_3 = 0$). Then

$$-1 = \tilde{F}_d^2 = F_d^2 - \frac{3}{4}$$

gives $F_d^2 = -\frac{1}{4}$. We have

$$K_W F_d = \left(K_Z + \frac{1}{3} D_1\right)\left(\tilde{F}_d + \frac{1}{4}(3F_1 + 2F_2 + F_3)\right) = -1.$$

Let D_1 and D_2 be respectively the (-3)- and (-2)-curves intersecting $\tilde{H}$. Since $\tilde{H} D_1 = \tilde{H} D_2 = 1$, we have

$$\tilde{H} = p^* H - \frac{1}{3} D_1 - \frac{1}{2} D_2,$$

and thus

$$-1 = \tilde{H}^2 = H^2 - \frac{1}{3} - \frac{1}{2},$$

and $H^2 = -\frac{1}{6}$. Moreover,

$$K_W H = \left(K_Z + \frac{1}{3} D_1\right)\left(\tilde{H} + \frac{1}{3} D_1 + \frac{1}{2} D_2\right) = -1 + \frac{1}{3} + \frac{1}{3} - \frac{1}{3} = -\frac{2}{3}.$$

We therefore compute

$$c_1^2(\mathcal{W}) = -\frac{2}{3} + 2\cdot 2 + \left(-2 - 1 - \frac{2}{3}\right) + \frac{1}{4}\left(2 - \frac{1}{4} - \frac{1}{6}\right) - 4 + (2+1+1) + \frac{1}{2}(1+0+0) = \frac{9}{16},$$

and thus $c_1^2(\mathcal{W}) = 3c_2(\mathcal{W}) = \frac{9}{16}$. □

Remark 19. In [12], Deraux obtains four different orbifold ball-quotient structures on surfaces birational to A/G_{48}. Among these, only the fourth one, W', is invariant by the involution σ, the obstruction being the divisor E in [12], which creates an asymmetry, unless it has weight 1. The orbifold $\mathcal{W}$ we just described can be seen as the quotient of W' by the involution σ.

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V. Koziarz
Univ. Bordeaux
IMB, CNRS, UMR 5251
F-33400 Talence
France

vincent.koziarz@math.u-bordeaux.fr

C. Rito
Permanent address:
Universidade de Trás-os-Montes e Alto Douro, UTAD
Quinta de Prados
5000-801 Vila Real
Portugal

www.utad.pt; crito@utad.pt

Temporary address:
Departamento de Matemática
Faculdade de Ciências da Universidade do Porto
Rua do Campo Alegre 687
4169-007 Porto
Portugal

www.fc.up.pt; crito@fc.up.pt

X. Roulleau
Aix-Marseille Université
CNRS, Centrale Marseille, I2M UMR 7373
13453 Marseille
France

Xavier.Roulleau@univ-amu.fr