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Maximal differential uniformity polynomials

by

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1. Introduction. Throughout this paper, n is a positive integer and $q = 2^n$. For a polynomial $f \in \mathbb{F}_q[x]$ we define the *differential uniformity* $\delta(f)$ following Nyberg [6]:

$$\delta(f) := \max_{(\alpha,\beta)\in\mathbb{F}_q^*\times\mathbb{F}_q} \sharp\{x \in \mathbb{F}_q \mid f(x+\alpha)+f(x)=\beta\}.$$

When $\delta(f) = 2$ the associated functions $f : \mathbb{F}_q \to \mathbb{F}_q$ are called *APN* (Almost Perfectly Nonlinear). These functions have been extensively studied as they offer good resistance against differential attacks (see [2]). Among them, those which are APN over infinitely many extensions of $\mathbb{F}_q$ have attracted special attention.

In the opposite direction, the third author [10] proved that most polynomials $f \in \mathbb{F}_q[x]$ of degree $m \equiv 0$ or $3 \pmod 4$ have differential uniformity equal to $m-1$ or $m-2$, the largest possible for polynomials of degree m. Precisely, he proved that for a given integer $m > 4$ such that $m \equiv 0 \pmod 4$ (respectively $m \equiv 3 \pmod 4$), if $\delta_0 = m-2$ (respectively $\delta_0 = m-1$) then

$$\lim_{n\to\infty} \frac{\sharp\{f \in \mathbb{F}_{2^n}[x] \mid \deg(f) = m,\ \delta(f) = \delta_0\}}{\sharp\{f \in \mathbb{F}_{2^n}[x] \mid \deg(f) = m\}} = 1.$$

The first two authors [1] extended this result to the second order differential uniformity.

The following conjecture is also stated in [10].

Conjecture 1.1. *For a given integer $m > 4$, there exists $\varepsilon_m > 0$ such that for all sufficiently large n, if f is a polynomial of degree m over $\mathbb{F}_{2^n}$,*

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then for at least $\varepsilon_m 2^{2n}$ *values of* $(\alpha,\beta)\in\mathbb{F}_{2^n}^*\times\mathbb{F}_{2^n}$ *we have*

$$\sharp\{x\in\mathbb{F}_q \mid f(x+\alpha)+f(x)=\beta\}=\delta(f).$$

Moreover, it was proved in [10] that all polynomials f of degree 7 have maximal differential uniformity (that is, $\delta(f)=6$) if n is large enough.

The aim of this paper is to exhibit an infinite set $\mathcal{M}$ (defined below) of integers m such that every polynomial $f\in\mathbb{F}_{2^n}[x]$ of degree m has maximal differential uniformity if n is large enough, that is, $\delta(f)$ is equal to the degree of $D_\alpha f(x)=f(x+\alpha)+f(x)$, the derivative of f with respect to α. We stress that, for $m\in\mathcal{M}$, our results are much stronger than those of [10] as we prove maximality of differential uniformity for all polynomials of degree m, as opposed to most of them.

Definition (Definition 3.10 and Proposition 3.11). We denote by $\mathcal{M}$ the set of odd integers m such that the unique polynomial g satisfying $g(x(x+1))=D_1(x^m)$ has distinct critical values.

An integer m belongs to $\mathcal{M}$ if and only if for any ζ_1 and ζ_2 in $\overline{\mathbb{F}}_2\setminus\{1\}$,

$$\zeta_1^{m-1}=\zeta_2^{m-1}=\left(\frac{1+\zeta_1}{1+\zeta_2}\right)^{m-1}=1 \implies \zeta_1=\zeta_2 \text{ or } \zeta_1=\zeta_2^{-1}.$$

Now we can state our main results.

Theorem (Theorems 5.3 and 5.7). *Let* $m\in\mathcal{M}$ *be such that* $m\equiv 7 \pmod 8$. *Then for* n *sufficiently large, for all polynomials* $f\in\mathbb{F}_{2^n}[x]$ *of degree* m *we have* $\delta(f)=m-1$. *Furthermore, Conjecture* 1.1 *is true for such integers* m.

For example, we will prove that this theorem applies for $m\in\{7,23,39,47,55,79,87,95,111,119,135,143,159,167,175,191,199\}$ (see Example 3.16). We also provide explicit infinite families of such integers m, namely $m=2\ell^{2k+1}+1$ for $k\geq 0$ and $\ell\in\{3,11,19,23,43,47,59,67,71,79,83,103,107,131,139,151,163,167,179,191,199\}$ (see Corollary 5.4).

When m is congruent to 3 modulo 8, we also obtain some results but we have conditions on the parity of n or we have to remove some polynomials.

Theorem (Theorem 5.5). *Let* $m\in\mathcal{M}$ *be such that* $m\geq 7$ *and* $m\equiv 3 \pmod 8$.

- (i) *For* n *even and sufficiently large and for all* $f\in\mathbb{F}_{2^n}[x]$ *of degree* m, *we have* $\delta(f)=m-1$.
- (ii) *For* n *sufficiently large and for all* $f=\sum_{i=0}^m a_{m-i}x^i\in\mathbb{F}_{2^n}[x]$ *of degree* m *such that* $a_1^2+a_0a_2\neq 0$, *we have* $\delta(f)=m-1$.

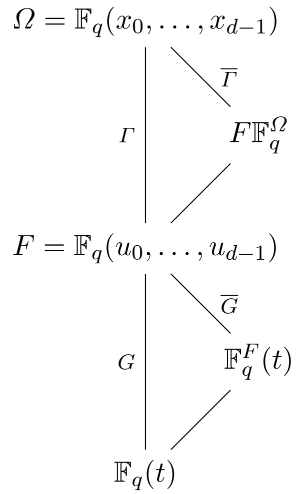
We also provide infinite families of integers $m\equiv 3 \pmod 8$ for which this theorem applies, namely $m=2\ell^k+1$ for $k\geq 1$ and $\ell\in\{17,41,97,113,137,193\}$

and $m = 2\ell^{2k} + 1$ for $k \geq 1$ and $\ell \in \{23, 47, 71, 79, 103, 151, 167, 191, 199\}$ (see Corollary 5.6).

Let us explain the strategy of the proofs of the above theorems, which has important similarities to that of [10] and [1]. For simplicity we consider in this sketch the case where m is congruent to 7 modulo 8.

If $f \in \mathbb{F}_q[x]$ has degree m and $\alpha \in \mathbb{F}_q^*$, we introduce the unique polynomial $L_\alpha f$ of degree $d = (m-1)/2$ such that $L_\alpha f(x(x+\alpha)) = D_\alpha f(x)$ (see Proposition 2.3). We consider the splitting field F of the polynomial $L_\alpha f(x) - t$ over the field $\mathbb{F}_q(t)$ with t transcendental over $\mathbb{F}_q$ and let $\mathbb{F}_q^F$ be the algebraic closure of $\mathbb{F}_q$ in F. The Galois groups $G = \mathrm{Gal}(F/\mathbb{F}_q(t))$ and $\overline{G} = \mathrm{Gal}(F/\mathbb{F}_q^F(t))$ are respectively the *arithmetic* and *geometric monodromy groups* of $L_\alpha f$.

If $u_0, \ldots, u_{d-1}$ are the roots of $L_\alpha f(x) = t$, then we will denote by x_i a root of $x^2 + \alpha x = u_i$. So the $2d$ elements $x_0, x_0+\alpha, \ldots, x_{d-1}, x_{d-1}+\alpha$ are the solutions of $D_\alpha f(x) = t$. We consider $\Omega = \mathbb{F}_q(x_0, \ldots, x_{d-1})$, the compositum of the fields $F(x_i)$, and $\mathbb{F}_q^\Omega$, the algebraic closure of $\mathbb{F}_q$ in Ω. We also set $\Gamma = \mathrm{Gal}(\Omega/F)$ and $\overline{\Gamma} = \mathrm{Gal}(\Omega/F\mathbb{F}_q^\Omega)$. Then we have the following diagram:



When the integer m belongs to $\mathcal{M}$ and is congruent to 7 modulo 8 we prove that for n sufficiently large and for any $f \in \mathbb{F}_{2^n}[x]$ of degree m, there exists α in $\mathbb{F}_{2^n}^*$ such that

(1) $L_\alpha f$ is Morse,
(2) the equation $x^2 + \alpha x = b_1/b_0$ has a solution in $\mathbb{F}_{2^n}$.

Now, condition (1) implies by Proposition 4.1 that the extension $F/\mathbb{F}_q(t)$ is regular. Conditions (1) and (2) imply by Proposition 4.6 that the extension Ω/F is regular. This enables us to apply the Chebotarev density theorem (see Proposition 5.1) to obtain, for n sufficiently large depending only on m, the existence of $\beta \in \mathbb{F}_{2^n}$ such that the polynomial $D_\alpha f(x)+\beta$ splits in $\mathbb{F}_{2^n}[x]$ with no repeated factors. The differential uniformity of f is thus equal to the degree of $D_\alpha f$.

The paper is organized as follows. Section 2 is devoted to the study of the operator L_α. Section 3 provides a detailed exposition of Morse polynomials in even characteristic. According to the appendix by Geyer in [5], Morse polynomials in this context are polynomials of odd degree satisfying two conditions: their critical points are nondegenerate and their critical values are distinct. The first condition leads to the study of the number of α such that the resultant of the derivative $(L_\alpha f)'$ and the second Hasse–Schmidt derivative $(L_\alpha f)^{[2]}$ does not vanish (Proposition 3.2). We give upper bounds for the number of exceptions in terms of m.

By contrast, we need additional requirements on m to guarantee that for enough α the polynomial $L_\alpha f$ has distinct critical values (see Proposition 3.6). Precisely, we will assume that $L_1(x^m)$ has distinct critical values, that is, $m \in \mathcal{M}$ (Definition 3.10). We complete Section 3 by exhibiting some families of infinitely many integers belonging to $\mathcal{M}$.

Section 4 is devoted to the study of the Galois groups G, $\overline{G}$, Γ and $\overline{\Gamma}$. We prove in Proposition 4.6 that if the equation $x^2+\alpha x = b_1/b_0$ has a solution in $\mathbb{F}_{2^n}$, i.e. $\mathrm{Tr}_{\mathbb{F}_{2^n}/\mathbb{F}_2}\big(\frac{b_1}{b_0\alpha^2}\big) = 0$, then the extension Ω/F is regular. The different expressions of b_1/b_0 we have obtained in Lemma 2.5, depending on the congruence class of m modulo 8, induce differences in the treatment.

Section 5 deals with the Chebotarev density theorem and contains the statements and proofs of the main results.

Let us stress the main difference between the common approach of [10] and [1] and the approach of the present paper. For simplicity, we assume again that $m \equiv 7 \pmod 8$. In [10] and [1], one of the key steps is to fix $\alpha_1,\dots,\alpha_k$ in $\mathbb{F}_{2^n}$ and to obtain a lower bound depending on n for the number of polynomials f in $\mathbb{F}_{2^n}[x]$ such that at least one of the $L_{\alpha_i} f$ is Morse. By contrast, we prove here that for n sufficiently large and for any f of degree m in $\mathbb{F}_{2^n}[x]$ there exists α such that $L_\alpha f$ is Morse.

2. The associated polynomial $L_\alpha f$.

Let $f \in \mathbb{F}_q[x]$ be a polynomial of degree $m \ge 7$ (the cases where $m < 7$ are handled in [10]) and $\alpha \in \mathbb{F}_q^*$. The derivative of a polynomial $f \in \mathbb{F}_q[x]$ along α is defined by

$$D_\alpha f(x) = f(x) + f(x+\alpha).$$

If we set $f = \sum_{k=0}^m a_{m-k}x^k$, a straightforward computation shows that $D_\alpha f = \sum_{k=0}^m c_{m-k}x^k$ where $c_k = a_k + \sum_{i=m-k}^m a_{m-i}\binom{i}{m-k}\alpha^{i-m+k}$. As we work over an even characteristic field, we have $c_0 = a_0 + a_0 = 0$, $c_1 = m\alpha a_0$ and $c_2 = (m-1)\alpha a_1 + \binom{m}{2}\alpha^2 a_0$. We deduce the following proposition.

Proposition 2.1. *Let $f \in \mathbb{F}_q[x]$ have degree m. If m is odd then the degree of $D_\alpha f$ is $m-1$. If m is even then the degree of $D_\alpha f$ is less than or equal to $m-2$, and equal to $m-2$ if and only if $a_1 + a_0\alpha\binom{m}{2} \neq 0$.*

In the whole paper, we will associate to any integer m the following integer d.

Definition 2.2. Let m be an integer. Suppressing the dependence on m in notation, we set $d = (m-1)/2$ if m is odd and $d = (m-2)/2$ if m is even.

2.1. Existence of $L_\alpha f$

Proposition 2.3. *Let $\alpha \in \mathbb{F}_q^*$ and let $f \in \mathbb{F}_q[x]$ have degree m. Then there exists a unique $g \in \mathbb{F}_q[x]$ of degree less than or equal to d such that*

$$D_\alpha f(x) = g(x(x+\alpha)).$$

Furthermore, the map $L_\alpha : f \mapsto g$ is linear and its restriction to the subspace of polynomials of degree at most m is surjective onto the subspace of polynomials of degree at most d.

Proof. The proof is similar to that of [1, Proposition 2.2] dealing with the set Λ_k of roots of multiplicity k of $D_\alpha f$ and noticing that $x \mapsto x+\alpha$ is an involution of each set Λ_k. The surjectivity of L_α follows from the fact that the kernel of the restriction of L_α to the space of polynomials of degree at most m is the subspace of polynomials $g(x(x+\alpha))$ where $g \in \mathbb{F}_q[x]$ has degree at most $[m/2]$ (see [1, Lemma 2.3]). ∎

2.2. The coefficients b_i of $L_\alpha f$. Let $f = \sum_{i=0}^{m} a_{m-i}x^i \in \mathbb{F}_q[x]$ be a polynomial of degree m and $L_\alpha f = \sum_{i=0}^{d} b_{d-i}x^i$ be the associated polynomial of degree d when m is odd and of degree less than or equal to d otherwise (see Proposition 2.1). To obtain information on the coefficients b_i, one can consider the triangular linear system with coefficients 1 on the diagonal arising by equating the coefficients of $x^{2d}, x^{2d-2}, \dots, x^2, x^0$ in $g(x(x+\alpha))$ and in $D_\alpha f$. Note that this approach proves again the unicity of g claimed in Proposition 2.3.

More precisely, a necessary condition for the term $b_s x^t$ to appear in $g(x(x+\alpha))$ is that $d-t \le s \le d-t/2$. In this case, it appears with the coefficient $\binom{d-s}{t-d+s}\alpha^{2(d-s)-t}$. So for each integer k between 0 and d, equating the coefficient of $x^{2(d-k)}$ in $g(x(x+\alpha))$ and in $D_\alpha f(x)$ gives

$$(1)\qquad \sum_{s=\max\{0,2k-d\}}^{k} \binom{d-s}{2k-2s}\alpha^{2k-2s} b_s = \sum_{i=2d-2k+1}^{m} \binom{i}{2d-2k}\alpha^{i-2d+2k} a_{m-i}.$$

We consider the polynomial ring $\mathbb{F}_2[\alpha, a_0, \dots, a_m]$ where $\alpha, a_0, \dots, a_m$ are indeterminates, equipped with the degree w such that $w(\alpha) = 1$ and

$w(a_j) = j$. This means that the monomial $\alpha^{d_\alpha} a_0^{d_0} a_1^{d_1} a_2^{d_2} \dots a_m^{d_m}$ has degree $d_\alpha + d_1 + 2d_2 + \dots + md_m$. Then using the triangular system obtained from (1) and induction on k one can prove the following homogeneity result.

LEMMA 2.4. *For all integers i such that $0 \le i \le d$ the coefficient b_i is in $\mathbb{F}_2[\alpha, a_0, \dots, a_m]$ and is a homogeneous polynomial of degree $2i+1$ if m is odd and of degree $2i+2$ if m is even, when considering the degree w such that $w(\alpha) = 1$ and $w(a_j) = j$.*

The relations (1) also provide expressions of the first coefficients $b_0, b_1, \dots$ of $L_\alpha f$ depending on the congruence class of m modulo 8, as made explicit in the next lemma which will be needed in the proof of Theorem 5.3. Note that formulas for b_1/b_0 appeared in [10] as well, but the last two had misprints.

LEMMA 2.5. *Let m be an integer. If $m \equiv 0 \pmod 4$ then $b_0 = a_1\alpha$, and if $m \equiv 3 \pmod 4$ then $b_0 = a_0\alpha$. Moreover, we have the following expressions for b_1/b_0 depending on the congruence class of m:*

$m \pmod 8$	b_1/b_0
3	$\alpha^2 + \frac{a_1\alpha + a_2}{a_0}$
7	$\frac{a_1\alpha + a_2}{a_0}$
0	$\frac{a_2\alpha + a_3}{a_1}$
4	$\alpha^2 + \frac{a_0\alpha^3 + a_2\alpha + a_3}{a_1}$

3. For almost every α the polynomial $L_\alpha f$ is Morse. We will focus now on polynomials f of degree $m \equiv 3 \pmod 4$ and thus, for nonzero α, on polynomials $L_\alpha f$ of odd degree $d = (m-1)/2$.

3.1. Morse polynomials in even characteristic. We consider the following notion of Morse polynomial given in all characteristics by Geyer in an appendix to [5].

DEFINITION 3.1. Let K be a field of characteristic $p \ge 0$. We say that a polynomial g over K is *Morse* if the following three conditions hold:

(a) the critical points of g, i.e. the zeroes of g', are nondegenerate,
(b) the critical values of g are distinct, i.e. $g'(\tau) = g'(\eta) = 0$ and $g(\tau) = g(\eta)$ imply $\tau = \eta$,
(c) if $p > 0$, then the degree of g is not divisible by p.

These conditions are chosen in such a way that g corresponds to a covering with maximum Galois group, that is, $\mathrm{Gal}(g(t) - x, K(x))$ is the symmetric group $\mathfrak{S}_d$ where d is the degree of g (see [5, Proposition 4.2]). For $p > 0$, the locus of non-Morse polynomials is described in the same appendix.

Let us sum up the situation for $p = 2$. In this case one has to introduce the Hasse–Schmidt derivative $g^{[2]}$ which is defined by the equality $g(t+u) \equiv$

$g(t)+g'(t)u+g^{[2]}(t)u^2 \pmod{u^3}$ where u and t are independent variables. If $g=\sum_{i=0}^{d} b_{d-i}x^i$ is a degree d polynomial in $\mathbb{F}_q[x]$ with q a power of 2, then condition (a) above is fulfilled if and only if g' and $g^{[2]}$ have no common roots, that is, if and only if the resultant

$$R := \operatorname{Res}(g', g^{[2]}) \in \mathbb{F}_2[b_0, \ldots, b_d]$$

does not vanish. And condition (b) is fulfilled if and only if

$$\Pi(g) := \prod_{i\neq j} (g(\tau_i) - g(\tau_j))$$

does not vanish, where $\tau_1, \ldots, \tau_{[(d-1)/2]}$ are the (double) roots of g'. Using the theorem on symmetric functions, one can obtain an expression of $\Pi(g)$ in terms of the coefficients $b_0, \ldots, b_d$ of g.

In order to calculate the second order Hasse–Schmidt derivative, we will make use of the following Lucas theorem about binomial coefficients (see for instance [4, introduction]). For p a prime number, write $m = m_0 + m_1 p + m_2 p^2 + \cdots + m_r p^r$ and $k = k_0 + k_1 p + k_2 p^2 + \cdots + k_r p^r$. Then

$$\binom{m}{k} \equiv \binom{m_0}{k_0}\binom{m_1}{k_1}\cdots\binom{m_r}{k_r} \pmod{p}.$$

3.2. Condition (a). In order to bound the number of α such that the critical values of $L_\alpha f$ are nondegenerate, in this subsection we study $\operatorname{Res}((L_\alpha f)', (L_\alpha f)^{[2]}) \in \mathbb{F}_2[a_0, \ldots, a_m]$.

We will need three lemmas. Lemma 3.3 enables us to study $\tilde{R} := \operatorname{Res}((D_\alpha f)', (D_\alpha f)^{[2]})$ rather than $\operatorname{Res}((L_\alpha f)', (L_\alpha f)^{[2]})$. Then Lemma 3.4 gives a result about the homogeneity and the degree of this polynomial if it is nonzero. To prove its nonnullity we evaluate it at $a_0 = 1$, $a_1 = \cdots = a_m = 0$, which amounts to determining in Lemma 3.5 if the polynomial x^m has nondegenerate critical points.

PROPOSITION 3.2. *Let $m \geq 7$ be such that $m \equiv 3 \pmod 4$ and let $f(x) = \sum_{k=0}^{m} a_{m-k}x^k \in \mathbb{F}_q[x]$ be of degree m. Then the critical points of $L_\alpha f$ are nondegenerate except for at most $m(m-3)$ values of $\alpha \in \overline{\mathbb{F}}_2$.*

LEMMA 3.3. *Let $f \in \mathbb{F}_q[x]$. For all $\alpha \in \mathbb{F}_q^*$ the polynomials $(L_\alpha f)'$ and $(L_\alpha f)^{[2]}$ have a common root in $\overline{\mathbb{F}}_2$ if and only if $(D_\alpha f)'$ and $(D_\alpha f)^{[2]}$ have a common root in $\overline{\mathbb{F}}_2$.*

Proof. Since $D_\alpha f = L_\alpha f \circ T_\alpha$ where $T_\alpha(x) := x(x+\alpha)$, we can prove the following two equalities:

$$(D_\alpha f)' = \alpha (L_\alpha f)' \circ T_\alpha,$$
$$(D_\alpha f)^{[2]} = (L_\alpha f \circ T_\alpha)^{[2]} = (L_\alpha f)' \circ T_\alpha + \alpha^2 (L_\alpha f)^{[2]} \circ T_\alpha.$$

The result follows. ∎

Lemma 3.4. *Let $m \geq 7$ be such that $m \equiv 3 \pmod 4$ and let $f = \sum_{k=0}^{m} a_{m-k}x^k \in \mathbb{F}_2[a_0, \ldots, a_m][x]$. Consider the degree w defined by $w(\alpha) = 1$ and $w(a_i) = i$ for any i and consider also the degree $\tilde{w}$ defined by $\tilde{w}(\alpha) = 0$ and $\tilde{w}(a_i) = 1$. Then the resultant $\mathrm{Res}((D_\alpha f)', (D_\alpha f)^{[2]})$ in the variable x, if nonzero, is a homogeneous polynomial in $\mathbb{F}_2[a_0, \ldots, a_m, \alpha]$ of degree $m(m-3)$ when considering the degree w, and a homogeneous polynomial of degree $2(m-3)$ when considering the degree $\tilde{w}$.*

Proof. As $f(x) = \sum_{k=0}^{m} a_{m-k}x^k$ and $f(x+\alpha) = \sum_{k=0}^{m} a_{m-k}(x+\alpha)^k$, these two polynomials are homogeneous of degree m for the degree w such that $w(\alpha) = 1$, $w(a_i) = i$ and $w(x) = 1$. It follows that $(D_\alpha f)'$ and $(D_\alpha f)^{[2]}$ are homogeneous of degree respectively $m-1$ and $m-2$ for the degree w. Using the formulae for $D_\alpha f$ given in Section 2, we have

$$D_\alpha f(x) = \alpha a_0 x^{m-1} + a_0\alpha^2 x^{m-2} + (a_0\alpha^3 + a_1\alpha^2 + a_2\alpha)x^{m-3} + \cdots .$$

The polynomial $(D_\alpha f)'$ has degree $m-3$ in x since m is odd and its leading coefficient is $a_0\alpha^2$. The polynomial $(D_\alpha f)^{[2]}$ has also degree $m-3$ in x since it can be shown that $(x^k)^{[2]} = \binom{k}{2}x^{k-2}$ using the binomial theorem, the above Lucas theorem and the congruence class of m. Its leading coefficient is $a_0\alpha$.

Thus we can write

$$(D_\alpha f)' = \sum_{i=0}^{m-3} d_i x^{m-3-i} \quad \text{and} \quad (D_\alpha f)^{[2]} = \sum_{i=0}^{m-3} e_i x^{m-3-i}$$

where $d_i, e_i \in \mathbb{F}_2[a_0, \ldots, a_m, \alpha]$ are such that $w(d_i) = i+2$ and $w(e_i) = i+1$. Then the resultant $\mathrm{Res}((D_\alpha f)', (D_\alpha f)^{[2]})$ in x, if nonzero, is a homogeneous polynomial of $\mathbb{F}_2[a_0, \ldots, a_m, \alpha]$ of degree $m(m-3)$ for the degree w. For the second homogeneity result claimed, note that this resultant is a sum of $2(m-3)$ products of the coefficients d_i and e_i, and each one of these coefficients is a linear combination of $a_0, \ldots, a_m$. ∎

Lemma 3.5. *Let $m \geq 7$ be such that $m \equiv 3 \pmod 4$ and let $f = x^m$. For all $\alpha \in \mathbb{F}_q^*$ the critical points of $L_\alpha f$ are nondegenerate.*

Proof. Using Lemma 3.3 we look for the common roots of $(D_\alpha f)'$ and $(D_\alpha f)^{[2]}$. We compute $(D_\alpha f)' = (x+\alpha)^{m-1} + x^{m-1}$ and $(D_\alpha f)^{[2]} = (x+\alpha)^{m-2} + x^{m-2}$. Hence, if $\omega \in \overline{\mathbb{F}}_2$ were a common root of $(D_\alpha f)'$ and $(D_\alpha f)^{[2]}$ then we would have $((\omega+\alpha)/\omega)^{m-1} = ((\omega+\alpha)/\omega)^{m-2} = 1$, and so $\alpha = 0$. ∎

Proof of Proposition 3.2. Lemma 3.3 enables us to study $\tilde{R} := \mathrm{Res}((D_\alpha f)', (D_\alpha f)^{[2]})$ rather than $\mathrm{Res}((L_\alpha f)', (L_\alpha f)^{[2]})$. Using the homogeneity results given by Lemma 3.4 we know that there is at most one term in $\tilde{R}$ of degree at least $m(m-3)$ in α, precisely $a_0^{2(m-3)}\alpha^{m(m-3)}$. We study whether this term appears or not.

By Lemma 3.5, for nonzero α the critical points of $L_\alpha(x^m)$ are nondegenerate, so $\tilde{R}(a_0 = 1, a_1 = 0, \dots, a_m = 0, \alpha = 1) \neq 0$ and this term does appear. Choosing $f \in \mathbb{F}_q[x]$ of degree m amounts to choosing coefficients $a_0, \dots, a_m$ in $\mathbb{F}_q$ with $a_0 \neq 0$. Thus we can consider $\tilde{R}$ as a nonzero polynomial in α of degree $m(m-3)$ which has at most $m(m-3)$ roots. ■

3.3. Condition (b). We use a similar strategy to prove that for almost every choice of α the polynomial $L_\alpha f$ has distinct critical values: we use a homogeneity result and we study the case of $L_\alpha(x^m)$. As this is a key point in our approach, we give equivalent conditions for $L_\alpha(x^m)$ to have distinct critical values. Recall that we work with $m \equiv 3 \pmod 4$ and that we set $d = (m-1)/2$.

Proposition 3.6. *Let $m \geq 7$ be such that $m \equiv 3 \pmod 4$.*

(i) *If there exists $\alpha \in \overline{\mathbb{F}}_2^*$ such that $L_\alpha(x^m)$ has distinct critical values then it has distinct critical values for any $\alpha \in \overline{\mathbb{F}}_2^*$.*
(ii) *Suppose that for any $\alpha \in \overline{\mathbb{F}}_2^*$ (or equivalently for $\alpha = 1$), $L_\alpha(x^m)$ has distinct critical values. Let $f \in \mathbb{F}_q[x]$ be of degree m. Then $L_\alpha f$ has distinct critical values except for at most $(5m-1)(m-3)(m-7)/64$ values of $\alpha \in \overline{\mathbb{F}}_2$.*

Proof. Let $\alpha \in \overline{\mathbb{F}}_2^*$ be such that $L_\alpha(x^m)$ has distinct critical values. Now let $\alpha' \in \overline{\mathbb{F}}_2^*$ and let us show that $L_{\alpha'}(x^m)$ has distinct critical values. We use the characterization given by Lemma 3.7: Suppose that $(\tau, \eta) \in (\overline{\mathbb{F}}_2)^2$ are such that

$$\tau^{m-1} + (\tau + \alpha')^{m-1} = \eta^{m-1} + (\eta + \alpha')^{m-1} = 0, \tag{2}$$
$$\tau^m + (\tau + \alpha')^m = \eta^m + (\eta + \alpha')^m. \tag{3}$$

Mutliplying (2) by $(\alpha/\alpha')^{m-1}$ and (3) by $(\alpha/\alpha')^m$, we find that $\frac{\alpha}{\alpha'}\eta$ is in $\{\frac{\alpha}{\alpha'}\tau, \frac{\alpha}{\alpha'}\tau + \alpha\}$, i.e. $\eta \in \{\tau, \tau + \alpha'\}$, which gives the result.

To prove (ii) we follow the strategy of the proof of Proposition 3.2. Consider $f = \sum_{i=0}^m a_{m-i}x^i \in \mathbb{F}_2[a_0, \dots, a_m][x]$ and $L_\alpha f = \sum_{i=0}^d b_{d-i}x^i \in \mathbb{F}_2[b_0, \dots, b_d, \alpha][x]$. By Lemma 3.8, setting $N = d\binom{(d-1)/2}{2}$ we can see $b_0^N \times \Pi(L_\alpha f)$ as a polynomial in $\mathbb{F}_2[a_0, \dots, a_m, \alpha]$. Now we use the homogeneity result of Lemma 3.8 to deduce that this last polynomial has at most one term of degree at least $(5d+2)\binom{(d-1)/2}{2}$ in α. Precisely, this term can only be $a_0^{(d+2)\binom{(d-1)/2}{2}}\alpha^{(5d+2)\binom{(d-1)/2}{2}}$.

In order to know if this term appears or not, we evaluate this polynomial at $a_0 = 1$ and $a_i = 0$ for all $i > 0$, which amounts to determining if the polynomial $L_\alpha(x^m)$ has distinct critical values, which is true by hypothesis. Now fix $f \in \mathbb{F}_q[x]$ of degree m and see $b_0^N \times \Pi(L_\alpha f)$ as a polynomial in $\mathbb{F}_2[\alpha]$. So we know its degree and thus $L_\alpha f$ has distinct critical values except for at most $(5d+2)\binom{(d-1)/2}{2}$ values of $\alpha \in \overline{\mathbb{F}}_2$. Then we conclude using the relation between m and d. ∎

The following lemma gives a condition on $D_\alpha f$ for $L_\alpha f$ to have distinct critical values.

Lemma 3.7. *Let $f \in \mathbb{F}_q[x]$. For all $\alpha \in \mathbb{F}_q^*$ the polynomial $L_\alpha f$ has distinct critical values if and only if for all $(\tau,\eta) \in (\overline{\mathbb{F}}_2)^2$, $(D_\alpha f)'(\tau) = (D_\alpha f)'(\eta) = 0$ and $D_\alpha f(\tau) = D_\alpha f(\eta) \Rightarrow \tau = \eta$ or $\tau = \eta + \alpha$.*

Proof. We have $L_\alpha f \circ T_\alpha = D_\alpha f$, so $(D_\alpha f)' = \alpha (L_\alpha f)' \circ T_\alpha$ where $T_\alpha(x) = x(x+\alpha)$. The result follows by noticing that $T_\alpha(\tau) = T_\alpha(\eta)$ if and only if $\tau \in \{\eta, \eta+\alpha\}$. ∎

Lemma 3.8. *Let $m \geq 7$ be such that $m \equiv 3 \pmod 4$ and set $N = d\binom{(d-1)/2}{2}$. Consider the polynomials $f = \sum_{k=0}^m a_{m-k}x^k \in \mathbb{F}_2[a_0,\dots,a_m][x]$ and $L_\alpha f = \sum_{k=0}^d b_{d-k}x^k \in \mathbb{F}_2[b_0,\dots,b_d,\alpha][x]$. Then $b_0^N \times \Pi(L_\alpha f)$ is a polynomial in $\mathbb{F}_2[a_0,\dots,a_m,\alpha]$ each of whose terms contains a product of $(d+2)\binom{(d-1)/2}{2}$ terms a_i. This polynomial is also homogeneous of degree $(5d+2)\binom{(d-1)/2}{2}$ when considering the weight w such that $w(\alpha) = 1$ and $w(a_i) = i$.*

Proof. Let $\tau_1,\dots,\tau_{(d-1)/2}$ be the double roots of the polynomial $(L_\alpha f)'$, and $\Pi(L_\alpha f) = \prod_{i\neq j}(L_\alpha f(\tau_i) - L_\alpha f(\tau_j))$. Then

$$\Pi(L_\alpha f) = \prod_{i<j}\Big(\sum_{k=0}^{d} b_{d-k}^2(\tau_i^{2k} + \tau_j^{2k})\Big).$$

So $\Pi(L_\alpha f)$ is a homogeneous polynomial of degree $2d\binom{(d-1)/2}{2}$ when considering the weight w such that $w(b_i) = i$ for all i and $w(\tau_j) = 1$ for all j. We also know that $\Pi(L_\alpha f) \in \mathbb{F}_2[b_0,\dots,b_d,\tau_1^2,\dots,\tau_{(d-1)/2}^2]$, and each term of $\Pi(L_\alpha f)$ contains a product of exactly $\binom{(d-1)/2}{2}$ terms b_i^2. Moreover, using the invariance under the action of $\mathfrak{S}_{(d-1)/2}$ and the theorem of symmetric functions, we find that $\Pi(L_\alpha f) \in \mathbb{F}_2[b_0,\dots,b_d,\sigma_1,\dots,\sigma_{(d-1)/2}]$ where $\sigma_1 = \sum \tau_i^2$, $\sigma_2 = \sum_{i<j}\tau_i^2\tau_j^2, \dots$. Using $(L_\alpha f)' = b_0\prod_{i=1}^{(d-1)/2}(x^2+\tau_i^2)$ it follows that $\Pi(L_\alpha f) \in \mathbb{F}_2[b_0,\dots,b_d,b_2/b_0,b_4/b_0,\dots,b_{d-1}/b_0]$. The denominator is at worst b_0^N (this happens if the τ_i are the only terms contributing to the degree, and if they only give rise to terms b_2/b_0). We deduce that $b_0^N \times \Pi(L_\alpha f)$ is a polynomial in the b_i, and that each term is a product of $(d+2)\binom{(d-1)/2}{2}$ indeterminates b_i. Furthermore, it is a homogeneous polynomial of degree $2d\binom{(d-1)/2}{2}$ when considering the weight w such that $w(b_i) = i$ for all i.

By Lemma 2.4, b_i is a homogeneous polynomial of $\mathbb{F}_2[a_0,\dots,a_m,\alpha]$ of degree $2i+1$ when considering the weight w such that $w(a_i) = i$ and $w(\alpha) = 1$.

We conclude that $b_0^N \times \Pi(L_\alpha f)$ is a homogeneous polynomial of degree $2 \times 2d\binom{(d-1)/2}{2} + (d+2)\binom{(d-1)/2}{2}$. ∎

Finally we reach the goal of this section: Propositions 3.2 and 3.6 enable us to bound the number of α such that $L_\alpha f$ is Morse.

THEOREM 3.9. *Let $m \geq 7$ be such that $m \equiv 3 \pmod 4$ and the polynomial $L_1(x^m)$ has distinct critical values. Then for all $f \in \mathbb{F}_{2^n}[x]$ of degree m the number of elements α in $\mathbb{F}_{2^n}^*$ such that $L_\alpha f$ is Morse is at least*

$$2^n - 1 - \tfrac{1}{64}(m-3)(5m^2+28m+7).$$

Proof. Let $f \in \mathbb{F}_{2^n}[x]$ of degree m and let $\alpha \in \mathbb{F}_{2^n}^*$. The polynomial $L_\alpha f$ is Morse if conditions (a)–(c) of Definition 3.1 hold. As $m \equiv 3 \pmod 4$, condition (c) is satisfied. Indeed, $D_\alpha f$ has degree $m-1$ by Proposition 2.1 and thus $L_\alpha f$ has odd degree $(m-1)/2$. Moreover, (a) fails for at most $m(m-3)$ values of α by Proposition 3.2. Furthermore, (b) fails for at most $(5m-1)(m-3)(m-7)/64$ values of α by Proposition 3.6. Thus $L_\alpha f$ is not Morse for at most $m(m-3) + (5m-1)(m-3)(m-7)/64$ values of α. ∎

3.4. Conditions for $L_1(x^m)$ to have distinct critical values. Condition (b) which is essential for the proofs of our main results leads by Proposition 3.6 to study for which exponents m the polynomial $L_\alpha(x^m)$ has distinct critical values. By Proposition 3.6(i) we are reduced to considering the polynomial $L_1(x^m)$. Then it is natural to introduce the following set $\mathcal{M}$ and to look for practical characterizations.

DEFINITION 3.10. Let $\mathcal{M}$ be the set of odd integers m such that the polynomial $L_1(x^m)$ has distinct critical values or equivalently $L_\alpha(x^m)$ has distinct critical values for any $\alpha \in \overline{\mathbb{F}}_2^*$.

Lemma 3.7 reduces the study of the critical values of $L_\alpha(x^m)$ to the study of equations involving $D_\alpha(x^m)$ and $(D_\alpha(x^m))' = x^{m-1} + (x+\alpha)^{m-1}$ for odd m.

The following proposition characterizes the elements of $\mathcal{M}$ in terms of roots of unity.

PROPOSITION 3.11. *Let $m \geq 7$ be an odd integer. For any $\alpha \in \overline{\mathbb{F}}_2^*$, the polynomial $L_\alpha(x^m)$ has distinct critical values if and only if the following condition is satisfied: for $\zeta_1, \zeta_2 \in \overline{\mathbb{F}}_2 \setminus \{1\}$,*

$$\zeta_1^{m-1} = \zeta_2^{m-1} = \left(\frac{1+\zeta_1}{1+\zeta_2}\right)^{m-1} = 1 \implies \zeta_1 = \zeta_2 \text{ or } \zeta_1 = \zeta_2^{-1}.$$

Proof. We use Lemma 3.7 to relate to the equations of Lemma 3.12. With the expressions of x_i and x_j obtained, we notice that $x_i = x_j + \alpha$ if and only if $\zeta_1\zeta_2 = 1$. ∎

Lemma 3.12. *Let $m \geq 7$ be odd and $\alpha \in \mathbb{F}_q^*$. Two distinct elements x_i and x_j in $\overline{\mathbb{F}}_2$ satisfy*

$$(\diamond) \qquad \begin{aligned} &x_i^{m-1} = (x_i+\alpha)^{m-1}, \quad x_j^{m-1} = (x_j+\alpha)^{m-1} \quad \textit{and} \\ &x_i^m + (x_i+\alpha)^m = x_j^m + (x_j+\alpha)^m \end{aligned}$$

if and only if

$$x_i = \frac{\zeta_1(1+\zeta_2)}{\zeta_1+\zeta_2}\alpha \quad \textit{and} \quad x_j = \frac{(1+\zeta_2)}{\zeta_1+\zeta_2}\alpha$$

for some distinct $\zeta_1, \zeta_2 \in \overline{\mathbb{F}}_2 \setminus \{1\}$ satisfying $\zeta_1^{m-1} = \zeta_2^{m-1} = \left(\frac{1+\zeta_1}{1+\zeta_2}\right)^{m-1} = 1$.

Proof. Suppose that x_i and x_j satisfy $(\diamond)$. We notice that they cannot be 0 or α, so we can set $\zeta_1 = x_i/x_j$ and $\zeta_2 = (x_i+\alpha)/(x_j+\alpha)$. As $x_i \neq x_j$, we have $\zeta_1 \neq \zeta_2$. Replacing $(x_i+\alpha)^{m-1}$ by x_i^{m-1} and $(x_j+\alpha)^{m-1}$ by x_j^{m-1} in $(\diamond)$ we obtain $\zeta_1^{m-1} = 1$. Replacing x_i^{m-1} by $(x_i+\alpha)^{m-1}$ and x_j^{m-1} by $(x_j+\alpha)^{m-1}$ in $(\diamond)$ we obtain $\zeta_2^{m-1} = 1$. Replacing x_i by $\zeta_1 x_j$ and $x_i+\alpha$ by $\zeta_2(x_j+\alpha)$ in the left hand side of the third equation in $(\diamond)$, we obtain $(1+\zeta_1)x_j^m = (1+\zeta_2)(x_j+\alpha)^m$, so $(1+\zeta_1)/(1+\zeta_2) = (x_j+\alpha)/x_j$, and $((1+\zeta_1)/(1+\zeta_2))^{m-1} = 1$. To obtain the claimed expressions of x_i and x_j, one can replace x_j by $\zeta_1^{-1}x_i$ in $x_i+\alpha = \zeta_2(x_j+\alpha)$. The converse follows from straightforward computations. ∎

Example 3.13. It is straightforward to see that the integers $m = 2^k+1$ for $k \geq 1$ belong to $\mathcal{M}$ since 1 is the only root of $x^{2^k}+1$.

Remark 3.14. As a consequence of Proposition 3.11 an odd integer m belongs to $\mathcal{M}$ if and only if $2(m-1)+1$ does. This implies that if an odd integer m belongs to $\mathcal{M}$ then for all $k \geq 0$ the integer $2^k(m-1)+1$ does. We also notice that if an integer m (not necessarily odd) satisfies the condition of Proposition 3.11 then $2(m-1)+1 \in \mathcal{M}$.

Example 3.15. As the polynomial x^3-1 has exactly two roots ζ and ζ^{-1} different from the unity, we can deduce that $m = 4$ satisfies the condition of Proposition 3.11. Thus according to the above remark, $2^k3+1 \in \mathcal{M}$ for all $k \geq 1$.

Example 3.16. Proposition 3.11 also provides us with a method of checking if an odd integer m belongs to $\mathcal{M}$. For a fixed odd integer m, write $m-1 = t2^s$ with t odd. Hence the $(m-1)$th roots of unity are exactly the tth roots of unity in characteristic two. Consider the smallest integer n such that $2^n \equiv 1 \pmod{t}$ and compute the list of the tth roots of unity distinct from 1 in $\mathbb{F}_{2^n}$. Then check for ζ_1 and ζ_2 in this list if $\left(\frac{1+\zeta_1}{1+\zeta_2}\right)^t = 1$ implies $\zeta_1 = \zeta_2$ or $\zeta_1 = \zeta_2^{-1}$ using an exhaustive method. For example, using the open source computer algebra system SAGE we have determined that the only odd integers less than 200 which do not belong to $\mathcal{M}$ are 15, 29, 31, 43, 57, 61, 63, 71, 85, 91, 99, 103, 113, 121, 125, 127, 141, 147, 151, 155, 169, 171, 179, 181, 183, 187 and 197.

Below we give some infinite families of good exponents.

EXAMPLE 3.17. Let us prove that for any $k \geq 0$ the integers $m = 2^k + 2$ satisfy the conditions of Proposition 3.11. First notice that if ζ is a $(m-1)$th root of unity then $(1+\zeta)^{2^k+1} = \zeta + \zeta^{-1}$. As a consequence, if ζ_1 and ζ_2 are two $(m-1)$th roots of unity such that $\left(\frac{1+\zeta_1}{1+\zeta_2}\right)^{m-1} = 1$ then

$$\zeta_2\big((1+\zeta_1)^{2^k+1} + (1+\zeta_2)^{2^k+1}\big) = \zeta_2^2 + (\zeta_1 + \zeta_1^{-1})\zeta_2 + 1.$$

But this is equal to zero, so ζ_2 is equal to ζ_1 or ζ_1^{-1}.

EXAMPLE 3.18. Applying Remark 3.14 to the previous example we deduce that $2^k + 2^s + 1 \in \mathcal{M}$ for any k and s satisfying $k \geq s \geq 1$.

EXAMPLE 3.19. If $m = 2^k - 1$ with $k \geq 4$, we notice that for any choice of a $(2^{k-1}-1)$th root of unity ζ_1, we also have $(1+\zeta_1)^{2^{k-1}-1} = 1$. So any couple (ζ_1, ζ_2) of $(2^{k-1}-1)$th roots of unity such that $\zeta_1 \neq \zeta_2$ and $\zeta_1\zeta_2 \neq 1$ will satisfy the hypothesis $\zeta_1^{m-1} = \zeta_2^{m-1} = \left(\frac{1+\zeta_1}{1+\zeta_2}\right)^{m-1} = 1$ but not the conclusion. In this case $L_\alpha(x^m)$ does not have distinct critical values, so $m \notin \mathcal{M}$.

The following result will be our main tool for obtaining infinite families of good exponents with convenient congruence class. Indeed, this result combined with the characterization of the set $\mathcal{M}$ given in Proposition 3.11 will provide us the families of good exponents specified in Proposition 5.2(iii) and exploited in Corollaries 5.4 and 5.6.

PROPOSITION 3.20. *Let p, ℓ be distinct primes with $\ell \neq 2$, $p^{\ell-1} \not\equiv 1 \pmod{\ell^2}$ and such that if $\zeta_1, \zeta_2 \neq 1$ are ℓth roots of unity in characteristic p such that $(\zeta_1+1)/(\zeta_2+1)$ is also an ℓth root of unity, then $\zeta_1 = \zeta_2$ or $\zeta_1 = \zeta_2^{-1}$. Then, for any $k \geq 2$, if $\zeta_1, \zeta_2 \neq 1$ are ℓ^kth roots of unity in characteristic p such that $(\zeta_1+1)/(\zeta_2+1)$ is also an ℓ^kth root of unity, then $\zeta_1 = \zeta_2$ or $\zeta_1 = \zeta_2^{-1}$.*

Proof. Induction on k. The case $k = 1$ is the hypothesis.

Assume now that ζ_1 has order exactly ℓ^k, $k \geq 2$, and let $\mathbb{F}_q = \mathbb{F}_p(\zeta_1)$. Because we have assumed that $p^{\ell-1} \not\equiv 1 \pmod{\ell^2}$, we know that the order of $p \pmod{\ell^k}$ is ℓ times the order of $p \pmod{\ell^{k-1}}$. Let $\mathbb{F}_r = \mathbb{F}_p(\zeta_1^\ell)$. It follows that $[\mathbb{F}_q : \mathbb{F}_p] = \ell[\mathbb{F}_r : \mathbb{F}_p]$. Then $q = r^\ell$ and the minimal polynomial of ζ_1 over $\mathbb{F}_r$ is $x^\ell - \alpha_1$, where $\alpha_1 = \zeta_1^\ell$ has order ℓ^{k-1}. In particular $\mathrm{N}\,\zeta_1 = \alpha_1$, $\mathrm{Tr}\,\zeta_1 = 0$ and $\mathrm{N}(1+\zeta_1) = 1 + \alpha_1$ where N, Tr are respectively the norm and trace in $\mathbb{F}_q/\mathbb{F}_r$, and the last equality follows by evaluating $x^\ell - \alpha_1$ at $x = -1$.

Assume first that ζ_2 has order exactly ℓ^k too and $\zeta_3 = (\zeta_1+1)/(\zeta_2+1)$ is also an ℓ^kth root of unity and write $\zeta_i^\ell = \alpha_i$, $i = 2,3$, so the α_i are ℓ^{k-1}th roots of unity. As before, we get $\mathrm{N}\,\zeta_i = \alpha_i$, $i = 2,3$, and $\mathrm{N}(1+\zeta_2) = 1+\alpha_2$. Taking norms, we get $\alpha_3 = (\alpha_1+1)/(\alpha_2+1)$, so by induction $\alpha_1 = \alpha_2$ or $\alpha_1 = \alpha_2^{-1}$.

If $\alpha_1 = \alpha_2$, then $\alpha_3 = 1$ and either $\zeta_1 = \zeta_2$ as desired, or $\zeta_1 = \omega\zeta_2$ with ω of order ℓ. In the latter case we get $(1+\omega\zeta_2)/(1+\zeta_2) = \omega^j$ for some $j = 0,1,\ldots,\ell-1$. If $j \neq 1$, we can solve the equation for ζ_2 and get $\zeta_2 \in \mathbb{F}_p(\omega)$, which is a contradiction. If $j = 1$ we get $\omega = 1$, again a contradiction.

If $\alpha_1 = \alpha_2^{-1}$, then $\alpha_3 = \alpha_1$ and either $\zeta_1 = \zeta_2^{-1}$ as desired, or $\zeta_1 = \omega\zeta_2^{-1}$ with ω of order ℓ. In the latter case we get $(1+\zeta_1)/(1+\omega\zeta_1^{-1}) = \omega^j\zeta_1$ for some $j = 0,1,\ldots,\ell-1$. This gives, for $j \neq 0$, $\zeta_1 \in \mathbb{F}_p(\omega)$, which is a contradiction. For $j = 0$, this gives $\omega = 1$, again a contradiction.

Finally, assume that ζ_2 has order smaller than ℓ^k, so $\zeta_2 \in \mathbb{F}_r$. We write our equation as $\zeta_1 + 1 = \zeta_3(\zeta_2+1)$. First note that ζ_3 cannot be in $\mathbb{F}_r$, since ζ_1 is not in $\mathbb{F}_r$, so $\mathrm{Tr}\,\zeta_1 = \mathrm{Tr}\,\zeta_3 = 0$, so taking the trace of our equation gives $1 = 0(\zeta_2+1) = 0$, a contradiction. ∎

Example 3.21. We verified by computer calculation that the hypothesis of this proposition holds when $p = 2$ and $\ell < 200$ except for $\ell = 7, 31, 73, 89, 127$. For example the case $\ell = 3$ follows from Example 3.15. These computations will enable us to exhibit the examples of Corollaries 5.4 and 5.6.

4. Regular extensions. Let $n \geq 1$ and set $q = 2^n$. Let t be an element transcendental over $\mathbb{F}_q$ and K an extension field of $\mathbb{F}_q(t)$. Recall that the extension $K/\mathbb{F}_q(t)$ is said to be *regular* if it is separable and $\mathbb{F}_q$ is algebraically closed in K, i.e. $\mathbb{F}_q^K = \mathbb{F}_q$ where $\mathbb{F}_q^K$ is the algebraic closure of $\mathbb{F}_q$ in K.

Let $\alpha \in \mathbb{F}_q^*$, let m be an integer and $d = (m-1)/2$ if m is odd and $d = (m-2)/2$ if m is even. Fix $f \in \mathbb{F}_q[x]$ of degree m such that $L_\alpha f$ has degree exactly d. Furthermore, suppose that d is odd, which is equivalent to $m \equiv 0$ or $3 \pmod 4$.

4.1. First floor: monodromy. We consider the *arithmetic monodromy group* G of the polynomial $L_\alpha f$. It is the Galois group of the extension $F/\mathbb{F}_q(t)$ where F is the splitting field of $L_\alpha f(x) - t$ over $\mathbb{F}_q(t)$. Consider also $\overline{G} := \mathrm{Gal}(F/\mathbb{F}_q^F(t))$, the *geometric monodromy group* of $L_\alpha f$. The groups G and $\overline{G}$ are transitive subgroups of the symmetric group $\mathfrak{S}_d$ and $\overline{G} \lhd G$.

Proposition 4.1. *Let $f \in \mathbb{F}_q[x]$ be such that the polynomial $L_\alpha f$ is Morse and has* (*odd*) *degree d.*

- (i) *Let u be a root of $L_\alpha f(x) - t$ in F. Then, for each place $\wp$ of F above the place ∞ at infinity of $\mathbb{F}_q(t)$, u has a simple pole at $\wp$.*
- (ii) $\mathrm{Gal}(F/\mathbb{F}_q(t)) = \mathfrak{S}_d$ *and the extension $F/\mathbb{F}_q(t)$ is regular.*

Proof. If $v_\wp$ is the valuation at $\wp$, we have $v_\wp(L_\alpha f(u)) = v_\wp(t)$ and by definition of the ramification index $e(\wp|\infty)$ we have $v_\wp(t) = e(\wp|\infty)v_\infty(t) = -e(\wp|\infty)$. Since d is supposed to be odd, it is prime to the characteristic of $\mathbb{F}_q(t)$, and so, by [9, proof of Theorem 4.4.5], we have $e(\wp|\infty) = d$. Hence, $v_\wp(L_\alpha f(u)) = -d$, which implies that $v_\wp(u) = -1$ and thus u has a simple pole at $\wp$.

The analogue of the Hilbert theorem given by Serre [9, Theorem 4.4.5] and detailed in even characteristic in the appendix of Geyer in [5] shows that the geometric monodromy group $\mathrm{Gal}(F/\mathbb{F}_q^F(t))$ of $L_\alpha f$ is $\mathfrak{S}_d$. But it is contained in the arithmetic monodromy group $\mathrm{Gal}(F/\mathbb{F}_q(t))$ which is also a subgroup of $\mathfrak{S}_d$. So they are equal and $\mathbb{F}_q^F = \mathbb{F}_q$. ∎

A consequence of the first part of the above proposition is that $L_\alpha f(x)-t$ has only simple roots; let us call them $u_0, \ldots, u_{d-1}$.

4.2. Second floor. Let x_i be such that $x_i^2 + \alpha x_i = u_i$. Hence $D_\alpha f(x_i) = t$. Consider $\Omega = \mathbb{F}_q(x_0, \ldots, x_{d-1})$, the compositum of the fields $F(x_i)$, and $\mathbb{F}_q^\Omega F$, the compositum of F and $\mathbb{F}_q^\Omega$. Let $\Gamma = \mathrm{Gal}(\Omega/F)$ and $\overline{\Gamma} = \mathrm{Gal}(\Omega/\mathbb{F}_q^\Omega F)$.

The following statement appears in [10].

LEMMA 4.2. *Suppose that $L_\alpha f$ is Morse and has degree d. If $\emptyset \subsetneq J \subsetneq \{0, \ldots, d-1\}$ then $\sum_{j\in J} u_j$ has a pole at a place of F over the place ∞ of $\mathbb{F}_q(t)$.*

Proof. To obtain a contradiction, fix $j_0 \in J$ and $j_1 \in \{0, \ldots, d-1\} \setminus J$, and suppose that $\sum_{j\in J} u_j$ has no pole at places above ∞. Then it has no pole at all, and so it is constant. Recall that $\mathrm{Gal}(F/\mathbb{F}_q(t)) = \mathfrak{S}_d$ by Proposition 4.1. Applying to $\sum_{j\in J} u_j$ the automorphism corresponding to the transposition $(j_0 j_1) \in \mathfrak{S}_d$ one obtains

$$\sum_{j\in J\setminus\{j_0\}} u_j + u_{j_0} = \sum_{j\in J\setminus\{j_0\}} u_j + u_{j_1},$$

which leads to $u_{j_0} = u_{j_1}$, a contradiction. ∎

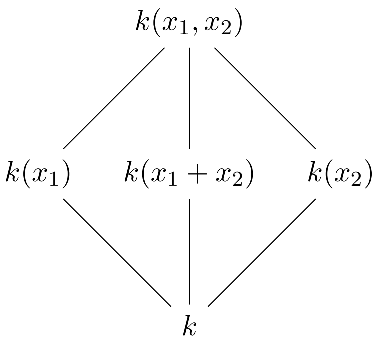
LEMMA 4.3. *Suppose that $L_\alpha f$ is Morse and has degree d. Let $\widetilde{F}$ be F or $\mathbb{F}_q^\Omega F$. Let J be a nonempty proper subset of $\{0, \ldots, d-1\}$. Then*

$$\sum_{j\in J} x_j \notin \widetilde{F}.$$

Proof. To obtain a contradiction, suppose that $\sum_{j\in J} x_j \in \widetilde{F}$. By Lemma 4.2 we know that there exists a place $\wp$ of F above ∞ such that $\sum_{j\in J} u_j$ has a pole at $\wp$. Moreover, this pole is simple as for all $j \in \{0, \ldots, d-1\}$ the root u_j has a simple pole by Proposition 4.1. Now consider $A = \sum_{j\in J} x_j$ and $B = \sum_{j\in J} x_j + \alpha$. If A (and thus B) belongs to $\widetilde{F}$, one can consider the valuation of A and B at $\wp$. As $A.B = \sum_{j\in J} u_j$, either A or B has a pole. Since A and B differ by a constant, A has a pole if and only if B has a pole. So both have a pole and the order of multiplicity is the same. Then we obtain $2v_\wp(A) = -1$, a contradiction. ∎

LEMMA 4.4. *Let $k(x_1)$ and $k(x_2)$ be two Artin–Schreier extensions of a field k of characteristic 2. Suppose that $x_i^2 + \alpha x_i = w_i$ with α and w_i in k^*. Then $k(x_1) = k(x_2)$ if and only if $x_1 + x_2 \in k$.*

Moreover, if $x_1 + x_2 \notin k$ then $k(x_1, x_2)$ is a degree 4 extension of k, and the three fields lying between k and $k(x_1, x_2)$ are those of the following diagram.



Proof. For the first assertion, see [1, proof of Lemma 4.1]. If $x_1+x_2 \notin k$, we can use $[k(x_1)(x_2) : k(x_1)] = 2$ to prove $[k(x_1, x_2) : k] = 4$. We deduce that

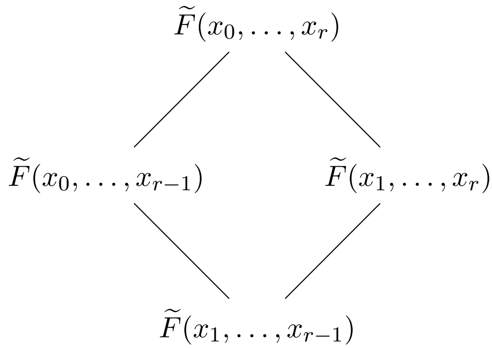
$$\mathrm{Gal}(k(x_1, x_2)/k) = (\mathbb{Z}/2\mathbb{Z})^2.$$

The field $k(x_1 + x_2)$ is a subextension since $x_1 + x_2$ is a root of $x^2 + \alpha x = w_1 + w_2$. It remains to prove that $k(x_1 + x_2)$ is different from $k(x_1)$ (and $k(x_2)$). According to the first statement of the lemma, it is sufficient to check that $x_1 + (x_1 + x_2) \notin k$. ∎

PROPOSITION 4.5. *Suppose that $L_\alpha f$ is Morse and has degree d. Let $\widetilde{F}$ be F or $\mathbb{F}_q^\Omega F$. Let r be an integer such that $0 \le r \le d-2$. Then*

(i) *$\widetilde{F}(x_0, \ldots, x_r)$ is an extension of degree 2^{r+1} of $\widetilde{F}$,*
(ii) *$\mathrm{Gal}(\widetilde{F}(x_0, \ldots, x_r)/\widetilde{F}) = (\mathbb{Z}/2\mathbb{Z})^{r+1}$, and*
(iii) *there are $2^{r+1}-1$ quadratic extensions of $\widetilde{F}$ between $\widetilde{F}$ and $\widetilde{F}(x_0, \ldots, x_r)$, namely $\widetilde{F}(\sum_{j\in J} x_j)$ with nonempty $J \subset \{0, \ldots, r\}$.*

Proof. We proceed by induction. The case $r = 0$ is trivial and the case $r = 1$ is given by Lemma 4.4. Assuming that the conclusion holds for $r-1$ with $1 \le r \le d-2$, we will prove it for r. The main idea is to consider the extensions of the diagram



and to apply Lemma 4.4.

We first prove that $x_0+x_r\notin\widetilde{F}(x_1,\dots,x_{r-1})$. Otherwise we would have the quadratic extension $\widetilde{F}(x_0+x_r)$ between $\widetilde{F}$ and $\widetilde{F}(x_1,\dots,x_{r-1})$. By the induction hypothesis, there would exist $J\subset\{1,\dots,r-1\}$ such that $\widetilde{F}(x_0+x_r)=\widetilde{F}(\sum_{j\in J}x_j)$. By Lemma 4.4 again we would have $x_0+x_r+\sum_{j\in J}x_j\in\widetilde{F}$, in contradiction with Lemma 4.3.

Next, we can apply Lemma 4.4 with $k=\widetilde{F}(x_1,\dots,x_{r-1})$ to deduce that $\widetilde{F}(x_0,\dots,x_r)$ is a quadratic extension of both $\widetilde{F}(x_1,\dots,x_r)$ and $\widetilde{F}(x_0,\dots,x_{r-1})$. It follows that $[\widetilde{F}(x_0,\dots,x_r):\widetilde{F}]=2^{r+1}$. Furthermore, we can define 2^{r+1} different $\widetilde{F}$-automorphisms of $\widetilde{F}(x_0,\dots,x_r)$ by sending x_i to x_i or to $x_i+\alpha$. So, all the elements of the Galois group $\mathrm{Gal}(\widetilde{F}(x_0,\dots,x_r)/\widetilde{F})$ have order dividing 2, and thus this group is certainly $(\mathbb{Z}/2\mathbb{Z})^{r+1}$.

For any nonempty subset $J\subset\{0,\dots,r\}$ we see that $\sum_{j\in J}x_j$ is a root of $x^2+\alpha x=\sum_{j\in J}u_j$, and we know from Lemma 4.3 that $\sum_{j\in J}x_j\notin\widetilde{F}$. We obtain this way $2^{r+1}-1$ different quadratic extensions between $\widetilde{F}$ and $\widetilde{F}(x_0,\dots,x_r)$. Indeed, if $\widetilde{F}(\sum_{j\in J_1}x_j)=\widetilde{F}(\sum_{j\in J_2}x_j)$ then $\sum_{j\in J_1}x_j+\sum_{j\in J_2}x_j\in\widetilde{F}$, which leads to $J_1=J_2$ by using Lemma 4.3. Finally, these $2^{r+1}-1$ quadratic extensions are the only ones. Indeed, the quadratic extensions between $\widetilde{F}$ and $\widetilde{F}(x_0,\dots,x_r)$ are in correspondence with the subgroups of $(\mathbb{Z}/2\mathbb{Z})^{r+1}$ of index 2. These subgroups are the hyperplanes of $(\mathbb{Z}/2\mathbb{Z})^{r+1}$ and there are $2^{r+1}-1$ of them. ∎

PROPOSITION 4.6. *Suppose that $L_\alpha f=\sum_{k=0}^{d}b_{d-k}x^k$ is Morse and has degree d. Let $\widetilde{F}$ be F or $F\mathbb{F}_q^{\Omega}$. If there exists $x\in\mathbb{F}_q$ such that $x^2+\alpha x=b_1/b_0$ then $\mathrm{Gal}(\widetilde{F}(x_0,\dots,x_{d-1})/\widetilde{F})=(\mathbb{Z}/2\mathbb{Z})^{d-1}$ and thus the extensions Ω/F and $\Omega/\mathbb{F}_q(t)$ are regular.*

Proof. As Proposition 4.5 already gives

$$\mathrm{Gal}(\widetilde{F}(x_0,\dots,x_{d-2})/\widetilde{F})=(\mathbb{Z}/2\mathbb{Z})^{d-1},$$

it remains to study the extension $\widetilde{F}(x_0,\dots,x_{d-1})/\widetilde{F}(x_0,\dots,x_{d-2})$.

Using $\sum_{i=0}^{d-1} u_i = b_1/b_0$ and the linearity of $x \mapsto x^2 + \alpha x$, we see that in any case the equation $x^2 + \alpha x = b_1/b_0$ has two solutions in $\overline{\mathbb{F}}_q$, namely $\sum_{i=0}^{d-1} x_i$ and $\alpha + \sum_{i=0}^{d-1} x_i$. With our hypothesis we deduce that $\sum_{i=0}^{d-1} x_i \in \mathbb{F}_q$, hence $\widetilde{F}(x_0, \ldots, x_{d-1}) = \widetilde{F}(x_0, \ldots, x_{d-2})$ and the result about the Galois group follows. Thus we have proved that $\Gamma = \overline{\Gamma}$ and so Ω/F is regular. Proposition 4.1 shows that $F/\mathbb{F}_q(t)$ is regular, which yields the regularity of $\Omega/\mathbb{F}_q(t)$. ∎

5. Main results. The main ingredient of the proof of our main results is the Chebotarev density theorem. The next proposition summarizes its contribution in our context.

PROPOSITION 5.1. *Let $m \geq 7$ be such that $m \equiv 3 \pmod 4$. Then there exists N depending only on m such that for all $n \geq N$, if we set $q = 2^n$ then for all $f \in \mathbb{F}_q[x]$ of degree m, and all α in $\mathbb{F}_q^*$ such that the extension $\Omega/\mathbb{F}_q(t)$ is regular, there exists $\beta \in \mathbb{F}_q$ such that the polynomial $D_\alpha f(x) + \beta$ splits in $\mathbb{F}_q[x]$ with no repeated factors.*

Proof. As $m \equiv 3 \pmod 4$, by Proposition 2.1 the polynomial $L_\alpha f$ has degree exactly $d = (m-1)/2$, which is odd by our hypothesis on m, and thus $F/\mathbb{F}_q(t)$ is separable. Since Ω/F is also separable, we infer that $\Omega/\mathbb{F}_q(t)$ is separable and thus Galois.

Since the extension $\Omega/\mathbb{F}_q(t)$ is supposed to be regular, by an application of the Chebotarev theorem (see [3, Theorem 1] which follows from [8, Proposition 4.6.8]) the number $N(S)$ of places v of $\mathbb{F}_q(t)$ of degree 1 unramified in Ω and such that the Artin symbol $\left(\frac{\Omega/\mathbb{F}_q(t)}{v}\right)$ is equal to the conjugacy class of $\mathrm{Gal}(\Omega/\mathbb{F}_q(t))$ consisting of the identity element satisfies

$$N(S) \geq \frac{q}{d_\Omega} - 2\left(\left(1 + \frac{g_\Omega}{d_\Omega}\right)q^{1/2} + q^{1/4} + 1 + \frac{g_\Omega}{d_\Omega}\right)$$

where $d_\Omega := [\Omega : \mathbb{F}_q(t)]$ and g_Ω is the genus of Ω.

But we have seen that

$$G = \mathrm{Gal}(F/\mathbb{F}_q(t)) \subset \mathfrak{S}_d \quad \text{and} \quad \Gamma = \mathrm{Gal}(\Omega/F)$$

is a group of order bounded by 2^d, thus $d_\Omega \leq d!2^d$. Moreover, one can obtain an upper bound on g_Ω depending only on d using [7, Lemma 14], namely $g_\Omega \leq \frac{1}{2}(\deg D_\alpha f - 3)d_\Omega + 1$, i.e.

$$g_\Omega \leq (d!2^d) \times (d - 3/2) + 1.$$

Thus if q is sufficiently large we will have $N(S) \geq 1$, which concludes the proof. ∎

Since our methods require the degree m of the polynomials to belong to the set $\mathcal{M}$ defined in Definition 3.10, we list some infinite subsets of $\mathcal{M}$ we have pointed out in Subsection 3.4.

Proposition 5.2. *The following integers m belong to the set $\mathcal{M}$:*

(i) $m = 2^k + 1$ *for* $k \geq 1$.
(ii) $m = 2^k + 2^s + 1$ *for* $k \geq s \geq 1$.
(iii) $m = 2^s \ell^k + 1$ *for* $k, s \geq 1$ *and for ℓ an odd prime such that* $2^{\ell-1} \not\equiv 1 \pmod{\ell^2}$ *and* $m' := \ell + 1$ *satisfies the condition of Proposition* 3.11.

Proof. The first two assertions are proved respectively in Examples 3.13 and 3.18. If ℓ is as in (iii) then Proposition 3.20 in the case of characteristic two tells us that $\ell^k + 1$ also satisfies the condition of Proposition 3.11. Now use Remark 3.14 to deduce that $2^s \ell^k + 1$ satisfies the condition of Proposition 3.11. For $s \geq 1$ it is odd and so it belongs to $\mathcal{M}$. ∎

Now we can state and prove our main results which establish for some polynomials f the maximality of the differential uniformity $\delta(f)$ defined in Section 1 by $\delta(f) = \max_{(\alpha,\beta) \in \mathbb{F}_q^* \times \mathbb{F}_q} \sharp\{x \in \mathbb{F}_q \mid f(x+\alpha) + f(x) = \beta\}$.

Theorem 5.3. *Let $m \in \mathcal{M}$ be such that* $m \equiv 7 \pmod 8$. *Then for n sufficiently large, for all polynomials $f \in \mathbb{F}_{2^n}[x]$ of degree m we have $\delta(f) = m - 1$.*

Proof. We fix $m \in \mathcal{M}$ such that $m \equiv 7 \pmod 8$. Let us prove that for n sufficiently large and for any polynomial $f = \sum_{i=0}^{m} a_{m-i} x^i$ in $\mathbb{F}_{2^n}[x]$ of degree m, there exists α in $\mathbb{F}_{2^n}^*$ such that

- $L_\alpha f$ is Morse,
- the equation $x^2 + \alpha x = b_1/b_0$ has a solution in $\mathbb{F}_{2^n}$, where $L_\alpha f = \sum_{i=0}^{d} b_{d-i} x^i$.

By Theorem 3.9, for all $f \in \mathbb{F}_{2^n}[x]$ of degree m, the number of elements α in $\mathbb{F}_{2^n}^*$ such that $L_\alpha f$ is Morse is at least $2^n - \frac{1}{64}(m-3)(5m^2 + 28m + 7)$.

Moreover, by Hilbert's Theorem 90, the equation $x^2 + \alpha x = b_1/b_0$ has a solution in $\mathbb{F}_{2^n}$ if and only if $\mathrm{Tr}_{\mathbb{F}_{2^n}/\mathbb{F}_2}\big(\frac{b_1}{b_0 \alpha^2}\big) = 0$. By Lemma 2.5 this is equivalent to $\mathrm{Tr}_{\mathbb{F}_{2^n}/\mathbb{F}_2}\big(\frac{a_1^2 + a_0 a_2}{a_0^2 \alpha^2}\big) = 0$. If $a_1^2 + a_0 a_2 = 0$, every choice of $\alpha \in \mathbb{F}_{2^n}^*$ is convenient. Otherwise the map sending α to $\frac{a_1^2 + a_0 a_2}{a_0^2 \alpha^2}$ is a permutation of $\mathbb{F}_{2^n}^*$ and then $2^{n-1} - 1$ values of α are convenient.

Hence as soon as $2^{n-1} > \frac{1}{64}(m-3)(5m^2 + 28m + 7) + 1$ we will have for any $f \in \mathbb{F}_{2^n}[x]$ of degree m the existence of α in $\mathbb{F}_{2^n}^*$ satisfying the two conditions. Now, these conditions imply by Proposition 4.6 that the extension $\Omega/\mathbb{F}_{2^n}(t)$ is regular.

Finally, we can apply Proposition 5.1 to obtain, for n sufficiently large depending only on m, the existence of $\beta \in \mathbb{F}_{2^n}$ such that $D_\alpha f(x) + \beta$ splits in $\mathbb{F}_{2^n}[x]$ with no repeated factors. Then $\delta(f) = m - 1$. ∎

To be concrete, using Proposition 5.2, the computations of Example 3.21 and taking into account the congruences of m we present in the following corollary some families of infinitely many integers for which the conclusion of Theorem 5.3 holds.

Corollary 5.4. *Let ℓ be a prime congruent to* 3 *modulo* 4 *such that* $2^{\ell-1} \not\equiv 1 \pmod{\ell^2}$ *and* $\ell+1$ *satisfies the condition of Proposition* 3.11 (*for example,* $\ell \in \{3, 11, 19, 23, 43, 47, 59, 67, 71, 79, 83, 103, 107, 131, 139, 151, 163, 167, 179, 191, 199, \ldots\}$). *Set* $m = 2\ell^{2k+1} + 1$ *with* $k \geq 0$. *Then for* n *sufficiently large, for all polynomials* $f \in \mathbb{F}_{2^n}[x]$ *of degree* m *we have* $\delta(f) = m-1$.

When m is congruent to 3 modulo 8, we also obtain some results but we have conditions on the parity of n or we have to remove some polynomials.

Theorem 5.5. *Let* $m \in \mathcal{M}$ *be such that* $m \geq 7$ *and* $m \equiv 3 \pmod 8$.

(i) *For* n *even and sufficiently large and for all* $f \in \mathbb{F}_{2^n}[x]$ *of degree* m, *we have* $\delta(f) = m - 1$.
(ii) *For* n *sufficiently large and for all* $f = \sum_{i=0}^{m} a_{m-i}x^i \in \mathbb{F}_{2^n}[x]$ *of degree* m *such that* $a_1^2 + a_0a_2 \neq 0$, *we have* $\delta(f) = m - 1$.

Proof. The proof is similar to the one of Theorem 5.3. The main difference comes from the expression of b_1/b_0 when $m \equiv 3 \pmod 8$. According to Lemma 2.5, we have $\mathrm{Tr}_{\mathbb{F}_{2^n}/\mathbb{F}_2}\big(\frac{b_1}{b_0\alpha^2}\big) = 0$ if and only if $\mathrm{Tr}_{\mathbb{F}_{2^n}/\mathbb{F}_2}\big(\frac{a_1^2+a_0a_2}{a_0^2\alpha^2}\big) = n$. The arguments of the above proof apply except when $a_1^2 + a_0a_2 = 0$ and n is odd. ∎

We remark that one cannot expect better in the case where $m \equiv 3 \pmod 8$, $a_1^2 + a_0a_2 = 0$ and n odd since Theorem 2(iii) of [10] shows that $\delta(f) < m - 1$ in this case.

Again using Proposition 5.2 and the computations of Example 3.21 we obtain the following corollary.

Corollary 5.6. *Let* ℓ *be an odd prime such that* $2^{\ell-1} \not\equiv 1 \pmod{\ell^2}$ *and* $\ell + 1$ *satisfies the condition of Proposition* 3.11.

(i) *If* $\ell \equiv 1 \pmod 8$ *then the conclusion of Theorem* 5.5 *holds for* $m = 2\ell^k + 1$ *with* $k \geq 1$ (*for example if* $\ell \in \{17, 41, 97, 113, 137, 193, \ldots\}$).
(ii) *If* $\ell \equiv 7 \pmod 8$ *then the conclusion of Theorem* 5.5 *holds for* $m = 2\ell^{2k+1}+1$ *with* $k \geq 0$ (*for example if* $\ell \in \{23, 47, 71, 79, 103, 151, 167, 191, 199, \ldots\}$).

Finally, we prove Conjecture 1.1 when $m \equiv 7 \pmod 8$.

Theorem 5.7. *For a given* $m \in \mathcal{M}$ *such that* $m \equiv 7 \pmod 8$, *there exists* $\varepsilon_m > 0$ *such that for all sufficiently large* n, *if* f *is a polynomial of degree* m *over* $\mathbb{F}_{2^n}$, *then for at least* $\varepsilon_m 2^{2n}$ *values of* $(\alpha, \beta) \in \mathbb{F}_{2^n}^* \times \mathbb{F}_{2^n}$ *we have* $\sharp\{x \in \mathbb{F}_q \mid f(x + \alpha) + f(x) = \beta\} = \delta(f) = m - 1$.

Proof. We follow the strategy in the proofs above. The point is to give lower bounds for the number of choices of α and β. We have shown the existence of a polynomial P of degree 3 such that for any n and $f \in \mathbb{F}_{2^n}[x]$ there exist at least $2^{n-1}+P(m)$ elements α such that the extension $\Omega/\mathbb{F}_{2^n}(t)$ is regular (see the proof of Theorem 5.3). Thus for any $\gamma_m < 1/2$, for n sufficiently large, there exists $\gamma_m 2^n$ suitable choices of α. For such a choice of α, the Chebotarev theorem used in the proof of Proposition 5.1 guarantees the existence of $\frac{1}{d!2^d}2^n + Q(2^{n/4})$ elements β such that $D_\alpha f(x) + \beta$ has $\delta(f)$ solutions where Q is a polynomial of degree 2. Thus for any $\gamma'_m < 1/d!2^d$, for n sufficiently large, there exist $2^n\gamma'_m$ suitable choices of β. Hence we obtain the result for any $\varepsilon_m < 1/d!2^{d+1}$. ∎

Observe that the proof of Theorem 5.7 provides explicit values of ε_m, namely any ε_m between 0 and $1/d!2^{d+1}$ with $d = (m-1)/2$. Observe also that, for $m \equiv 3 \pmod 8$, the same strategy leads to a proof of an analogue of this theorem for polynomials f such that $a_1^2 + a_0a_2 \neq 0$ or a proof of another analogue for even n.

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