

# MAXIMAL CURVES WITH RESPECT TO QUADRATIC EXTENSIONS OVER FINITE FIELDS

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ABSTRACT. We propose a detailed study of a canonical bound which relates the numbers of rational points of a curve over a finite field with that over its quadratic extension. Alternative proofs which make a connection with the variance enable to obtain optimal refinements.

We focus on the curves reaching the bound, which we call Hallouin-Perret-maximal curves. We provide different characterizations and stress natural links with the curves which attain the Ihara bound. As consequences, we establish the list of such curves with low genus and we outline a maximality result which involves the Suzuki curves.

At last we determine which polynomials correspond to the Jacobian of a Hallouin-Perret-maximal curve of genus 2.

## 1. INTRODUCTION

Throughout the whole paper we consider an absolutely irreducible smooth projective algebraic curve  $X$  (just called curve from now on) of genus  $g$  and defined over the finite field  $\mathbb{F}_q$ . In the context of estimating the number  $\#X(\mathbb{F}_{q^n})$  of rational points of  $X$  over  $\mathbb{F}_{q^n}$ , we propose a detailed study of an inequality implicit in the work of Ihara in [16] and highlighted by Hallouin and Perret in [7]. Indeed, this inequality canonically appears in [7] among a series of meaningful bounds obtained as consequences of non-negativity of a series of Gram determinants.

Let us sketch the method developed in [7]. The Neron-Severi group of the surface  $X \times X$  can be quotiented by numerical equivalence and thus tensorised to obtain a real vector space  $\text{Num}(X \times X)_{\mathbb{R}}$  equipped by the intersection pairing. As a consequence of the Hodge-index theorem the intersection pairing is negative definite on the vector space orthogonal to the plane generated by the classes of the horizontal and vertical fibres. We denote by  $\wp$  the orthogonal projection onto this subspace. For an integer  $k$  we thus consider  $\gamma_k$  the push-down by  $\wp_*$  of the class of the graph of the  $k$ -th iterated Frobenius that we normalize by  $\sqrt{q}^k$ . The non-negativity of the determinant of the Gram matrix  $\text{Gram}(\gamma_0, \gamma_1)$  expresses exactly the Weil inequality, as explained in Subsection 2.2 in [7]:

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**Theorem.** (Weil, [36]) *Let  $X$  be a curve defined over  $\mathbb{F}_q$  of genus  $g$ . We have*

$$(1) \quad | \#X(\mathbb{F}_q) - (q + 1) | \leq 2g\sqrt{q}.$$

Next, the non-negativity of the Gram matrix  $\text{Gram}(\gamma_0, \gamma_1, \gamma_2)$  together with the arithmetic constraint  $\#X(\mathbb{F}_{q^2}) \geq \#X(\mathbb{F}_q)$  yields to the Ihara bound, as explained in Subsection 3.3 in [7]:

**Theorem.** (Ihara, [16]) *Let  $X$  be a curve defined over  $\mathbb{F}_q$  of genus  $g \geq 1$ . We have*

$$(2) \quad \#X(\mathbb{F}_q) - (q + 1) \leq \frac{\sqrt{(8q + 1)g^2 + (4q^2 - 4q)g} - g}{2}.$$

Meanwhile, Hallouin and Perret have also noticed that the non-negativity of the determinant of the Gram matrix  $\text{Gram}(\gamma_0, \gamma_1, \gamma_2)$  leads to the following inequality:

**Theorem.** (Hallouin and Perret, Proposition 12 in [7]) *Let  $X$  be a curve defined over  $\mathbb{F}_q$  of genus  $g \geq 1$ . We have*

$$(3) \quad \#X(\mathbb{F}_{q^2}) - (q^2 + 1) \leq 2gq - \frac{1}{g}(\#X(\mathbb{F}_q) - (q + 1))^2.$$

This is the inequality we aim to study in our work. A first interpretation provided in [7] is that the inequality (3) improves the Weil bound for  $\#X(\mathbb{F}_{q^2})$  all the more as  $\#X(\mathbb{F}_q)$  is far from  $\#\mathbb{P}^1(\mathbb{F}_q) = q + 1$ . Our first contribution in this paper is to provide alternative and elementary proofs of the inequality (3). In particular we make a link with the statistical variance of the real parts of the reciprocal roots of the  $L$ -polynomial of  $X$  (that is of the numerator polynomial of the zeta function of  $X$ ). This way, the inequality (3) appears as a consequence of the positivity of the variance. As another consequence, we can complete inequality (3) with a lower bound.

**Theorem.** (Theorem 3.2, Corollary 3.3 and Proposition 3.4) *Let  $X$  be a curve of genus  $g \geq 2$  defined over  $\mathbb{F}_q$ . We denote by  $\alpha_1, \dots, \alpha_g$  the real parts of its Frobenius eigenvalues, that we consider as a statistical sample whose mean is given by  $\bar{\alpha} := \frac{1}{g} \sum_{j=1}^g \alpha_j$  and whose variance equals*

$$V(\alpha) := \frac{1}{g} \sum_{j=1}^g (\alpha_j - \bar{\alpha})^2.$$

(i) *The difference between the right hand side and the left hand side of the inequality (3) is a multiple of the variance of the  $\alpha_j$ 's:*

$$2gq - \frac{1}{g}(\#X(\mathbb{F}_q) - (q + 1))^2 - \#X(\mathbb{F}_{q^2}) + (q^2 + 1) = 4gV(\alpha).$$

(ii) *We deduce*

$$\left| \#X(\mathbb{F}_{q^2}) - (q^2 + 1) + \frac{1}{g}(\#X(\mathbb{F}_q) - (q + 1))^2 \right| \leq 2gq.$$

(iii) If  $g$  is odd then we have

$$-2q \left( g - \frac{2}{g} \right) \leq \#X(\mathbb{F}_{q^2}) - (q^2 + 1) + \frac{1}{g} (\#X(\mathbb{F}_q) - (q + 1))^2.$$

Let us now stress a link with a series of inequalities involving the coefficients  $a_1$  and  $a_2$  of a  $q$ -Weil polynomial  $T^{2g} + a_1T^{2g-1} + a_2T^{2g-2} + \dots + a_2q^{g-2}T^2 + a_1q^{g-1}T + q^g$ . We recall that to be a  $q$ -Weil polynomial is a well-known necessary condition to be the characteristic polynomial of an abelian variety defined over  $\mathbb{F}_q$  as explained in Section 3.2.

When  $g = 2$  (respectively  $g = 3, g = 4, g = 5$ ) Rück (respectively Haloui, Haloui and Singh, Sohn) have stated an upper bound on the coefficient  $a_2$  in [26] (respectively [8], [9], [31]). One can also consult the work of Marseglia (see [22]) for a survey and recent progresses on the subject. Actually, computations naturally related to the inequality (3) lead to the following generalization of these four upper bounds for any value of  $g \geq 2$  as well as to a lower bound on  $a_2$ .

**Theorem.** (Theorem 3.9 and Proposition 3.7) We consider  $g \geq 2$ .

(i) If  $T^{2g} + a_1T^{2g-1} + a_2T^{2g-2} + \dots + a_2q^{g-2}T^2 + a_1q^{g-1}T + q^g$  is a  $q$ -Weil polynomial then

$$(4) \quad a_2 \leq \frac{a_1^2(g-1)}{2g} + gq \text{ and}$$

$$(5) \quad a_2 \geq \begin{cases} \frac{g-1}{2g}a_1^2 - gq & \text{when } g \geq 2 \text{ is even,} \\ \frac{g-1}{2g}a_1^2 + \frac{(2-g^2)}{g}q & \text{when } g \geq 3 \text{ is odd.} \end{cases}$$

(ii) Let  $X$  be a curve defined over  $\mathbb{F}_q$  of genus  $g \geq 2$  and  $L_X(T) = 1 + a_1T + a_2T^2 + \dots + a_2q^{g-2}T^{2g-2} + a_1q^{g-1}T^{2g-1} + q^gT^{2g}$  be its  $L$ -polynomial. Then the inequalities (4) and (5) obviously hold true.

Moreover the bound (4) is reached if and only (3) is an equality.

The case of equality in (3) will thus receive special attention in our work. Let us point out another meaningful motivation to study this case of equality. One could relate with the words of Serre on page 96 in [29]: “it would be natural for curves to ask for many points, not only over  $\mathbb{F}_q$ , but also over several given extensions of  $\mathbb{F}_q$ ”. Indeed, if we fix the values of  $g$ ,  $q$  and  $N_1$ , among the curves of genus  $g > 0$  with  $N_1$  points over  $\mathbb{F}_q$ , those which reach equality in (3) have the largest number of points over  $\mathbb{F}_{q^2}$ .

**Definition 1.1.** A curve  $X$  defined over  $\mathbb{F}_q$  of genus  $g > 0$  is said to be a Hallouin-Perret-maximal curve (or HP-maximal curve for short) if

$$(6) \quad \#X(\mathbb{F}_{q^2}) - (q^2 + 1) = 2gq - \frac{1}{g} (\#X(\mathbb{F}_q) - (q + 1))^2.$$

We will thus provide different characterizations of HP-maximal curves. These characterizations will prove useful to identify families of HP-maximal curves such as the Weil-maximal or Weil-minimal curves (see Example 2.3), to establish the stability of the notion of HP-maximality by coverings (see

Proposition 2.11) or to obtain an upper bound of the genus of a HP-maximal curve depending only on  $q$  (see Proposition 2.10).

**Proposition.** *(Proposition 2.1 and Proposition 2.9) Let  $X$  be a curve of genus  $g \geq 1$  defined over  $\mathbb{F}_q$ . We denote by  $\alpha_1, \dots, \alpha_g$  the real parts of the reciprocal roots of the  $L$ -polynomial of  $X$ . The following assertions are equivalent*

- (i)  $X$  is a Hallouin-Perret-maximal curve,
- (ii) all the  $\alpha_j$ 's are equal, that is the zeta function of  $X$  is of the form

$$Z_X(T) = \frac{(1 - 2\alpha T + qT^2)^g}{(1 - T)(1 - qT)}$$

- (in this case,  $2\alpha$  is an integer and we have  $2\alpha = \frac{q+1-\#X(\mathbb{F}_q)}{g}$ ),
- (iii) the number of rational points of the Jacobian  $\text{Jac}(X)$  of  $X$  attains the upper bound (7) given in [1]

$$\#\text{Jac}(X)(\mathbb{F}_q) = (q + 1 + \tau_{\text{Jac}(X)}/g)^g$$

where  $\tau_{\text{Jac}(X)}$  stands for the opposite of the trace of the Jacobian of  $X$ , that is  $\tau_{\text{Jac}(X)} := -2 \sum_{j=1}^g \alpha_j$ .

The inequality (3) is strongly linked to the Ihara bound. For instance, one can guess an elementary proof of (3) in the original proof of Ihara of inequality (2) based upon Cauchy-Schwarz inequality. So it is quite natural that we can relate the curves reaching the two bounds. To this end, we recall that we have obviously  $\#X(\mathbb{F}_{q^2}) \geq \#X(\mathbb{F}_q)$ . We will follow [23] and we will call Diophantine-stable curve (with respect to the extension  $\mathbb{F}_{q^2}/\mathbb{F}_q$ ) a curve  $X$  such that  $\#X(\mathbb{F}_{q^2}) = \#X(\mathbb{F}_q)$ .

**Proposition.** *(Proposition 4.1) Let  $X$  be a curve of genus  $g \geq 1$ . The curve  $X$  is Ihara-maximal (i.e. the Ihara inequality (2) becomes an equality) if and only if  $X$  is both a Hallouin-Perret-maximal curve and a Diophantine-stable curve with respect to the extension  $\mathbb{F}_{q^2}/\mathbb{F}_q$ .*

We will obtain in Section 4 the list of Ihara-maximal curves of low genera  $g$  and also for low values of  $q$ . Precisely, we give in Table 1 the complete list up to isomorphism of Ihara-maximal curves when  $g \leq 18$ , and in Remark 4.6 when  $q \leq 13$ , except (in both cases) when  $g = q = 7$ . For these values, we know that there exists at least one Ihara-maximal curve, but we do not know if there is unicity.

Whereas it is essentially a reformulation of a result of Fuhrmann and Torres (Theorem 2 in [6]) we think it is worthwhile to highlight the following theorem which provides an analogue of a theorem of Rück and Stichtenoth where the Suzuki curves play the role of Hermitian curves as explained in Subsection 4.2.

**Theorem.** *(Theorem 4.4, reformulation of Theorem 2 in [6]) We consider  $t \geq 1$  and  $q = 2^{2t+1}$ . Let  $X$  be a curve defined over  $\mathbb{F}_q$ . Suppose that  $X$*

has genus  $g = \frac{\sqrt{q}(q-1)}{\sqrt{2}}$ . Then  $X$  is Ihara-maximal if and only if  $X$  is  $\mathbb{F}_q$ -isomorphic to the Suzuki curve  $S$  which is the non-singular model of the curve of equation  $y^q - y = x^{q_0}(x^q - x)$  where  $q_0 = 2^t$ .

At last, it is noteworthy that the Jacobian of a HP-maximal curve is a power of a simple abelian variety (see Proposition 2.5). In the same direction, we will determine among the polynomials  $(T^2 + aT + q)^2 \in \mathbb{Z}[T]$  which ones correspond to the Jacobian of a genus-2 HP-maximal curve  $X$ , and we will characterize when the Jacobian of  $X$  is simple or splits into the power of an ordinary or supersingular elliptic curve (Theorem 5.1). This classification work will be the aim of Section 5. As a consequence we will deduce the following existence result.

**Proposition.** *(Proposition 5.2) Over any finite field there exists a non-elliptic Hallouin-Perret-maximal curve.*

## 2. DIFFERENT CHARACTERIZATIONS OF HALLOUIN-PERRET-MAXIMALITY

Let  $X$  be a curve defined over  $\mathbb{F}_q$  of genus  $g \geq 1$  and  $N_n = \#X(\mathbb{F}_{q^n})$  be its number of rational points over  $\mathbb{F}_{q^n}$ . It is well-known that the zeta function of  $X$

$$Z_{X, \mathbb{F}_q}(T) := \exp \left( \sum_{n=1}^{\infty} \frac{N_n T^n}{n} \right)$$

is a rational function

$$Z_{X, \mathbb{F}_q}(T) = \frac{L_X(T)}{(1-T)(1-qT)}$$

where  $L_X(T)$ , called the  $L$ -polynomial of  $X$ , is a polynomial in  $\mathbb{Z}[T]$  of degree  $2g$ . It has the form

$$L_X(T) = \prod_{j=1}^g (1 - \omega_j T)(1 - \bar{\omega}_j T)$$

where the inverse roots  $\omega_j$  of  $L_X(T)$  are algebraic integers such that  $|\omega_j| = \sqrt{q}$  (the so-called Riemann Hypothesis for curves over finite fields, proved by Weil, see [36]). We also denote by  $\alpha_j$  the real part of  $\omega_j$ , so that the  $L$ -polynomial of  $X$  can be written

$$L_X(T) = \prod_{j=1}^g (1 - 2\alpha_j T + qT^2)$$

where  $|\alpha_j| \leq \sqrt{q}$ .

Let us recall now the link with the characteristic polynomial of the Jacobian of  $X$ . Let  $A$  be an abelian variety of dimension  $g$  defined over  $\mathbb{F}_q$ . The absolute Galois group  $\text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$  is topologically generated by the automorphism  $x \mapsto x^q$  and corresponds to the so-called Frobenius endomorphism of  $A$ . It induces an endomorphism on the Tate module  $T_\ell(A)$  of  $A$  (where  $\ell$

is any prime number distinct from the characteristic of  $\mathbb{F}_q$ ). The characteristic polynomial  $f_A$  of  $A$  is defined as the characteristic polynomial of the Frobenius seen as an endomorphism on the  $\mathbb{Q}_\ell$ -vector space  $T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ . It is a classic result that  $f_A$  is a monic polynomial of degree  $2g$  with integer coefficients whose roots have all modulus  $\sqrt{q}$  (the Riemann Hypothesis for abelian varieties also proved by Weil, see [36]).

It is also a standard result that the  $L$ -polynomial of  $X$  is the reciprocal polynomial of the characteristic polynomial of the Jacobian  $\text{Jac}(X)$  of  $X$ :

$$f_{\text{Jac}(X)}(T) = T^{2g} L_X(1/T) = \prod_{j=1}^{2g} (T - \omega_j)(T - \bar{\omega}_j) = \prod_{j=1}^g (T^2 - 2\alpha_j T + q).$$

So we will sometimes refer to the  $\omega_j$ 's as the Frobenius eigenvalues of  $X$ .

**2.1. Characterization by the zeta function.** An important point in our work is the following proposition which characterizes the HP-maximal curves in terms of the real parts of their Frobenius eigenvalues.

**Proposition 2.1.** *Let  $X$  be a curve of genus  $g$  defined over  $\mathbb{F}_q$ . We denote by  $\alpha_1, \dots, \alpha_g$  the real parts of its Frobenius eigenvalues. The curve  $X$  is a Hallowin-Perret-maximal curve if and only if  $\alpha_1 = \dots = \alpha_g$ , that is if and only if its zeta function is of the form*

$$Z_X(T) = \frac{(1 - 2\alpha T + qT^2)^g}{(1 - T)(1 - qT)}$$

where  $\alpha$  is the common value of all the  $\alpha_j$ . In this case,  $2\alpha$  is an integer and we have  $2\alpha = \frac{q+1-\#X(\mathbb{F}_q)}{g}$ .

*Proof.* It is well-known that we have:

$$\#X(\mathbb{F}_q) = q + 1 - \sum_{j=1}^g (\omega_j + \bar{\omega}_j) \quad \text{and} \quad \#X(\mathbb{F}_{q^2}) = q^2 + 1 - \sum_{j=1}^g (\omega_j^2 + \bar{\omega}_j^2).$$

Saying that  $X$  is a HP-maximal curve is thus equivalent to saying that

$$-\sum_{j=1}^g (\omega_j^2 + \bar{\omega}_j^2) = 2gq - \frac{1}{g} \left( -\sum_{j=1}^g (\omega_j + \bar{\omega}_j) \right)^2.$$

But  $\omega_j + \bar{\omega}_j = 2\alpha_j$ , so  $\omega_j^2 + \bar{\omega}_j^2 = 4\alpha_j^2 - 2q$  and thus the equality comes down to  $g \sum_{j=1}^g \alpha_j^2 = \left( \sum_{j=1}^g \alpha_j \right)^2$ . But Cauchy-Schwarz equality holds if and only if all the  $\alpha_j$ 's are equal, that is if and only if  $L_X$  has the claimed form.

If  $\alpha$  denotes the common value of the  $\alpha_j$ 's, by the relation above we get  $\alpha = \frac{q+1-\#X(\mathbb{F}_q)}{2g}$ . Therefore  $2\alpha$  is both a rational number and an algebraic integer, because it is the sum of the algebraic integers  $\omega_1$  and  $\bar{\omega}_1$ . We deduce that  $2\alpha$  is an integer.  $\square$

**Remark 2.2.** It is straightforward that any elliptic curve  $E$  over  $\mathbb{F}_q$  is a HP-maximal curve since  $N_1 := \#E(\mathbb{F}_q) = q + 1 - (\omega + \bar{\omega})$  and  $N_2 := \#E(\mathbb{F}_{q^2}) = q^2 + 1 - (\omega^2 + \bar{\omega}^2)$  and thus  $(N_1 - (q + 1))^2 = (\omega + \bar{\omega})^2 = q^2 + 1 + 2q - N_2$ .

Meanwhile the characterization of Proposition 2.1 gives again immediately that any elliptic curve is a HP-maximal curve.

**Remark 2.3.** As a consequence of the previous proposition, a Weil-maximal curve i.e. a curve which reaches the Weil upper bound (respectively a Weil-minimal curve i.e. a curve which reaches the Weil lower bound) over  $\mathbb{F}_q$  is a HP-maximal curve. Indeed, if  $X$  is a Weil-maximal curve over  $\mathbb{F}_q$  of genus  $g$  then  $\sharp X(\mathbb{F}_q) = q + 1 + 2g\sqrt{q}$  and  $q$  must be a square. If we still denote by  $\omega_1, \bar{\omega}_1, \dots, \omega_g, \bar{\omega}_g$  its Frobenius eigenvalues, then we can write  $\sharp X(\mathbb{F}_q) = q + 1 - \sum_{j=1}^g (\omega_j + \bar{\omega}_j)$  and we thus obtain that  $\sum_{j=1}^g (\omega_j + \bar{\omega}_j) = -2g\sqrt{q}$ . But the Riemann Hypothesis says that  $|\omega_j| = \sqrt{q}$  for any  $j = 1, \dots, g$  which implies that all the  $\omega_j$  are equal to  $-\sqrt{q}$  and then  $X$  is a HP-maximal curve. In the same way, if  $X$  is Weil-minimal then all the  $\omega_j$  are equal to  $\sqrt{q}$  and then  $X$  is also a HP-maximal curve.

**Remark 2.4.** A HP-maximal curve of genus  $g \geq 2$  is not necessarily Weil-maximal nor Weil-minimal: the curve  $X$  of genus 2 defined over  $\mathbb{F}_3$  by the equation  $y^2 = (-1 - x - x^3)(1 - x + x^3)$  verifies  $N_1 := \sharp X(\mathbb{F}_3) = 2$  and  $N_2 := \sharp X(\mathbb{F}_9) = 20$  so  $X$  is a HP-maximal curve but is neither Weil-maximal nor Weil-minimal.

## 2.2. Geometric condition.

**Proposition 2.5.** *Let  $X$  be a curve of genus  $g$  defined over  $\mathbb{F}_q$ . If  $X$  is a Halluin-Perret-maximal curve then the Jacobian of  $X$  is  $\mathbb{F}_q$ -isogenous to a power of a  $\mathbb{F}_q$ -simple abelian variety.*

*Proof.* Let  $f_{\text{Jac}(X)}$  be the characteristic polynomial of the Jacobian of  $X$

$$f_{\text{Jac}(X)} = \prod_{j=1}^g (T^2 - 2\alpha_j T + q)$$

where the  $\alpha_j$ 's are the real parts of its Frobenius eigenvalues (and so  $|\alpha_j| \leq \sqrt{q}$  by the Riemann Hypothesis). By Proposition 2.1 we know that if  $X$  is a HP-maximal curve then the  $\alpha_j$ 's are all equal. If we denote by  $\alpha$  this common value, it amounts to saying that  $f_{\text{Jac}(X)}$  has the form

$$f_{\text{Jac}(X)} = (T^2 - 2\alpha T + q)^g.$$

But it is well-known that any abelian variety  $A$  over  $\mathbb{F}_q$  can factor uniquely, up to  $\mathbb{F}_q$ -isogeny, into a product of powers of non- $\mathbb{F}_q$ -isogenous  $\mathbb{F}_q$ -simple abelian varieties. And Tate stated in Theorem 2 (e) in [33] that  $A$  is  $\mathbb{F}_q$ -isogenous to a power of a  $\mathbb{F}_q$ -simple abelian variety if and only if its characteristic polynomial is a power of a  $\mathbb{Q}$ -irreducible polynomial.

Since the reduced discriminant of  $T^2 - 2\alpha T + q$  is equal to  $\alpha^2 - q$ , which is non-positive,  $f_{\text{Jac}(X)}$  is a power of a  $\mathbb{Q}$ -irreducible polynomial. The result follows.  $\square$

**Remark 2.6.** The reciprocal of Proposition 2.5 is false as we find many counterexamples in the database [19]. For instance, one can consider the curve  $X$  of genus 2 defined over  $\mathbb{F}_5$  by the equation  $y^2 = x^5 + 3x$ . Its Jacobian is simple as its characteristic polynomial  $1 + 25x^4$  is irreducible

in  $\mathbb{Q}[x]$ . But it turns out that we have  $N_1 = 2$  and  $N_2 = 26$  so  $X$  is not a HP-maximal curve.

**Remark 2.7.** The Jacobian of a Hallouin-Perret-maximal curve can be a simple abelian variety. Indeed, the LMFDB database [19] provides<sup>1</sup> the equations of many hyperelliptic curves of genus 2 defined over  $\mathbb{F}_{49}$  such that  $N_1 = 36$  and  $N_2 = 2500$ , for example the curve of equation  $y^2 = (3a+3)x^6 + (6a+1)x^5 + (2a+2)x^4 + (5a+6)x^3 + (5a+3)x^2 + 3x + 3a + 4$  where  $a$  is a root of the Conway polynomial defining  $\mathbb{F}_{49}$ . If we consider such a curve  $X$ , one can easily check that the equality (6) is verified, so  $X$  is a HP-maximal curve. Moreover, by Proposition 2.1 we recover that the  $L$ -polynomial of  $X$  is  $L_X(T) = (1 - 7T + 49T^2)^2$ . So, if there existed an elliptic curve  $E$  defined over  $\mathbb{F}_{49}$  with  $L$ -polynomial  $L_E(T) = 1 - (\omega + \bar{\omega})T + qT^2 = 1 - 7T + 49T^2$  we would have  $\#E(\mathbb{F}_{49}) = 49 + 1 - (\omega + \bar{\omega}) = 43$ . But according to the LMFDB database once again, such an elliptic curve does not exist. So  $X$  is an example of a HP-maximal curve whose Jacobian is a simple abelian surface.

**Remark 2.8.** Let  $X$  be a HP-maximal curve of genus  $g$  defined over  $\mathbb{F}_q$  with  $q = p^a$  and let  $f_{\text{Jac}(X)} = (T^2 - 2\alpha T + q)^g$  be its characteristic polynomial where  $\alpha = \frac{q+1-\#X(\mathbb{F}_q)}{2g}$ . The Deuring-Waterhouse theorem (Theorem 4.1 in [35]) characterizes the isogeny classes of elliptic curves over a finite field in terms of their characteristic polynomial. It enables to state that the Jacobian of  $X$  is  $\mathbb{F}_q$ -isogenous to a power of an elliptic curve if and only if one of the following conditions is satisfied:

- (i)  $(2\alpha, p) = 1$  ;
- (ii)  $a$  is even and  $2\alpha = \pm 2\sqrt{q}$  ;
- (iii)  $a$  is even and  $p \not\equiv 1 \pmod{3}$  and  $2\alpha = \pm\sqrt{q}$  ;
- (iv)  $a$  is odd and  $p = 2$  or  $3$  and  $2\alpha = \pm p^{\frac{a+1}{2}}$  ;
- (v) either (v.i)  $a$  is odd or (v.ii)  $a$  is even and  $p \not\equiv 1 \pmod{4}$  and  $\alpha = 0$ .

**2.3. Characterization by the number of points of the Jacobian.** Let  $A$  be an abelian variety of dimension  $g$  defined over  $\mathbb{F}_q$  with characteristic polynomial  $f_A$  whose roots are denoted by  $\omega_1, \bar{\omega}_1, \dots, \omega_g, \bar{\omega}_g$ .

If we set  $\tau_A := -\sum_{j=1}^g (\omega_j + \bar{\omega}_j) = -2\sum_{j=1}^g \alpha_j$  for the opposite of the trace of the Frobenius on  $A$  (where the  $\alpha_j$ 's are the real parts of the complex numbers  $\omega_j$ 's), then Aubry, Haloui and Lachaud proved in Theorem 2.1 of [1] the following bound on the number of rational points on  $A$ :

$$(7) \quad \#A(\mathbb{F}_q) \leq (q + 1 + \tau_A/g)^g$$

with equality if and only if the  $\alpha_j$ 's are equal. Theorem 2.1 of [1] together with Proposition 2.1 lead to the following statement.

**Proposition 2.9.** *Let  $X$  be a curve of genus  $g$  defined over  $\mathbb{F}_q$ . Then  $X$  is a Hallouin-Perret-maximal curve if and only if the number of rational points of the Jacobian  $\text{Jac}(X)$  of  $X$  attains the upper bound (7), namely*

$$(8) \quad \#\text{Jac}(X)(\mathbb{F}_q) = (q + 1 + \tau_{\text{Jac}(X)}/g)^g.$$

<sup>1</sup>[https://www.lmfdb.org/Variety/Abelian/Fq/2/49/ao\\_fr](https://www.lmfdb.org/Variety/Abelian/Fq/2/49/ao_fr)

*Proof.* Let  $X$  be a curve of genus  $g$  defined over  $\mathbb{F}_q$  and let  $\alpha_1, \dots, \alpha_g$  be the real parts of the inverse roots  $\omega_j$  of its  $L$ -polynomial  $L_X(T)$ . Now consider the Jacobian  $\text{Jac}(X)$  of  $X$  whose characteristic polynomial  $f_{\text{Jac}(X)}$  is the reciprocal polynomial of the  $L$ -polynomial  $L_X(T)$  of  $X$ . By Proposition 2.1 we have that  $X$  is a HP-maximal curve if and only if the  $\alpha_j$  are all equal. Now Theorem 2.1 of [1] ensures that it is equivalent to saying that  $\text{Jac}(X)$  attains the upper bound (7).  $\square$

**2.4. Genus of a Hallouin-Perret-maximal curve.** As a consequence of Ihara's results in [16] a curve defined over  $\mathbb{F}_q$  cannot be Weil-maximal if its genus is large with respect to  $q$  (see also Theorem 5.1.1 in [29]). Precisely, if  $X$  is a Weil-maximal curve defined over  $\mathbb{F}_q$  of genus  $g$  then  $g \leq \frac{q-\sqrt{q}}{2}$ . In Proposition 2.10 we will obtain that the genus of a HP-maximal curve defined over  $\mathbb{F}_q$  is also bounded by a function of  $q$ .

The Frobenius angles of a curve are defined to be the arguments of the complex inverse roots  $\omega_j$  of the  $L$ -polynomial of the curve. Elkies, Howe and Ritzenthaler have given in [5] an explicit upper bound on the genus of a curve depending on the set of the Frobenius angles. More precisely they have proved in Theorem 1.1 of [5] that if a non-empty set  $S \subset [0, \pi]$  of cardinal  $s$  contains all non-negative Frobenius angles of  $X$ , then the genus  $g$  of  $X$  satisfies  $g \leq 23s^2q^{2s} \log q$  and  $g < (\sqrt{q} + 1)^{2r} \left(\frac{1+q^{-r}}{2}\right)$  where  $r = 1/2$  if  $S = \{0\}$  and  $r = \sharp(S \cap \{\pi\}) + 2 \sum_{\theta \in S \setminus \{0, \pi\}} \lceil \frac{\pi}{2\theta} \rceil$  otherwise.

As a consequence we obtain the following bounds.

**Proposition 2.10.** *Let  $X$  be a Hallouin-Perret-maximal curve defined over  $\mathbb{F}_q$  of genus  $g$ . Then*

$$g \leq 23q^2 \log q.$$

Moreover, when we write  $L_X(T) = (1 + aT + qT^2)^g$  then

- (i) If  $0 < a < 2\sqrt{q}$  then  $g < (\sqrt{q} + 1)^4 \left(\frac{q^2+1}{2q^2}\right)$ .
- (ii) If  $a = 2\sqrt{q}$  then  $g \leq \frac{q-\sqrt{q}}{2}$ .
- (iii) If  $a = -2\sqrt{q}$  then  $g \leq \frac{(\sqrt{q}+1)^2}{2\sqrt{q}}$ .

*Proof.* By Proposition 2.1 we know that if  $X$  is a HP-maximal curve then  $f_{\text{Jac}(X)}$  only admits one root  $\omega$  and its conjugate. With our usual conventions we have  $a = -2\alpha = -(\omega + \bar{\omega})$ . The theorem of Elkies, Howe and Ritzenthaler quoted above applies easily as  $S$  is reduced to one element. So  $s = 1$  and we obtain the general bound  $g \leq 23q^2 \log q$ .

(i) When  $0 < a < 2\sqrt{q}$  we know that the real part  $\alpha$  of  $\omega$  lies in the open interval  $(-\sqrt{q}, 0)$ . So the Frobenius angle  $\theta$  which is defined as the unique non-negative argument of the inverse roots of  $L_X$  satisfies  $\pi/2 < \theta < \pi$ . We deduce  $\frac{1}{2} < \frac{\pi}{2\theta} < 1$ , so by definition we know that  $r = 2 \lceil \frac{\pi}{2\theta} \rceil = 2$  and Theorem 1.1 of [5] ensures that  $g < (\sqrt{q} + 1)^4 \left(\frac{q^2+1}{2q^2}\right)$ .

(ii) If  $a = 2\sqrt{q}$  then  $\omega = -\sqrt{q}$  and we recognize the case where the curve is Weil-maximal. The bound on  $g$  is thus a consequence of the results of Ihara in [16] (see also Theorem 5.1.1 in [29]).

(iii) If  $a = -2\sqrt{q}$  then  $\omega = \sqrt{q}$  and  $\theta = 0$ . (This time the curve is Weil-minimal.) As  $S$  is reduced to the set  $\{0\}$ , by definition we have that  $r = 1/2$  and Theorem 1.1 of [5] gives  $g \leq \frac{(\sqrt{q}+1)^2}{2\sqrt{q}}$ .  $\square$

**2.5. In coverings.** Let  $Y \rightarrow X$  be a non-constant morphism of curves over  $\mathbb{F}_q$ . We know that if  $Y$  attains the Weil upper bound (or the Weil lower bound), the same holds for  $X$  (see Theorem 5.2.1. of [29]). More generally, it is proved in Corollary 13 of [3] that if  $Y \rightarrow X$  is a finite flat morphism between two varieties over a finite field, then the reciprocal polynomial of the characteristic polynomial of the Frobenius endomorphism on the  $i$ -th étale cohomology group of  $X$  divides that of  $Y$ . In particular, if  $Y \rightarrow X$  is a non-constant morphism of curves, the  $L$ -polynomial  $L_X(T)$  of  $X$  divides the  $L$ -polynomial  $L_Y(T)$  in  $\mathbb{Z}[T]$ . We can deduce the following statement.

**Proposition 2.11.** *Let  $Y \rightarrow X$  be a non-constant morphism of curves over  $\mathbb{F}_q$ . If  $Y$  is a Hallouin-Perret-maximal curve then  $X$  is also a Hallouin-Perret-maximal curve.*

*Proof.* If  $Y$  is a HP-maximal curve, then by Proposition 2.1 the real parts of the eigenvalues of the Frobenius on  $Y$  are all equal. Since  $L_X(T)$  divides  $L_Y(T)$  the same holds for  $X$ .  $\square$

### 3. HP-DEFECT OF A CURVE

In this section we introduce the HP-defect of a curve over a finite field as a measure of how far the curve is from being Hallouin-Perret-maximal, that is of how far the inequality (3) is from the case of equality.

**3.1. Definition and first properties.** To define the HP-defect we naturally consider the difference between the right hand side and the left hand side of the inequality (3) and we normalize in order to work with integers.

**Definition 3.1.** Let  $X$  be a curve of genus  $g \geq 1$  defined over  $\mathbb{F}_q$ . The HP-defect of  $X$ , denoted by  $\delta_{HP}(X)$ , is defined by

$$\delta_{HP}(X) = 2qq^2 - (\sharp X(\mathbb{F}_q) - (q+1))^2 - g(\sharp X(\mathbb{F}_{q^2}) - (q^2+1)).$$

The following theorem aims to give different expressions of  $\delta_{HP}$  in terms of the real parts  $\alpha_j$ 's of the eigenvalues  $\omega_j$ 's of the Frobenius endomorphism on  $X$ . It will prove useful to give alternative and elementary proofs of (3) and to obtain other bounds on  $\delta_{HP}$ .

As we will provide an interpretation of  $\delta_{HP}$  in terms of statistical tools, we briefly clarify the context. When we consider a statistical sample  $z = (z_1, \dots, z_g) \in \mathbb{R}^g$ , one naturally associates its mean  $\bar{z}$  defined by  $\bar{z} := (z_1 + \dots + z_g)/g$ . The variance is another important numerical value associated to a sample which is defined as a measure of dispersion around the mean by the equality  $V(z) := \sum_{j=1}^g (z_j - \bar{z})^2/g$ .

**Theorem 3.2.** *Let  $X$  be a curve of genus  $g \geq 1$  defined over  $\mathbb{F}_q$  and let  $\alpha_1, \dots, \alpha_g$  be the real parts of its Frobenius eigenvalues. Then we have:*

(i)

$$\delta_{HP}(X) = 4 \left( g \sum_{j=1}^g \alpha_j^2 - \left( \sum_{j=1}^g \alpha_j \right)^2 \right).$$

(ii) If we set  $\sigma_1 = \sum_{j=1}^g \alpha_j$  and  $\sigma_2 = \sum_{1 \leq j < k \leq g} \alpha_j \alpha_k$  then we have

$$\delta_{HP}(X) = 4 \left( (g-1)\sigma_1^2 - 2g\sigma_2 \right) \text{ and}$$

(iii)

$$\delta_{HP}(X) = 4 \sum_{1 \leq j < k \leq g} (\alpha_j - \alpha_k)^2.$$

(iv) If we consider  $(\alpha_j)_{1 \leq j \leq g}$  as a statistical sample whose mean is given by  $\bar{\alpha} := \frac{1}{g} \sum_{j=1}^g \alpha_j$  and whose variance equals  $V(\alpha) := \frac{1}{g} \sum_{j=1}^g (\alpha_j - \bar{\alpha})^2$ , then we have

$$\delta_{HP}(X) = 4g^2 V(\alpha).$$

*Proof.* (i) If we still denote by  $\omega_j$ 's the Frobenius eigenvalues of  $X$ , we get from the definition of the HP-defect:

$$\delta_{HP}(X) = 2qg^2 - \left( - \sum_{j=1}^g (\omega_j + \bar{\omega}_j) \right)^2 - g \left( - \sum_{j=1}^g (\omega_j^2 + \bar{\omega}_j^2) \right)$$

which implies that

$$\delta_{HP}(X) = 2qg^2 - 4 \left( \sum_{j=1}^g \alpha_j \right)^2 + g \sum_{j=1}^g (4\alpha_j^2 - 2q)$$

and the result follows.

(ii) and (iii) These points are direct consequences of point (i).

(iv) If we factorize the equality (i) by  $4g^2$  we get  $4g^2 \left( \frac{1}{g} \sum_{j=1}^g \alpha_j^2 - \left( \frac{1}{g} \sum_{j=1}^g \alpha_j \right)^2 \right)$ ,

and we recognize  $\overline{\alpha^2} - \bar{\alpha}^2$ , that is a classical formulation of the variance of  $\alpha$ . Note that the equality (iii) is also a well-known expression for the variance of  $\alpha$  (see Section 2.4 in [24] for example).  $\square$

The previous theorem gives alternative and elementary proofs of the Hallouin-Perret bound (3) which simply reads  $\delta_{HP}(X) \geq 0$ . It also provides an interpretation of this bound in terms of non-negativity of the variance of the sample  $(\alpha_j)$ . From points (i) or (iv) one can also immediately deduce an upper bound on  $\delta_{HP}(X)$  and then a symmetric formulation which completes inequality (3).

**Corollary 3.3.** *Let  $X$  be a curve of genus  $g \geq 2$  defined over  $\mathbb{F}_q$ . We have*

$$\delta_{HP}(X) \in [0, 4qg^2]$$

that is

$$(9) \quad \left| \#X(\mathbb{F}_{q^2}) - (q^2 + 1) + \frac{1}{g} (\#X(\mathbb{F}_q) - (q + 1))^2 \right| \leq 2qg.$$

Let us exploit the variance formulation of  $\delta_{HP}(X)$ . It is a classical statistical problem to give bounds for the variance of a sample  $(\alpha_j)_{1 \leq j \leq g}$  of elements in a range  $[\alpha_{\min}; \alpha_{\max}]$ . In [17] (see Lemma 1) Kaiblinger and Spangl state that we always have

$$V(\alpha) \leq \frac{1}{4}(\alpha_{\max} - \alpha_{\min})^2,$$

and that, if  $g$  is odd the bound can be improved this way

$$V(\alpha) \leq \frac{1}{4} \left( 1 - \frac{1}{g^2} \right) (\alpha_{\max} - \alpha_{\min})^2.$$

Lemma 1 in [17] also indicates that when  $g$  is even this upper bound on  $V(\alpha)$  is reached if and only if half of the values  $\alpha_j$  equal  $-\sqrt{q}$  whereas the other half equal  $\sqrt{q}$ . When  $g$  is odd, the upper bound is reached if and only if  $(g-1)/2$  or  $(g+1)/2$  values equal  $-\sqrt{q}$  whereas the others equal  $\sqrt{q}$ . We deduce the following proposition.

**Proposition 3.4.** *Let  $X$  be a curve of genus  $g \geq 2$  defined over  $\mathbb{F}_q$  and let  $\alpha_1, \dots, \alpha_g$  be the real parts of its Frobenius eigenvalues.*

(i) *If  $g$  is even then*

$$(10) \quad -2qg \leq \#X(\mathbb{F}_{q^2}) - (q^2 + 1) + \frac{1}{g} (\#X(\mathbb{F}_q) - (q + 1))^2$$

*and equality holds if and only if  $g/2$  values  $\alpha_j$  equal  $-\sqrt{q}$  whereas the others equal  $\sqrt{q}$ .*

(ii) *If  $g$  is odd then*

$$(11) \quad -2q \left( g - \frac{2}{g} \right) \leq \#X(\mathbb{F}_{q^2}) - (q^2 + 1) + \frac{1}{g} (\#X(\mathbb{F}_q) - (q + 1))^2$$

*and equality holds if and only if  $(g-1)/2$  or  $(g+1)/2$  values  $\alpha_j$  equal  $-\sqrt{q}$  whereas the others equal  $\sqrt{q}$ .*

Actually the inequality (10) is less accurate than the Weil bound applied to the curve considered over  $\mathbb{F}_{q^2}$ , but we indicate it for the sake of completeness. However, the bound (11) can sometimes yield valuable information. For example, consider a genus 3 curve defined over  $\mathbb{F}_4$  such that  $N_1 = 5$ . The Weil bound (1) yields  $N_2 \geq 5$  whereas (11) enables to deduce from the extra information about  $N_1$  that  $N_2 \geq 9$ . Actually the LMFDB database [19] provides the existence of such curves<sup>2</sup> with  $N_2 = 11$  (for example the curve of equation  $ax^4 + xy^3 + xz^3 + y^4 = 0$  where  $a \in \mathbb{F}_4$  satisfies  $a^2 + a + 1 = 0$ ). Moreover it indicates the existence of an abelian variety<sup>3</sup> whose associated virtual curve would have 9 rational points over  $\mathbb{F}_{16}$ . (For

<sup>2</sup>[https://www.lmfdb.org/Variety/Abelian/Fq/3/4/a\\_ad\\_ac](https://www.lmfdb.org/Variety/Abelian/Fq/3/4/a_ad_ac)

<sup>3</sup>[https://www.lmfdb.org/Variety/Abelian/Fq/3/4/a\\_ae\\_a](https://www.lmfdb.org/Variety/Abelian/Fq/3/4/a_ae_a)

reference, this abelian variety is defined over  $\mathbb{F}_4$ , has dimension 3, and its  $L$ -polynomial is given by  $(1 - 2T)^2(1 + 2T)^2(1 + 4T^2)$ .)

**Remark 3.5.** The necessary conditions on the  $\alpha_j$ 's established in point (i) of Proposition 3.4 help us to find, with the help of the LMFDB database [19], curves of genus 2 which reach the lower bound  $-2qg$ . Let us give some examples when  $g$  is even. For instance the (Diophantine-stable) hyperelliptic curve  $X$  of equation  $y^2 = x^5 + 4x$  defined over  $\mathbb{F}_5$  satisfies  $\#X(\mathbb{F}_q) = \#X(\mathbb{F}_{q^2}) = 6$  and we can check the equality  $\#X(\mathbb{F}_{q^2}) - (q^2 + 1) + \frac{1}{g}(\#X(\mathbb{F}_q) - (q + 1))^2 = -2qg$ . We have also found examples when  $q = 7$ ,  $q = 8$  or  $q = 11$ . For the case  $q = 9$  the necessary conditions lead us to  $N_1 = 10$ ,  $N_2 = 46$ , and to an isogeny class which does not contain any Jacobian according to the LMFDB database<sup>4</sup>. (For reference, this isogeny class is associated to the  $L$ -polynomial  $(1 - 3T)^2(1 + 3T)^2$ .) For the case  $q = 13$  one can find (at least) two hyperelliptic curves of genus 2 with equations  $y^2 = x^5 + 12x$  and  $y^2 = x^6 + 2x^3 + 8$  such that  $\alpha_1 = -\sqrt{13}$  and  $\alpha_2 = \sqrt{13}$ . We thus have  $N_1 = 14$ ,  $N_2 = 118$  and the lower bound is reached.

**Remark 3.6.** We are indebted to Christophe Ritzenthaler for pointing out the following example of a non-elliptic curve of odd genus which reaches the lower bound  $-2q(g - \frac{2}{g})$ . We take  $q = 49$  and we consider a generator  $a$  of the multiplicative group  $\mathbb{F}_q^*$ . Thus the curve of equation  $x^4 + a^{43}x^3y + a^{36}x^3z + a^{27}x^2y^2 + a^{10}x^2yz + a^{31}x^2z^2 + a^9xy^3 + a^{47}xy^2z + a^6xyz^2 + a^{19}xz^3 + a^9y^4 + 6y^3z + a^{46}y^2z^2 + a^{22}yz^3 + 6z^4 = 0$  has genus 3 and is such that  $N_1 = 2108$  whereas  $N_2 = 36$ , and so we can check  $\#X(\mathbb{F}_{q^2}) - (q^2 + 1) + \frac{1}{g}(\#X(\mathbb{F}_q) - (q + 1))^2 = -2q(g - \frac{2}{g})$ . The curve is obtained by twisting the curve of equation  $x^4 + y^4 + z^4 = 0$  so that it only changes one elliptic factor.

**3.2. Weil polynomials and Hallouin-Perret-maximal curves.** In this subsection we relate the topic of HP-maximal curves with the question of determining whether a polynomial  $f(T) = 1 + a_1T + a_2T^2 + \dots + a_2q^{g-2}T^{2g-2} + a_1q^{g-1}T^{2g-1} + q^gT^{2g} \in \mathbb{Z}[T]$  can be the reciprocal polynomial of the characteristic polynomial of a dimension  $g$  abelian variety over  $\mathbb{F}_q$ . We provide necessary conditions in terms of the two first coefficients  $a_1$  and  $a_2$  and in terms of  $\delta_{HP}$ .

Recall that a  $q$ -Weil number is an algebraic integer such that its image under every embedding has absolute value  $\sqrt{q}$  and that a monic polynomial of  $\mathbb{Z}[T]$  is called a  $q$ -Weil polynomial if all its roots are  $q$ -Weil numbers. The Honda-Tate theorem (see [32]) establishes a bijection between the simple abelian varieties over  $\mathbb{F}_q$  up to isogeny and the  $q$ -Weil numbers up to conjugation so that a well-known necessary condition for  $f$  is to be a  $q$ -Weil polynomial.

When  $g = 2$  Rück has proved (Theorem 1.1 in [26]) that a necessary condition for  $f$  to be a  $q$ -Weil polynomial is that  $a_2 \leq \frac{a_1^2}{4} + 2q$ . In the same direction, Haloui has proved (Theorem 1.1 in [8]) that for  $g = 3$  a necessary condition is that  $a_2 \leq \frac{a_1^2}{3} + 3q$ . Haloui and Singh have also proved (Theorem

<sup>4</sup>[https://www.lmfdb.org/Variety/Abelian/Fq/2/9/a\\_as](https://www.lmfdb.org/Variety/Abelian/Fq/2/9/a_as)

1.1 in [9]) that if  $g = 4$  a necessary condition is given by  $a_2 \leq \frac{3}{8}a_1^2 + 4q$ . Lastly, when  $g = 5$  Sohn has stated (Theorem 1.2 in [31]) that the condition becomes  $a_2 \leq \frac{2}{5}a_1^2 + 5q$ . We will propose a generalization of this series of necessary conditions for any  $g$ , as well as a lower bound for the coefficient  $a_2$ .

We start expliciting relations between the coefficients  $a_1, a_2$  and the defect  $\delta_{HP}(X)$  in the case where  $f(T)$  is the  $L$ -polynomial of a curve  $X$ .

**Proposition 3.7.** *Let  $X$  be a curve defined over  $\mathbb{F}_q$  of genus  $g \geq 2$  and with HP-defect  $\delta_{HP}$ . For any  $j \in \{1, \dots, g\}$  we also note  $\alpha_j$  for the real part of the inverse root  $\omega_j$  of the  $L$ -polynomial  $L_X(T) = 1 + a_1T + a_2T^2 + \dots + a_2q^{g-2}T^{2g-2} + a_1q^{g-1}T^{2g-1} + q^gT^{2g}$ . We have*

$$(12) \quad \sum_{j=1}^g \alpha_j^2 = \frac{a_1^2 + \delta_{HP}}{4g}.$$

We deduce

$$(13) \quad a_2 \leq \frac{a_1^2(g-1)}{2g} + gq$$

with equality if and only if  $X$  is a Hallouin-Perret-maximal curve. We also have the inequality

$$(14) \quad a_2 \geq \begin{cases} \frac{g-1}{2g}a_1^2 - gq & \text{when } g \geq 2 \text{ is even,} \\ \frac{g-1}{2g}a_1^2 + \frac{(2-g^2)}{g}q & \text{when } g \geq 3 \text{ is odd.} \end{cases}$$

*Proof.* For the first point we express equality ((i)) in Theorem 3.2 in terms of the coefficient  $a_1$  which satisfies  $a_1 = -2 \sum_{j=1}^g \alpha_j$ .

For the second point we start from equality ((ii)) in Theorem 3.2. We use  $a_1 = -2\sigma_1$  and  $a_2 = gq + 4\sigma_2$  to get

$$(15) \quad 2ga_2 + \delta_{HP} = (g-1)a_1^2 + 2g^2q$$

and thus  $a_2 = a_1^2(g-1)/2g + gq - \delta_{HP}/2g$ . But  $\delta_{HP}$  is non-negative and equals zero if and only if the curve is HP-maximal.

For the third point we just exploit the inequalities  $\delta_{HP} \leq 4g^2q$  when  $g$  is even and  $\delta_{HP} \leq 4(g^2-1)q$  when  $g$  is odd together with (15).  $\square$

**Remark 3.8.** Let us point out a geometric interpretation in the euclidean space  $\mathbb{R}^g$  if we associate to a curve  $X$  the point  $P_X$  of coordinates  $(\alpha_j)_{1 \leq j \leq g}$ . Thanks to the Riemann Hypothesis we know that this point belongs to the closed ball  $\overline{B}_\infty(0, \sqrt{q})$ . Now we fix the value of  $a_1$ . A way to translate Proposition (3.7) is to say that the point  $P_X$  belongs to the affine plane  $\mathcal{P}$  of equation  $\sum_{j=1}^g x_j = -\frac{a_1}{2}$  as well as to the sphere  $\mathcal{S}$  of equation  $\sum_{j=1}^g x_j^2 = \frac{a_1^2 + \delta}{4g}$ . Since the radius  $r := \sqrt{\frac{a_1^2 + \delta_{HP}}{4g}}$  of  $\mathcal{S}$  is greater than or equal to the distance  $d := \frac{|a_1|}{2\sqrt{g}}$  from the origin to the plane  $\mathcal{P}$ , we deduce that the intersection of  $\mathcal{S}$  and  $\mathcal{P}$  is nonempty. Moreover,  $d = r$  if and only if  $\delta_{HP} = 0$ . In other words,  $X$  is a HP-maximal curve if and only if  $\mathcal{P}$  is tangent to  $\mathcal{S}$ .

Until the end of this section, we no longer consider any algebraic curve  $X$ . We just assume that  $f$  is a  $q$ -Weil polynomial and we still denote by  $\alpha_j$  the real parts of its complex roots  $\omega_j$ . We still consider  $\tilde{\delta} := 4g^2V(\alpha)$  where  $V(\alpha)$  is the variance of the  $\alpha_j$ 's. The real number  $\tilde{\delta}$  no longer expresses in terms of the number of points of a curve over a finite field and over its quadratic extension, but the different results of Theorem 3.2 (with  $\tilde{\delta}$  in place of  $\delta_{HP}$ ) as well as the inequalities (13) and (14) remain valid. We thus obtain the following generalization of the results of Rück, Haloui, Haloui and Singh, and Sohn together with a lower bound on the coefficient  $a_2$ .

**Theorem 3.9.** *We consider  $g \geq 2$ . If  $T^{2g} + a_1T^{2g-1} + a_2T^{2g-2} + \dots + a_2q^{g-2}T^2 + a_1q^{g-1}T + q^g$  is a  $q$ -Weil polynomial then*

$$a_2 \leq \frac{a_1^2(g-1)}{2g} + gq$$

and

$$(16) \quad a_2 \geq \begin{cases} \frac{g-1}{2g}a_1^2 - gq & \text{when } g \geq 2 \text{ is even,} \\ \frac{g-1}{2g}a_1^2 + \frac{(2-g^2)}{g}q & \text{when } g \geq 3 \text{ is odd.} \end{cases}$$

**Remark 3.10.** For instance let us consider the case where  $g = 3$  and compare with the lower bound  $a_2 \geq 4\sqrt{q}|a_1| - 9q$  provided in [8]. The bound (16) reads  $a_2 \geq a_1^2/3 - 7q/3$  and thus is better if and only if  $a_1^2/3 - 7q/3 \geq 4\sqrt{q}|a_1| - 9q$ . Taking into account that  $|a_1| \leq 6\sqrt{q}$  (see Condition (1) in Theorem 1.1 in [8]) the condition becomes  $|a_1| \leq 2\sqrt{q}$ . This may occur as the example<sup>5</sup> already given above of an isogeny class of an abelian variety defined over  $\mathbb{F}_4$ , of dimension 3 and whose  $L$ -polynomial is given by  $(1 - 2T)^2(1 + 2T)^2(1 + 4T^2)$  leads to  $a_1 = 0$  and  $a_2 = -4$ . In this case our bound (16) yields  $a_2 \geq -28/3$  whereas the bound  $a_2 \geq 4\sqrt{q}|a_1| - 9q$  in [8] gives  $a_2 \geq -36$ .

To be fair, we stress that (16) only provides a lower bound on  $a_2$ , whereas the sets of inequalities given in Theorem 1.1 in [26], Theorem 1.1 in [8], Theorem 1.1 in [9], Theorem 1.2 in [31] and studied in depth by Marseglia in [22] completely characterize the  $q$ -Weil polynomials of small degrees.

#### 4. IHARA-MAXIMAL CURVES

This section is devoted to the study of curves whose number of rational points reaches the Ihara bound (2), which are called Ihara-maximal curves.

**4.1. A characterization of Ihara-maximal curves.** As stated in Proposition 4.1 below, the Ihara-maximal curves will appear as the curves which are both Hallouin-Perret-maximal and Diophantine-stable.

**Proposition 4.1.** *Let  $X$  be a curve of genus  $g \geq 1$  defined over  $\mathbb{F}_q$ . The following assertions are equivalent.*

- (i)  $X$  is a Ihara-maximal curve.

<sup>5</sup>[https://www.lmfdb.org/Variety/Abelian/Fq/3/4/a\\_ae\\_a](https://www.lmfdb.org/Variety/Abelian/Fq/3/4/a_ae_a)

(ii)  $X$  is both a Hallouin-Perret-maximal curve and a Diophantine-stable curve with respect to the extension  $\mathbb{F}_{q^2}/\mathbb{F}_q$ .

(iii)  $Z_X(T) = \frac{(1-2\alpha T+qT^2)^g}{(1-T)(1-qT)}$  where  $\alpha = \frac{1}{4} - \frac{\sqrt{(8q+1)g^2+(4q^2-4q)g}}{4g}$ .

*Proof.* We still write  $\alpha_1, \dots, \alpha_g$  for the real parts of the Frobenius eigenvalues of  $X$ . For the purpose of proving the first implication (i)  $\Rightarrow$  (ii), we recall that the original proof of Ihara (§2 in [16]) rests on three inequalities. More precisely, the Cauchy-Schwarz inequality  $g \sum_{j=1}^g \alpha_j^2 \geq (\sum_{j=1}^g \alpha_j)^2$  together with the arithmetic inequality  $N_1 \leq N_2$  which reads

$$N_1 \leq q^2 + 1 + 2qg - 4 \sum_{j=1}^g \alpha_j^2$$

lead to the following quadratic inequality in  $N_1$

$$(17) \quad N_1^2 - (2q + 2 - g)N_1 + (q + 1)^2 - (q^2 + 1)g - 2qg^2 \leq 0.$$

When  $X$  is Ihara-maximal, the three inequalities become equalities. But the equality  $N_1 = N_2$  defines the Diophantine-stability. And the Cauchy-Schwarz inequality becomes an equality if and only if the  $\alpha_j$ 's are all equal, which characterizes HP-maximal curves by Proposition 2.1.

To prove that (ii)  $\Rightarrow$  (iii) one can notice that  $N_2 - (q^2 + 1) = 2qg - \frac{1}{g}(N_1 - (q + 1))^2$  and  $N_1 = N_2$  imply that inequality (17) becomes an equality. The only non-negative solution is  $N_1 = q + 1 + \frac{1}{2} \left( \sqrt{(8q + 1)g^2 + (4q^2 - 4q)g} - g \right)$  and thus one can use Proposition 2.1 to obtain the claimed form for the zeta function of  $X$ .

Finally, if (iii) is verified we are again in the context of Proposition 2.1 and then the knowledge of the value  $\alpha = \frac{q+1-\#X(\mathbb{F}_q)}{2g}$  enables to determine  $N_1$  and to conclude that  $X$  is Ihara-maximal.  $\square$

**Remark 4.2.** As a straightforward application we identify a family of Ihara-maximal curves. Indeed, for  $t \geq 1$  and  $q = 2^{2t+1}$  let us consider the Deligne-Lusztig curve of Suzuki type (called Suzuki curve from now), associated to the Suzuki group  $Sz(q)$ , that is the curve defined over  $\mathbb{F}_q$  as the non-singular model  $S$  of the plane curve given by the equation  $y^q - y = x^{q_0}(x^q - x)$  where  $q_0 = 2^t$ . It is well-known (see [11], [12] or Proposition 4.3 of [10]) that this curve has genus  $g = \frac{\sqrt{q}(q-1)}{\sqrt{2}}$ , satisfies  $L_S(T) = (1 + 2q_0T + qT^2)^g$  and has  $q^2 + 1$  rational points over  $\mathbb{F}_q$  and over  $\mathbb{F}_{q^2}$ .

So this curve is Diophantine-stable, and Proposition (2.1) ensures the Hallouin-Perret-maximality. A Suzuki curve is thus Ihara-maximal.

**4.2. An analog of a theorem of Rück and Stichtenoth.** Ihara has proved in [16] that a Weil-maximal curve  $X$  defined over  $\mathbb{F}_q$  has a genus less than or equal to  $\frac{\sqrt{q}(\sqrt{q}-1)}{2}$ . Indeed, the Ihara bound  $N_2^* := q + 1 + (\sqrt{(8q + 1)g^2 + 4qg(q - 1)} - g)/2$  becomes sharper than the Weil-bound  $N_1^* := q + 1 + 2g\sqrt{q}$  for  $g > g_2 := \frac{\sqrt{q}(\sqrt{q}-1)}{2}$ . Hallouin and Perret have proposed an even sharper bound  $N_3^*$  (see [7] and [2]) valid when  $g \geq g_3 :=$

$\frac{\sqrt{q(q-1)}}{\sqrt{2}}$ , which implies that the genus of a Ihara-maximal curve is less than or equal to  $g_3$ .

And in this direction they provide (Theorem 14 of [7]) an increasing sequence  $(g_n)_{n \geq 1}$  of integers and a sequence of upper bounds  $N_n^*(g)$  such that  $N_n^*(g)$  is a valid bound for  $\#X(\mathbb{F}_q)$  when the genus  $g$  of the curve is greater than or equal to  $g_n$ . The bounds are proven to be sharper and sharper (see point 18 of Theorem 14 in [7]) and the following expression for  $g_n$  is established:  $g_n = \sqrt{q}^{n+1} \sum_{k=1}^n \frac{1}{\sqrt{q}^k} \cos\left(\frac{k\pi}{n+1}\right)$ .

Let us now come back to the case of the Weil bound. Rück and Stichtenoth have characterized the Weil-maximal curves of maximal genus the following way.

**Theorem 4.3. (Rück and Stichtenoth, main theorem in [27])** *We suppose that  $q$  is a square and we consider a curve  $X$  defined over  $\mathbb{F}_q$ . Suppose that  $X$  has genus  $g = \frac{\sqrt{q}(\sqrt{q}-1)}{2}$ . Then  $X$  is Weil-maximal if and only if  $X$  is  $\mathbb{F}_q$ -isomorphic to the Hermitian curve whose equation is  $y^{\sqrt{q}} + y = x^{\sqrt{q}+1}$ .*

We notice that it is possible to obtain an analogue of this result for Ihara-maximal curves by reformulating a maximality theorem of Fuhrmann and Torres (Theorem 2 in [6]) for Suzuki curves. This theorem asserts that if  $t \geq 1$ ,  $q = 2^{2t+1}$  and  $q_0 = 2^t$  then any curve of genus  $g = q_0(q-1)$  and such that  $\#X(\mathbb{F}_q) = q^2 + 1$  is isomorphic to the Suzuki curve. It is thus sufficient to verify (with a tedious but straightforward computation) that in this setting  $q^2 + 1$  equals the Ihara-bound  $q + 1 + (\sqrt{(8q+1)g^2 + 4qg(q-1)} - g)/2$  to obtain the following analogue of the Rück and Stichtenoth theorem.

**Theorem 4.4. (Reformulation of Theorem 2 in [6])** *We consider  $t \geq 1$  and  $q = 2^{2t+1}$ . Let  $X$  be a curve defined over  $\mathbb{F}_q$ . Suppose that  $X$  has genus  $g = \frac{\sqrt{q}(q-1)}{\sqrt{2}}$ . Then  $X$  is Ihara-maximal if and only if  $X$  is  $\mathbb{F}_q$ -isomorphic to the Suzuki curve  $S$  which is the non-singular model of the curve of equation  $y^q - y = x^{q_0}(x^q - x)$  where  $q_0 = 2^t$ .*

#### 4.3. Determination of Ihara-maximal curves for small values of $g$ or $q$ .

**Proposition 4.5.** *We give in the Table 1 the complete list, up to isomorphism, of Ihara-maximal curves defined over  $\mathbb{F}_q$  of genus  $g \leq 18$ , except for the case  $g = 7$ . In this case, we know that  $q = 7$  and that there exists at least one Ihara-maximal curve, but we do not know if there is unicity.*

We also provide in each case the number  $N_1 = N_2$  of rational points over  $\mathbb{F}_q$  (and  $\mathbb{F}_{q^2}$ ), the  $L$ -polynomial of the curve, an equation of an affine model of the curve and we indicate with a  $\times$  in the sixth column whether the curve is Weil-maximal.

*Proof.* We notice that if  $g < (q - \sqrt{q})/2$  then the Weil bound  $N_1^*$  (with the notations of Subsection 4.2) is sharper than the Ihara bound  $N_2^*$ , and so

$g$	$q$	$N_1 = N_2$	$L_X(T)$	Ihara-maximal curve	Weil-max
1	2	5	$1 + 2T + 2T^2$	$y^2 + y = x^3 + x$	
	3	7	$1 + 3T + 3T^2$	$y^2 = x^3 + 2x + 1$	
	4	9	$1 + 4T + 4T^2$	$x^3 + y^3 + z^3 = 0$	×
3	4	14	$(1 + 3T + 4T^2)^3$	$x^4 + x^2y + xy^3 + x + y^2 = 0$	
	4	14	$(1 + 3T + 4T^2)^3$	$x^4 + x^2y^2 + y^4 + x^2y + xy^2 + x^2 + xy + y^2 + 1 = 0$	
	9	28	$(1 + 3T)^6$	$x^4 + y^4 + z^4 = 0$	×
6	16	65	$(1 + 8T + 16T^2)^6$	$x^5 + y^5 + z^5 = 0$	×
7	7	36	$(1 + 4T + 7T^2)^7$	There exists at least one curve: the fibre product $y_1^3 = 5(x+2)(x+5)/x$ $y_2^3 = 3x^2(x+5)/(x+3)$	
10	25	126	$(1 + 10T + 25T^2)^{10}$	$x^6 + y^6 + z^6 = 0$	×
14	8	65	$(1 + 4T + 8T^2)^{14}$	$y^8 - y = x^2(x^8 - x)$	

TABLE 1. List (up to isomorphism) of all Ihara-maximal curves of genus  $g \leq 18$  except for  $g = 7$ .

a curve cannot be Ihara-maximal. It enables to discard these cases. Thus, with the help of a Python program we list all the couples  $(g, q)$  for which  $(q - \sqrt{q})/2 \leq g \leq 18$  and for which  $(8q + 1)g^2 + 4qg(q - 1)$  is a square. We also list the corresponding values of  $N_1 (= N_2)$ .

Some of these couples can be discarded because the Ihara bound is greater than a known upper bound of the maximal number  $N_q(g)$  of rational points of a curve of genus  $g$  over  $\mathbb{F}_q$ . We sum up in Table 2 these discarded couples with the helpful references (obtained for the most part thanks to ManyPoints [34]).

$g$	4	6	8	8	8	10	10	15	16	16	18	18
$q$	8	9	11	11	19	5	16	25	4	13	29	41
Ihara bound	29	40	56	29	88	36	87	161	45	102	204	270
Up. bound on $N_q(g)$	25	38	55	88	84	33	86	160	38	101	203	258
Reference	[28]	[14]	[18]	[18]	[30]	[14]	[14]	[14]	[30]	[18]	[14]	[29]

TABLE 2. List of discarded cases in proof of Proposition 4.5

Let us now treat the remaining cases with the help of Proposition 4.1 which ensures that a curve is Ihara-maximal if and only if it is both HP-maximal and Diophantine-stable. The case of curves of genus one is easily handled as such a curve is always HP-maximal (see Remark 2.2) and as Bars, Lario and Vrioni provide in Proposition 3.1 in [4] the only isomorphism classes of Diophantine-stable elliptic curves.

When  $g = 3$  and  $q = 4$ , we should have  $N_1 = N_2 = 14$ . In [4] the authors indicate the two only isomorphism classes of Diophantine-stable curves for the extension  $\mathbb{F}_{16}/\mathbb{F}_4$ .

When  $(g, q) = (3, 9), (6, 16)$  or  $(10, 25)$  we notice that  $g = \sqrt{q}(\sqrt{q} - 1)/2$  and that the candidate curve must be Weil-maximal. So by the Theorem of Rück and Stichtenoth quoted in Subsection 4.2 we know that we deal with an Hermitian curve.

If  $g = 14$  and  $q = 8$ , it is remarkable that  $g = \sqrt{q}(q - 1)/\sqrt{2}$ , so Theorem 4.4 ensures that in this case a Ihara-maximal curve is a Suzuki curve.

Finally, Özbudak, Temür and Yayla provide in [25] a fibre product of Kummer extensions which is an example of Ihara-maximal curve of genus 7 defined over  $\mathbb{F}_7$  and such that  $N_1 = 36$ .  $\square$

**Remark 4.6.** We can proceed in the same way to obtain the list of the Ihara-maximal curves defined over  $\mathbb{F}_q$  for  $q \leq 13$ . The point is to bound the possible values of  $g$  for a given value of  $q$ . As in the proof of Proposition 4.5 we know that  $(q - \sqrt{q})/2 \leq g$ . If we refer to the discussion of Subsection 4.2 and to Theorem 14 in [7], we know that a curve of genus  $g > \sqrt{q}(q - 1)/\sqrt{2}$  cannot be Ihara-maximal, so  $g \leq \sqrt{q}(q - 1)/\sqrt{2}$ . So this time we list all the couples  $(g, q)$  for which  $(q - \sqrt{q})/2 \leq g \leq \sqrt{q}(q - 1)/\sqrt{2}$ ,  $q \leq 13$  and for which  $(8q + 1)g^2 + 4qg(q - 1)$  is a square. In comparison with the proof of Proposition 4.5 the only additional case is that of the couple  $(g, q) = (25, 13)$  which would lead to a curve with 144 rational points. But it is known that in this context  $N_1 \leq 142$  (see [18]).

Thus the complete list, up to isomorphism, of Ihara-maximal curves defined over  $\mathbb{F}_q$  for  $q \leq 13$  of genus  $g \geq 1$ , except for the case  $q = 7$ , is given by the eight curves of Table 1 corresponding to the couples  $(g, q) \in \{(1, 2), (1, 3), (1, 4), (3, 4), (3, 9), (7, 7), (14, 8)\}$ .

**Remark 4.7.** So we have managed to discuss the existence of Ihara-maximal curves of genus  $g$  for any  $g \leq 18$  and any value of  $q$ . In this direction, the first case we do not know how to treat is the one of a possible curve of genus  $g = 19$ . With the method described above, we know that this curve shall be defined over  $\mathbb{F}_{19}$  and such that  $N_1 = N_2 = 153$ , but we are unable to determine whether there exists such a curve.

In the other direction, we know if there exists a Ihara-maximal curve defined over  $\mathbb{F}_q$  for any  $q \leq 13$ . To make further progress we would need to know if there exists a curve of genus 24 defined over  $\mathbb{F}_{16}$  with  $N_1 = N_2 = 161$ .

## 5. ABELIAN SURFACES ISOGENOUS TO JACOBIANS OF HALLOUIN-PERRET-MAXIMAL CURVES

We focus in this section on abelian surfaces which are isogenous to Jacobians of genus 2 HP-maximal curves. Recall that by Proposition 2.1, the  $L$ -polynomial of such a curve reads  $(1 + aT + qT^2)^2$ , and so the characteristic polynomial of the Jacobian of such a curve expresses  $(T^2 + aT + q)^2$ . The next result will answer the natural question: among the polynomials

$(T^2 + aT + q)^2$ , which ones do correspond to the characteristic polynomial of the Jacobian of a HP-maximal curve?

The proof will be based upon successive works which aim to describe the characteristic polynomials which can be associated to abelian and Jacobian surfaces. Maisner and Nart have characterized in [20] when a  $q$ -Weil polynomial of degree 4 corresponds to an abelian surface defined over  $\mathbb{F}_q$  and when this abelian surface is simple. Furthermore, Howe, Nart and Ritzenthaler have determined in [15] when simple abelian surfaces are isogenous to a Jacobian and also when the isogeny class of a square of an ordinary elliptic curve contains a Jacobian. Finally, we will make use of the characterization of abelian surfaces whose isogeny class contains the product of two supersingular elliptic curves given in characteristic 2 by Maisner and Nart (see [21]), in characteristic 3 by Howe (see [13]) and in greater characteristic by Howe, Nart and Ritzenthaler (see [15]). When the abelian surface is not simple, we use the characterization of Waterhouse (see [35]) of a polynomial of degree 2 arising as the characteristic polynomial of an elliptic curve. Putting all of this together, one obtains the following characterization of HP-maximal curves of genus 2 related to the structure of their Jacobians.

**Theorem 5.1.** *Let  $q = p^n$  be a power of a prime  $p$ . We consider a polynomial  $f(T) = (T^2 + aT + q)^2 \in \mathbb{Z}[T]$  (which amounts to saying that  $a \in \mathbb{Z}$ ). Then  $f(T)$  is the characteristic polynomial of a Jacobian of a Hallouin-Perret-maximal curve defined over  $\mathbb{F}_q$  if and only if one of the following conditions holds.*

### 1) Simple abelian surface case

- (1.1)  $n$  is even and  $p \equiv 1 \pmod{4}$  and  $a = 0$ .
- (1.2)  $n$  is even and  $p \equiv 1 \pmod{3}$  and  $a = \pm\sqrt{p^n}$ .

In these cases,  $f(T)$  is the characteristic polynomial of a simple abelian surface defined over  $\mathbb{F}_q$  which is supersingular and isogenous to the Jacobian of a Hallouin-Perret-maximal curve.

### 2) Split ordinary case

$$|a| \leq 2\sqrt{q}, (a, p) = 1 \text{ and } a^2 - 4q \notin \{-3, -4, -7\}.$$

In this case,  $f(T)$  is the characteristic polynomial of an abelian surface defined over  $\mathbb{F}_q$  which is isogenous to  $E \times E$  where  $E$  is an ordinary elliptic curve and its isogeny class contains the Jacobian of a Hallouin-Perret-maximal curve.

### 3) Split supersingular case

- (3.1)  $p = 2$  and  $n > 1$ :
  - (i)  $n$  odd and  $a = 0$ .
  - (ii)  $n$  odd and  $a = \pm\sqrt{2p^n}$ .
  - (iii)  $n$  even and  $a = 0$ .
  - (iv)  $n$  even and  $a = \pm p^{\frac{n}{2}}$ .
  - (v)  $n \geq 4$  even and  $a = \pm 2p^{\frac{n}{2}}$ .

(3.2)  $p = 3$ :

(i)  $n \geq 3$  odd and  $a = 0$ .

(ii)  $n$  even and  $a$  verifies one of the following conditions:

$$(a = 0) \quad \text{or} \quad \left(a = \pm p^{\frac{n}{2}}\right) \quad \text{or} \quad \left(a = \pm 2p^{\frac{n}{2}} \quad \text{and} \quad n \geq 4\right).$$

(3.3)  $p > 3$ :

(i)  $n$  even and  $a = \pm 2p^{\frac{n}{2}}$ .

(ii)  $n$  even,  $p \not\equiv 1 \pmod{3}$  and  $a = \pm p^{\frac{n}{2}}$ .

(iii)  $n$  odd and  $a = 0$ .

(iv)  $n$  even,  $p \not\equiv 1 \pmod{4}$  and  $a = 0$ .

In all these cases,  $f(T)$  is the characteristic polynomial of an abelian surface defined over  $\mathbb{F}_q$  which is isogenous to  $E \times E$  where  $E$  is a supersingular elliptic curve and its isogeny class contains the Jacobian of a Hallouin-Perret-maximal curve.

*Proof.* Throughout the proof we will consider a square polynomial  $f(T) = (T^2 + aT + q)^2 = T^4 + (2a)T^3 + (a^2 + 2q)T^2 + (2a)qT + q^2 \in \mathbb{Z}[T]$ . By Rück's result (see Theorem 1.1 of [26] or Lemma 2.1 of [20]) we know that  $f$  is a  $q$ -Weil polynomial if and only if  $|a| \leq 2\sqrt{q}$  and  $4|a|\sqrt{q} \leq a^2 + 4q$ .

(1) Let us first consider the case of simple abelian surfaces. Theorem 2.9 in [20] provides necessary and sufficient conditions for a  $q$ -Weil polynomial  $f(T) = T^4 + a_1T^3 + a_2T^2 + qa_1T + q^2 \in \mathbb{Z}[T]$  to be the characteristic polynomial of a simple abelian surface defined over  $\mathbb{F}_q$ .

This theorem classifies such surfaces in four families called mixed (M), ordinary (O) and supersingular (SS1) and (SS2). This classification involves the integer  $\Delta = a_1^2 - 4a_2 + 8q$ , but when  $f(T) = (T^2 + aT + q)^2$  we have  $\Delta = 4a^2 - 4(a^2 + 2q) + 8q = 0$ . In our case  $\Delta$  is a square and so the cases (M) and (O) are discarded. It is not possible to fulfill the conditions of the cases (SS1) or (SS2), except when  $a = 0$ ,  $n$  is even and  $p \equiv 1 \pmod{4}$  or when  $a = \pm\sqrt{q}$ ,  $n$  is even and  $p \equiv 1 \pmod{3}$ . For these values we check that  $f(T) = (T^2 + aT + q)^2$  does correspond to a Weil polynomial thanks to Rück's result.

Moreover, Howe, Nart and Ritzenthaler have given in Theorem 1.2 of [15] necessary conditions on a Weil polynomial to be the characteristic polynomial of a simple abelian surface defined over  $\mathbb{F}_q$  which is not isogenous to a Jacobian. Since  $a_2 = a^2 + 2q$  the condition  $a_2 < 0$  of Table 1.2 of [15] is never satisfied. We conclude that in the considered cases  $f(T)$  is indeed the characteristic polynomial of a simple abelian surface  $A$  isogenous to a Jacobian. This concludes the case (1) concerning simple abelian surfaces.

(2) If  $(a, q) = 1$  then by Deuring and Waterhouse the polynomial  $T^2 + aT + q$  is the characteristic polynomial of an ordinary elliptic curve  $E$  and thus  $(T^2 + aT + q)^2$  is the characteristic polynomial of an abelian surface isogenous to  $E \times E$ . Moreover, by Theorem 2.3 of [15],  $E \times E$  is isogenous to the Jacobian of a curve  $X$  if and only if  $a^2 - 4q$  is neither  $-3$  nor  $-4$  nor  $-7$ . In this case, we know that  $L_X(T) = (T^2 + aT + q)^2$  and so  $X$  is indeed a HP-maximal curve by Proposition 2.1.

(3) It remains to consider abelian surfaces which are isogenous to the product of two supersingular elliptic curves.

(3.1) We first treat the characteristic 2 case. Maisner and Nart have given in Table 1 (respectively Table 3) in [21] the list of the 6 (respectively 15) isogeny classes of abelian surfaces in characteristic 2 that contain the product of two supersingular elliptic curves over  $\mathbb{F}_{2^n}$  when  $n$  is odd (respectively even), together with the numbers of  $\mathbb{F}_{2^n}$ -isomorphism classes of supersingular curves of genus 2 whose Jacobian lies in each isogeny class. Theirs tables involve the couples  $(a_1, a_2) = (2a, a^2 + 2q)$ .

The cases in Table 1 and Table 3 for which the condition  $a_2 = \frac{a_1^2}{4} + 2q$  holds are the following:

$$\left\{ \begin{array}{ll} (a_1, a_2) = (0, 2q) & \text{for odd } n > 1 \quad (\text{so } a = 0), \\ (a_1, a_2) = (\pm 2\sqrt{2q}, 4q) & \text{for odd } n > 1 \quad (\text{so } a = \pm\sqrt{2q}), \\ (a_1, a_2) = (0, 2q) & \text{for even } n \quad (\text{so } a = 0), \\ (a_1, a_2) = (\pm 2\sqrt{q}, 3q) & \text{for even } n \quad (\text{so } a = \pm\sqrt{q}), \text{ and} \\ (a_1, a_2) = (\pm 4\sqrt{q}, 6q) & \text{for even } n > 2 \quad (\text{so } a = \pm 2\sqrt{q}). \end{array} \right.$$

So when  $n$  and  $a$  fulfill one of these conditions we know that there exists an elliptic curve  $E$  such that  $E \times E$  is isogenous to the Jacobian of a curve and admits  $f(T) = (T^2 + aT + q)^2$  as characteristic polynomial. Since  $a$  and  $p$  are not coprime, the elliptic curve  $E$  is supersingular according to Theorem 4.1 of [35].

(3.2) We now deal with the characteristic 3 case. In this context, Howe has determined all the polynomials that occur as characteristic polynomials of abelian surfaces and whose isogeny class contains the Jacobian of a curve of genus 2 (see Theorem 1.1 in [13]). We can identify among them the only polynomials which are square polynomials  $f(T) = (T^2 + aT + q)^2$ :

$$\left\{ \begin{array}{ll} f(T) = (T^2 + q)^2 & \text{for odd } n \geq 3 \quad (\text{so } a = 0), \\ f(T) = (T^2 + q)^2 & \text{for even } n \quad (\text{so } a = 0), \\ f(T) = (T^2 \pm \sqrt{q}T + q)^2 & \text{for even } n \quad (\text{so } a = \pm\sqrt{q}), \\ f(T) = (T^2 \pm 2\sqrt{q}T + q)^2 & \text{for even } n \geq 4 \quad (\text{so } a = \pm 2\sqrt{q}). \end{array} \right.$$

What is left is to show that in any case the corresponding abelian surface is not simple and that  $T^2 + aT + q$  is indeed the characteristic polynomial of a supersingular elliptic curve. To show that the abelian surface is not simple, it is sufficient to check that none of the conditions (1.1) and (1.2) of the theorem is satisfied. Now, according to Deuring-Waterhouse theorem (Theorem 4.1 in [35]), there is a natural bijection between the isogenous classes of elliptic curves over  $\mathbb{F}_q$  and the set of integers  $m$  with  $|m| \leq 2\sqrt{q}$  that satisfy one of the following conditions:

- (1)  $(q, m) = 1$ ;
- (2)  $q$  is a square and  $m = \pm 2\sqrt{q}$ ;
- (3)  $q$  is a square,  $p \not\equiv 1 \pmod{3}$ , and  $m = \pm\sqrt{q}$ ;
- (4)  $q$  is not a square,  $p = 2$  or  $3$ , and  $m = \pm\sqrt{pq}$ ;
- (5)  $q$  is not a square and  $m = 0$ ; or  $q$  is a square,  $p \not\equiv 1 \pmod{4}$ , and  $m = 0$ .

And this bijection is such that  $\sharp E(\mathbb{F}_q) = q + 1 - m$  for all curves  $E$  from the isogeny class corresponding to  $m$ . Furthermore only the first item does not correspond to supersingular elliptic curves. So we just have to check that  $|a| \leq 2\sqrt{q}$  and that the conditions of one of the cases (2), (3), (4) or (5) are satisfied.

(3.3) The case of characteristic  $p > 3$  will be a conclusion of works undertaken by Howe, Nart and Ritzenthaler. Indeed, in this setting, Theorem 2.4 of [15] asserts that there is a Jacobian isogenous to the product of two supersingular elliptic curves  $E_1$  and  $E_2$  of characteristic polynomials  $f_{E_1}(T) = T^2 - rT + q$  and  $f_{E_2}(T) = T^2 - sT + q$  if and only if  $r^2 = s^2$ . When  $E_1 = E_2$  this condition is certainly satisfied. Furthermore, if the Jacobian of a curve  $X$  is isogenous to  $E \times E$ , then the  $L$ -polynomial of  $X$  has form  $L_X(T) = (1 - 2\alpha T + qT^2)^2$  which implies by Proposition 2.1 that  $X$  is HP-maximal.

So we are reduced to discuss under which conditions there exists a supersingular elliptic curve  $E$ . The answer is given by the Deuring-Waterhouse theorem (Theorem 4.1 of [35]) already quoted above, namely by the conditions (2), (3), (4) and (5) of the list given in point (3.2).  $\square$

Recall that any curve of genus 1 is a HP-maximal curve (see Remark 2.2). As a consequence, there exists a HP-maximal curve whatever the base field. What about the question for non-elliptic curves? The previous theorem brings the following answer.

**Proposition 5.2.** *Over any finite field there exists a non-elliptic Hallouin-Perret-maximal curve.*

*More precisely, there exists a Hallouin-Perret-maximal curve of genus 2 defined over  $\mathbb{F}_q$  if and only if  $q > 2$ , and there exists a Hallouin-Perret-maximal curve of genus 3 over  $\mathbb{F}_2$ .*

*Proof.* Maisner, Nart and Howe have provided in [20] a complete description of the curves of genus 2 defined over  $\mathbb{F}_2$  up to  $\mathbb{F}_2$ -isomorphism and quadratic twist. None of the couples  $(a_1, a_2)$  given in Table 2 in [20] satisfies the condition  $a_2 = a_1/4 + 2q$ , so there is no HP-maximal curve of genus 2 over  $\mathbb{F}_2$ .

Now we assume that  $q > 2$ , and we first consider the case of characteristic 2. When  $q = 4$ , Table 7 of [20] shows (first row and last column) the existence of a curve such that  $a_1 = 0$  and  $a_2 = 8 = \frac{a_1^2}{4} + 2q$  so the existence of a HP-maximal curve of genus 2 over  $\mathbb{F}_4$  is established.

If  $q = 2^n$  with  $n \geq 3$ , the choice  $a = 3$  gives  $|a| \leq 2\sqrt{q}$  and fulfills the conditions  $(a, p) = 1$  and  $a^2 - 4q \notin \{-3, -4, -7\}$  of point (2) of Theorem 5.1. Thus the polynomial  $(T^2 + 3T + q)^2$  is the characteristic polynomial of an abelian surface (isogenous to  $E \times E$  where  $E$  is an ordinary elliptic curve) whose isogeny class contains the Jacobian of a HP-maximal curve of genus 2 defined over  $\mathbb{F}_q$ .

Finally in the case of characteristic greater than or equal to 3 one can take  $a = 2$ , so once again the conditions of point (2) of Theorem 5.1 are fulfilled, and so there exists a HP-maximal curve of genus 2 defined over  $\mathbb{F}_q$ .

For the remaining case  $q = 2$ , the LMFDB database (see [19]) provides a genus-3 HP-maximal curve, namely<sup>6</sup> the pointless curve defined by the equation  $x^4 + x^2y^2 + x^2yz + x^2z^2 + xy^2z + xyz^2 + y^4 + y^2z^2 + z^4 = 0$  which admits 14 points over  $\mathbb{F}_4$ .  $\square$

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<sup>6</sup>[https://www.lmfdb.org/Variety/Abelian/Fq/3/2/ad\\_j\\_an](https://www.lmfdb.org/Variety/Abelian/Fq/3/2/ad_j_an)

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