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MAXIMAL CURVES WITH RESPECT TO QUADRATIC EXTENSIONS OVER FINITE FIELDS

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Abstract. We propose a detailed study of a canonical bound which relates the numbers of rational points of a curve over a finite field with that over its quadratic extension. Alternative proofs which relate to the variance enable to complete the inequality in a symmetrical way and to obtain optimal refinements.

We focus on the curves reaching the bound, which we call Diophantine-maximal curves. We provide different characterizations and stress natural links with the curves which attain the Ihara bound. As consequences, we establish the list of such curves with low genus and we outline a maximality result which involves the Suzuki curves.

At last we determine which polynomials correspond to the Jacobian of a Diophantine-maximal curve of genus 2.

1. Introduction

Throughout the whole paper we consider an absolutely irreducible smooth projective algebraic curve X (just called curve from now on) of genus g and defined over the finite field $\mathbb{F}_q$. In the context of estimating the number $\sharp X(\mathbb{F}_{q^n})$ of rational points of X over $\mathbb{F}_{q^n}$, we propose a detailed study of an inequality hightlighted by Hallouin and Perret. This inequality canonically appears in [7] among a series of meaningful bounds obtained as consequences of non-negativity of a series of Gram determinants.

Let us sketch the method developped in [7]. The Neron-Severi group of the surface $X \times X$ can be quotiented by numerical equivalence and thus tensorised to obtain a real vector space $\mathrm{Num}(X \times X)_{\mathbb{R}}$ equipped by the intersection pairing. As a consequence of the Hodge-index theorem the intersection pairing is negative definite on the vector space orthogonal to the plane generated by the classes of the horizontal and vertical fibres. We denote by $\wp$ the orthogonal projection onto this subspace. For an integer k we thus consider γ_k the push-down by $\wp_*$ of the class of the graph of the k-th iterated Frobenius that we normalize by $\sqrt{q}^k$. The non-negativity of the

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determinant of the Gram matrix $\mathrm{Gram}(\gamma_0, \gamma_1)$ expresses exactly the Weil inequality, as explained in Subsection 2.2.1 in [7]:

Theorem. *(Weil, [34]) Let X be a curve defined over $\mathbb{F}_q$ of genus g. We have*

$$| \sharp X(\mathbb{F}_q) - (q+1) | \leq 2g\sqrt{q}. \tag{1}$$

Next, the non-negativity of the Gram matrix $\mathrm{Gram}(\gamma_0, \gamma_1, \gamma_2)$ together with the arithmetic constraint $\sharp X(\mathbb{F}_{q^2}) \geq \sharp X(\mathbb{F}_q)$ yields to the Ihara bound, as explained in Subsection 2.3 in [7]:

Theorem. *(Ihara, [16]) Let X be a curve defined over $\mathbb{F}_q$ of genus $g \geq 1$. We have*

$$\sharp X(\mathbb{F}_q) - (q+1) \ \leq \frac{\sqrt{(8q+1)g^2 + (4q^2-4q)g} - g}{2}. \tag{2}$$

Meanwhile, Hallouin and Perret have also noticed that the non-negativity of the determinant of the Gram matrix $\mathrm{Gram}(\gamma_0, \gamma_1, \gamma_2)$ leads to the following inequality:

Theorem. *(Hallouin and Perret, Proposition 12 in [7]) Let X be a curve defined over $\mathbb{F}_q$ of genus $g \geq 1$. We have*

$$\sharp X(\mathbb{F}_{q^2}) - (q^2+1) \leq 2gq - \frac{1}{g}\big(\sharp X(\mathbb{F}_q) - (q+1)\big)^2. \tag{3}$$

This is the inequality we aim to study in our work. A first interpretation provided in [7] is that the inequality (3) improves the Weil bound for $\sharp X(\mathbb{F}_{q^2})$ all the more as $\sharp X(\mathbb{F}_q)$ is far from $\sharp \mathbb{P}^1(\mathbb{F}_q) = q+1$. Our first contribution in this paper is to provide alternative and elementary proofs of the inequality (3). In particular we make a link with the statistical variance of the real parts of the reciprocal roots of the L-polynomial of X. This way, the inequality (3) appears as a consequence of the positivity of the variance. As another consequence, we can complete inequality (3) with a lower bound which leads to a symmetrical and new inequality.

Theorem. *(Theorem 3.2 and Corollary 3.3) Let X be a curve of genus $g \geq 2$ defined over $\mathbb{F}_q$. We denote by $\alpha_1, \ldots, \alpha_g$ the real parts of its Frobenius eigenvalues, that we consider as a statistical sample whose mean is given by $\overline{\alpha} := \frac{1}{g}\sum_{i=1}^{g} \alpha_i$ and whose variance equals $V(\alpha) := \frac{1}{g}\sum_{i=1}^{g} (\alpha_i - \overline{\alpha})^2$.*

(i) The difference between the right hand side and the left hand side of the inequality (3) is a multiple of the variance of the α_i's:

$$2gq - \frac{1}{g}\big(\sharp X(\mathbb{F}_q) - (q+1)\big)^2 - \sharp X(\mathbb{F}_{q^2}) + (q^2+1) = 4gV(\alpha).$$

(ii) We deduce

$$\left|\#X(\mathbb{F}_{q^2}) - (q^2+1) + \frac{1}{g}\left(\#X(\mathbb{F}_q) - (q+1)\right)^2\right| \leq 2qg.$$

Classical inequalities on the variance enable us to improve the previous lower bound for odd genus g in an optimal way (see Theorem 3.4).

Let us now stress a link with a series of inequalities involving the coefficients a_1 and a_2 of a q-Weil polynomial $T^{2g} + a_1T^{2g-1} + a_2T^{2g-2} + \cdots + a_2q^{g-2}T^2 + a_1q^{g-1}T + q^g$. We recall that to be a q-Weil polynomial is a well-known necessary condition to be the characteristic polynomial of an abelian variety defined over $\mathbb{F}_q$ as explained in Section 3.2.

When $g = 2$ (respectively $g = 3$, $g = 4$) Rück (respectively Haloui, Haloui and Singh) have proved that $a_2 \leq \frac{a_1^2(g-1)}{2g} + gq$ in [25] (respectively [8], [9]). Actually, computations naturally related to the inequality (3) lead to the following generalization of these three results for any value of $g \geq 2$.

Theorem. *(Theorem 3.9 and Proposition 3.7)*

(i) If $T^{2g}+a_1T^{2g-1}+a_2T^{2g-2}+\cdots+a_2q^{g-2}T^2+a_1q^{g-1}T+q^g$ is a q-Weil polynomial then

$$a_2 \leq \frac{a_1^2(g-1)}{2g} + gq. \tag{4}$$

(ii) Let X be a curve defined over $\mathbb{F}_q$ of genus $g \geq 2$ and $L_X(T) = 1 + a_1T + a_2T^2 + \cdots + a_2q^{g-2}T^{2g-2} + a_1q^{g-1}T^{2g-1} + q^gT^{2g}$ be its L-polynomial. Then the inequality (4) obviously holds true. Moreover the bound (4) is reached if and only (3) is an equality.

The case of equality in (3) will thus receive special attention in our work. Let us point out another meaningful motivation to study this case of equality. One could relate with the words of Serre on page 96 in [28]: “it would be natural for curves to ask for many points, not only over $\mathbb{F}_q$, but also over several given extensions of $\mathbb{F}_q$”. Indeed, if we fix the values of g, q and N_1, among the curves of genus $g > 0$ with N_1 points over $\mathbb{F}_q$, those which reach equality in (3) have the largest number of points over $\mathbb{F}_{q^2}$.

Recall that we have obviously $\sharp X(\mathbb{F}_{q^2}) \geq \sharp X(\mathbb{F}_q)$ and following [22] a curve satisfying $\sharp X(\mathbb{F}_{q^2}) = \sharp X(\mathbb{F}_q)$ is called a Diophantine-stable curve (with respect to the extension $\mathbb{F}_{q^2}/\mathbb{F}_q$). Inequality (3) gives an upper bound on $\sharp X(\mathbb{F}_{q^2})$ so we propose to call Diophantine-maximal with respect to the extension $\mathbb{F}_{q^2}/\mathbb{F}_q$ and with respect to Inequality (3) any curve reaching this bound. When the context is clear, we will only say Diophantine-maximal.

Definition 1.1. A curve X defined over $\mathbb{F}_q$ of genus $g > 0$ is said to be a Diophantine-maximal curve (or DM-curve for short) if

$$\sharp X(\mathbb{F}_{q^2}) - (q^2+1) = 2gq - \frac{1}{g}\big(\sharp X(\mathbb{F}_q) - (q+1)\big)^2. \tag{5}$$

We will thus provide different characterizations of DM-curves. These characterizations will prove useful to identify families of DM-curves such as the Weil-maximal or Weil-minimal curves (see Example 2.3), to establish the stability of the notion of Diophantine-maximality by coverings (see Proposition 2.11) or to obtain an upper bound of the genus of a DM-curve depending only on q (see Proposition 2.10).

Proposition. *(Proposition 2.1 and Proposition (2.9)) Let X be a curve of genus $g \geq 1$ defined over $\mathbb{F}_q$. We denote by $\alpha_1, \ldots, \alpha_g$ the real parts of the reciprocal roots of the L-polynomial of X. The following assertions are equivalent*

(i) X is a DM-curve,

(ii) all the α_j's are equal, that is the zeta function of X is of the form

$$Z_X(T) = \frac{(1-2\alpha T + qT^2)^g}{(1-T)(1-qT)}$$

(in this case, 2α is an integer and we have $2\alpha = \frac{q+1-\sharp X(\mathbb{F}_q)}{g}$),

(iii) the number of rational points of the Jacobian $\mathrm{Jac}(X)$ of X attains the upper bound (6) given in [1]

$$\sharp\, \mathrm{Jac}(X)(\mathbb{F}_q) = (q+1+\tau_{\mathrm{Jac}(X)}/g)^g$$

where $\tau_{\mathrm{Jac}(X)}$ stands for the opposite of the trace of the Jacobian of X, that is $\tau_{\mathrm{Jac}(X)} := -2\sum_{j=1}^{g}\alpha_j$.

The inequality (3) is strongly linked to the Ihara bound. For instance, one can guess an elementary proof of (3) in the original proof of Ihara of inequality (2) based upon Cauchy-Schwarz inequality. So it is quite natural that we can relate the curves reaching the two bounds.

Proposition. *(Proposition 4.1) Let X be a curve of genus $g \geq 1$. The curve X is Ihara-maximal (i.e. the Ihara inequality (2) becomes an equality) if and only if X is both a Diophantine-maximal curve and a Diophantine-stable curve.*

We will obtain in Section 4 the list of Ihara-maximal curves of low genera g and also for low values of q. Precisely, we give in Table 1 the complete list up to isomorphism of Ihara-maximal curves when $g \leq 18$, and in Remark

4.6 when $q \leq 13$, except (in both cases) when $g = q = 7$. For these values, we know that there exists at least one Ihara-maximal curve, but we do not know if there is unicity.

Whereas it is essentially a reformulation of a result of Fuhrmann and Torres (Theorem 2 in [6]) we think it is worthwhile to highlight the following theorem which provides an analogue of a theorem of Rück and Stichtenoth where the Suzuki curves play the role of Hermitian curves as explained in Subsection 4.2.

Theorem. *(Theorem 4.4, reformulation of Theorem 2 in [6]) We consider $t \geq 2$ and $q = 2^{2t+1}$. Let X be a curve defined over $\mathbb{F}_q$. Suppose that X has genus $g = \frac{\sqrt{q}(q-1)}{\sqrt{2}}$. Then X is Ihara-maximal if and only if X is $\mathbb{F}_q$-isomorphic to the Suzuki curve S which is the non-singular model of the curve of equation $y^q - y = x^{q_0}(x^q - x)$ where $q_0 = 2^t$.*

At last, it is noteworthy that the Jacobian of a DM-curve is a power of a simple abelian variety (see Proposition 2.5). In the same direction, we will determine among the polynomials $(T^2 + aT + q)^2 \in \mathbb{Z}[T]$ which ones correspond to the Jacobian of a genus-2 DM-curve X, and we will characterize when the Jacobian of X is simple or splits into the power of an ordinary or supersingular elliptic curve (Theorem 5.1). This classification work will be the aim of Section 5. As a consequence we will deduce the following existence result.

Proposition. *(Proposition 5.2) Over any finite field there exists a non-elliptic DM-curve.*

2. Different characterizations of Diophantine-maximality

Let X be a curve defined over $\mathbb{F}_q$ of genus $g \geq 1$ and $N_n = \sharp X(\mathbb{F}_{q^n})$ be its number of rational points over $\mathbb{F}_{q^n}$. It is well-known that the zeta function of X

$$Z_{X,\mathbb{F}_q}(T) := \exp\left(\sum_{n=1}^{\infty} \frac{N_n T^n}{n}\right)$$

is a rational function

$$Z_{X,\mathbb{F}_q}(T) = \frac{L_X(T)}{(1-T)(1-qT)}$$

where $L_X(T)$, called the L-polynomial of X, is a polynomial in $\mathbb{Z}[T]$ of degree $2g$. It has the form

$$L_X(T) = \prod_{j=1}^{g}(1-\omega_j T)(1-\overline{\omega}_j T)$$

where the inverse roots ω_j of $L_X(T)$ are algebraic integers such that $|\omega_j| = \sqrt{q}$ (the so-called Riemann Hypothesis for curves over finite fields). We also denote by α_j the real part of ω_j, so that the L-polynomial of X can be written

$$L_X(T) = \prod_{j=1}^{g}(1-2\alpha_j T + qT^2)$$

where $|\alpha_j| \leq \sqrt{q}$.

This polynomial can also be seen as the reciprocal polynomial of the characteristic polynomial $f_{\mathrm{Jac}(X)}$ of the Frobenius endomorphism acting on the Tate module of the Jacobian $\mathrm{Jac}(X)$ of X:

$$f_{\mathrm{Jac}(X)} = T^{2g}L_X(1/T) = \prod_{j=1}^{2g}(T-\omega_j)(T-\overline{\omega}_j) = \prod_{j=1}^{g}(T^2 - 2\alpha_j T + q).$$

So we will sometimes refer to the ω_j's as the Frobenius eigenvalues of X.

2.1. **Characterization by the zeta function.** An important point in our work is the following proposition which characterizes the DM-curves in terms of the real parts of its Frobenius eigenvalues.

Proposition 2.1. *Let X be a curve of genus g defined over $\mathbb{F}_q$. We denote by $\alpha_1, \ldots, \alpha_g$ the real parts of its Frobenius eigenvalues. The curve X is a DM-curve if and only if $\alpha_1 = \cdots = \alpha_g$, that is if and only if its zeta function is of the form*

$$Z_X(T) = \frac{(1-2\alpha T + qT^2)^g}{(1-T)(1-qT)}.$$

In this case, 2α is an integer and we have $2\alpha = \frac{q+1-\sharp X(\mathbb{F}_q)}{g}$.

Proof. It is well-known that we have:

$$\sharp X(\mathbb{F}_q) = q+1-\sum_{j=1}^{g}(\omega_j + \overline{\omega}_j) \quad \text{and} \quad \sharp X(\mathbb{F}_{q^2}) = q^2+1-\sum_{j=1}^{g}(\omega_j^2 + \overline{\omega}_j{}^2).$$

Saying that X is a DM-curve is thus equivalent to saying that

$$-\sum_{j=1}^{g}(\omega_j^2 + \overline{\omega}_j{}^2) = 2gq - \frac{1}{g}\Big(-\sum_{j=1}^{g}(\omega_j + \overline{\omega}_j)\Big)^2.$$

But $\omega_j+\overline{\omega}_j=2\alpha_j$, so $\omega_j^2+\overline{\omega}_j{}^2=4\alpha_j^2-2q$ and thus the equality comes down to $g\sum_{j=1}^g\alpha_j^2=\left(\sum_{j=1}^g\alpha_j\right)^2$. But Cauchy-Schwarz equality holds if and only if all the α_j's are equal, that is if and only L_X has the claimed form.

If α denotes the common value of the α_j's, by the relation above we get $\alpha=\frac{q+1-\sharp X(\mathbb{F}_q)}{2g}$. Therefore 2α is both a rational number and an algebraic integer, because it is the sum of the algebraic integers ω_1 and $\overline{\omega}_1$. We deduce that 2α is an integer. □

Remark 2.2. It is straighforward that any elliptic curve E over $\mathbb{F}_q$ is a DM-curve since $N_1:=\sharp E(\mathbb{F}_q)=q+1-(\omega+\overline{\omega})$ and $N_2:=\sharp E(\mathbb{F}_{q^2})=q^2+1-(\omega^2+\overline{\omega}^2)$ and thus $(N_1-(q+1))^2=(\omega+\overline{\omega})^2=q^2+1+2q-N_2$. Meanwhile the characterization of Proposition 2.1 gives again immediately that any elliptic curve is a DM-curve.

Remark 2.3. As a consequence of the previous proposition, a Weil-maximal curve i.e. a curve which reaches the Weil upper bound (respectively a Weil-minimal curve i.e. a curve which reaches the Weil lower bound) over $\mathbb{F}_q$ is a DM-curve. Indeed its only Frobenius eigenvalue is $\omega=-\sqrt{q}$ (respectively $\omega=\sqrt{q}$).

Remark 2.4. A DM-curve curve of genus $g\geq 2$ is not necessarily Weil-maximal nor Weil-minimal: the curve X of genus 2 defined over $\mathbb{F}_3$ by the equation $y^2=(-1-x-x^3)(1-x+x^3)$ verifies $N_1:=\sharp X(\mathbb{F}_3)=2$ and $N_2:=\sharp X(\mathbb{F}_9)=20$ so X is a DM-curve but is neither Weil-maximal nor Weil-minimal.

2.2. **Geometric condition.**

Proposition 2.5. *Let X be a curve of genus g defined over $\mathbb{F}_q$. If X is a DM-curve then the Jacobian of X is $\mathbb{F}_q$-isogenous to a power of a $\mathbb{F}_q$-simple abelian variety.*

Proof. Let $f_{\mathrm{Jac}(X)}$ be the characteristic polynomial of the Jacobian of X

$$f_{\mathrm{Jac}(X)}=\prod_{j=1}^{g}(T^2-2\alpha_jT+q)$$

where the α_j's are the real parts of its Frobenius eigenvalues. By Proposition 2.1 we know that if X is a DM-curve then the α_j's are equal, which amounts to saying that $f_{\mathrm{Jac}(X)}$ has the form

$$f_{\mathrm{Jac}(X)}=(T^2-2\alpha T+q)^g.$$

The discriminant of $T^2-2\alpha T+q$ is non-positive by the Riemann Hypothesis, so the polynomial $f_{\mathrm{Jac}(X)}$ is a power of a $\mathbb{Q}$-irreducible polynomial.

But it is well-known that any abelian variety A over $\mathbb{F}_q$ can factor uniquely, up to $\mathbb{F}_q$-isogeny, into a product of powers of non-$\mathbb{F}_q$-isogenous $\mathbb{F}_q$-simple abelian varieties. And Tate stated in Theorem 2 (e) in [31] that A is $\mathbb{F}_q$-isogenous to a power of a $\mathbb{F}_q$-simple abelian variety if and only if its characteristic polynomial is a power of a $\mathbb{Q}$-irreducible polynomial. So the result follows. □

Remark 2.6. The reciprocal is false as we find many counterexamples in the database [19]. For instance, one can consider the curve X of genus 2 defined over $\mathbb{F}_5$ by the equation $y^2 = x^5 + 3x$. Its Jacobian is simple as its characteristic polynomial $1 + 25x^4$ is irreducible in $\mathbb{Q}[x]$. But it turns out that we have $N_1 = 2$ and $N_2 = 26$ so X is not a DM-curve.

Remark 2.7. The LMFDB database ([19]) provides[1] the equations of many hyperelliptic curves of genus 2 defined over $\mathbb{F}_{49}$ such that $N_1 = 36$ and $N_2 = 2500$. If we consider such a curve X, one can easily check that the equality (5) is verified, so X is a DM-curve. By Proposition 2.1 we recover that the L-polynomial of X is $L_X(T) = (1 - 7T + 49T^2)^2$. If there existed an elliptic curve E defined over $\mathbb{F}_{49}$ with L-polynomial $L_E(T) = 1 - (\omega + \overline{\omega})T + qT^2 = 1 - 7T + 49T^2$ we would have $\sharp E(\mathbb{F}_{49}) = 49 + 1 - (\omega + \overline{\omega}) = 43$. But according to the LMFDB database once again, such an elliptic curve does not exist. So X is an example of a DM-curve whose Jacobian is a simple abelian surface.

Remark 2.8. Let X be a DM-curve of genus g defined over $\mathbb{F}_q$ with $q = p^a$ and let $f_{\mathrm{Jac}(X)} = (T^2 - 2\alpha T + q)^g$ be its characteristic polynomial where $\alpha = \frac{q+1-\sharp X(\mathbb{F}_q)}{2g}$. Following the study of the isogeny classes of elliptic curves over a finite field given by Waterhouse in Theorem 4.1 of [33], one can state that the Jacobian of X is $\mathbb{F}_q$-isogenous to a power of an elliptic curve if and only if some one of the following conditions is satisfied:

- (i) $(2\alpha, p) = 1$;
- (ii) a is even and $2\alpha = \pm 2\sqrt{q}$;
- (iii) a is even and $p \not\equiv 1 \pmod 3$ and $2\alpha = \pm\sqrt{q}$;
- (iv) a is odd and $p = 2$ or 3 and $2\alpha = \pm p^{\frac{a+1}{2}}$;
- (v) either (v.i) a is odd or (v.ii) a is even and $p \not\equiv 1 \pmod 4$ and $\alpha = 0$.

2.3. **Characterization by the number of points of the Jacobian.** Furthermore, the Jacobian of a DM-curve is maximal in the sense that it reaches the following upper bound provided by Aubry, Haloui and Lachaud

[1] `https://www.lmfdb.org/Variety/Abelian/Fq/2/49/ao_fr`

in Theorem 2.1 of [1] for the number of rational points on an abelian variety

$$\sharp A(\mathbb{F}_q) \leq (q+1+\tau_A/g)^g \tag{6}$$

where $\tau_A := -\sum_{j=1}^{g}(\omega_j+\overline{\omega}_j) = -2\sum_{j=1}^{g}\alpha_j$ is the opposite of the trace of A, if we denote by ω_j's the roots of the characteristic polynomial f_A of A and by α_j's their real parts.

Theorem 2.1 of [1] also states that there is equality if and only if the α_j's are equal. Since the characteristic polynomial $f_{\mathrm{Jac}(X)}$ of the Jacobian of X is the reciprocal polynomial of the L-polynomial $L_X(T)$ of X, Proposition 2.1 leads to the following statement.

Proposition 2.9. *Let X be a curve of genus g defined over $\mathbb{F}_q$. Then X is a DM-curve if and only if the number of rational points of the Jacobian* $\mathrm{Jac}(X)$ *of X attains the upper bound (6), namely*

$$\sharp\,\mathrm{Jac}(X)(\mathbb{F}_q) = (q+1+\tau_{\mathrm{Jac}(X)}/g)^g. \tag{7}$$

2.4. **Genus of a DM-curve.** Ihara has shown that a curve defined over $\mathbb{F}_q$ cannot be Weil-maximal if its genus is large with respect to q. Precisely, he has proven that if X is a Weil-maximal curve defined over $\mathbb{F}_q$ of genus g then $g \leq \frac{q-\sqrt{q}}{2}$. In this subsection we will see that the genus of a DM-curve defined over $\mathbb{F}_q$ is also bounded by a function of q.

Indeed Elkies, Howe and Ritzenthaler have given in [5] an explicit upper bound on the genus of curves whose Jacobians are isogenous to a product of powers of given abelian varieties expressed in terms of their Frobenius angles. Precisely they proved in Theorem 1.1. of [5] that, if S is a finite non-empty set of s real numbers θ with $0 \leq \theta \leq \pi$, and if the non-negative Frobenius angles of X all lie in S then the genus g of X satisfies $g \leq 23s^2q^{2s}\log q$ and $g < (\sqrt{q}+1)^{2r}(\frac{1+q^{-r}}{2})$ where $r = 1/2$ if $S = \{0\}$ and $r = \sharp(S\cap\{\pi\}) + 2\sum_{\theta\in S\setminus\{0,\pi\}}\lceil\frac{\pi}{2\theta}\rceil$ otherwise.

As a consequence we obtain the following bounds.

Proposition 2.10. *Let X be a DM-curve defined over $\mathbb{F}_q$ of genus g. Then*

$$g \leq 23q^2\log q.$$

Moreover, when we write $L_X(T) = (1+2\alpha T+qT^2)^g$ then

(i) If $2\alpha > 0$ then $g < (\sqrt{q}+1)^4(\frac{q^2+1}{2q^2})$.
(ii) If $2\alpha = 2\sqrt{q}$ then $g \leq \frac{q-\sqrt{q}}{2}$.
(iii) If $2\alpha = -2\sqrt{q}$ then $g \leq \frac{(\sqrt{q}+1)^2}{2\sqrt{q}}$.

Proof. By Proposition 2.1 we know that if X is a DM-curve then $f_{\mathrm{Jac}(X)}$ only admits one root ω and its conjugate, and we have $2\alpha = -(\omega+\overline{\omega})$.

The theorem of Elkies, Howe and Ritzenthaler quoted above applies easily as S is reduced to one element. So $s = 1$ and we obtain the general bound $g \leq 23q^2 \log q$.

(i) When $0 < 2\alpha < 2\sqrt{q}$ we know that the real part of ω lies between $-\sqrt{q}$ and 0, so the unique nonnegative Frobenius angle θ satisfies $-\pi/2 < \theta < 0$. We deduce $\frac{1}{2} < \frac{\pi}{2\theta} < 1$, so $r = 2\lceil \frac{\pi}{2\theta} \rceil = 2$.

(ii) If $2\alpha = 2\sqrt{q}$ then $\omega = -\sqrt{q}$ and we recognize the case where the curve is Serre-maximal which is solved by Ihara.

(iii) If $2\alpha = -2\sqrt{q}$ then $\omega = \sqrt{q}$ and $\theta = 0$. This time the curve is Serre-minimal. We deduce that $r = 1/2$ and Theorem 1.1 of [5] gives $g \leq \frac{(\sqrt{q}+1)^2}{2\sqrt{q}}$. □

2.5. **In coverings.** Let $Y \longrightarrow X$ be a non-constant morphism of curves over $\mathbb{F}_q$. We know that if Y attains the Weil upper bound (or the Weil lower bound), the same holds for X (see Theorem 5.2.1. of [28]). More generally, it is proved in Corollary 13 of [3] that if $Y \longrightarrow X$ is a finite flat morphism between two varieties over a finite field, then the reciprocal polynomial of the characteristic polynomial of the Frobenius endomorphism on the i-th étale cohomology group of X divides that of Y. In particular, if $Y \longrightarrow X$ is a non-constant morphism of curves, the L-polynomial $L_X(T)$ of X divides the L-polynomial $L_Y(T)$ in $\mathbb{Z}[T]$. We can deduce the following statement.

Proposition 2.11. *Let $Y \longrightarrow X$ be a non-constant morphism of curves over $\mathbb{F}_q$. If Y is a DM-curve then X is also a DM-curve.*

Proof. If Y is a DM-curve, then by Proposition 2.1 the real parts of the eigenvalues of the Frobenius on Y are all equal. Since $L_X(T)$ divides $L_Y(T)$ the same holds for X. □

3. DM-defect of a curve

In this section we introduce the DM-defect of a curve over a finite field as a measure of how far a curve is from being Diophantine-maximal, that is of how far the inequality (3) is from the case of equality.

3.1. **Definition and first properties.** To define the DM-defect we naturally consider the difference between the right hand side and the left hand side of the inequality (3) and we normalize in order to work with integers.

Definition 3.1. Let X be a curve of genus $g \geq 1$ defined over $\mathbb{F}_q$. The DM-defect of X, denoted by $\delta_{DM}(X)$, is defined by

$$\delta_{DM}(X) = 2qg^2 - (\sharp X(\mathbb{F}_q) - (q+1))^2 - g\left(\sharp X(\mathbb{F}_{q^2}) - (q^2+1)\right).$$

The following theorem aims to give different expressions of δ_{DM} in terms of the real parts α_j's of the eigenvalues ω_j's of the Frobenius endomorphism on X. It will prove useful to give alternative and elementary proofs of (3) and to obtain other bounds on δ_{DM}.

Theorem 3.2. *Let X be a curve of genus $g \geq 1$ defined over $\mathbb{F}_q$ and let $\alpha_1, \ldots, \alpha_g$ be the real parts of its Frobenius eigenvalues. Then we have:*

(i)

$$\delta_{DM}(X) = 4\left(g\sum_{i=1}^{g}\alpha_i^2 - \left(\sum_{i=1}^{g}\alpha_i\right)^2\right).$$

(ii) If we set $\sigma_1 = \sum_{i=1}^{g}\alpha_i$ and $\sigma_2 = \sum_{1\leq i<j\leq g}\alpha_i\alpha_j$ then we have

$$\delta_{DM}(X) = 4\left((g-1)\sigma_1^2 - 2g\sigma_2\right) \text{ and}$$

(iii)

$$\delta_{DM}(X) = 4\sum_{1\leq i<j\leq g}(\alpha_i - \alpha_j)^2.$$

(iv) If we consider $(\alpha_i)_{1\leq i\leq g}$ as a statistical sample whose mean is given by $E(\alpha) := \frac{1}{g}\sum_{i=1}^{g}\alpha_i$ and whose variance equals $V(\alpha) := \frac{1}{g}\sum_{i=1}^{g}(\alpha_i - E(\alpha))^2$, then we have

$$\delta_{DM}(X) = 4g^2V(\alpha).$$

Proof. (i) If we still denote by ω_j's the Frobenius eigenvalues of X, we get from the definition of the DM-defect:

$$\delta_{DM}(x) = 2qg^2 - \big(-\sum_{j=1}^{g}(\omega_j + \overline{\omega}_j)\big)^2 - g\big(-\sum_{j=1}^{g}(\omega_j^2 + \overline{\omega}_j{}^2)\big)$$

which implies that

$$\delta_{DM}(x) = 2qg^2 - 4\big(\sum_{j=1}^{g}\alpha_j\big)^2 + g\sum_{j=1}^{g}\big(4\alpha_j^2 - 2q\big)$$

and the result follows.

(ii) and (iii) are direct consequences of (i).

(iv) If we factorize the equality (i) by $4g^2$ we get $4g^2\left(\frac{1}{g}\sum_{i=1}^{g}\alpha_i^2 - \left(\frac{1}{g}\sum_{i=1}^{g}\alpha_i\right)^2\right)$, and we recognize $E(\alpha^2) - E(\alpha)^2$, that is a classical formulation of the variance of α. Note that the equality (iii) is also a well-known expression for the variance of α (see Section 2.4 in [23] for example). □

The previous theorem gives alternative and elementary proofs of the Hallouin-Perret bound (3) which simply reads $\delta_{DM}(X) \geq 0$. It also provides an interpretation of this bound in terms of non-negativity of the variance of the sample (α_j). From points (i) or (iv) one can also immediately deduce an upper bound on δ_{DM} and then a symmetric formulation which completes inequality (3).

Corollary 3.3. *Let X be a curve of genus $g \geq 2$ defined over $\mathbb{F}_q$. We have*

$$\delta_{DM}(X) \in [0, 4qg^2]$$

that is

$$\left| \#X(\mathbb{F}_{q^2}) - (q^2+1) + \frac{1}{g} \left(\#X(\mathbb{F}_q) - (q+1)\right)^2 \right| \leq 2qg. \tag{8}$$

Let us exploit the variance formulation of δ_{DM}. It is a classical statistical problem to give bounds for the variance of a sample $(\alpha_j)_{1 \leq j \leq g}$ of elements in a range $[\alpha_{\min}; \alpha_{\max}]$. In [17] (see Lemma 1) Kaiblinger and Spangl state that we always have

$$V(\alpha) \leq \frac{1}{4}(\alpha_{\max} - \alpha_{\min})^2,$$

and that, if g is odd the bound can be improved this way

$$V(\alpha) \leq \frac{1}{4}\left(1 - \frac{1}{g^2}\right)(\alpha_{\max} - \alpha_{\min})^2.$$

Lemma 1 in [17] also indicates that when g is even this upper bound on $V(\alpha)$ is reached if and only if half of the values α_j equal $-\sqrt{q}$ whereas the other half equal $\sqrt{q}$. When g is odd, the upper bound is reached if and only if $(g-1)/2$ or $(g+1)/2$ values equal $-\sqrt{q}$ whereas the others equal $\sqrt{q}$. We deduce the following theorem.

Theorem 3.4. *Let X be a curve of genus $g \geq 2$ defined over $\mathbb{F}_q$ and let $\alpha_1, \ldots, \alpha_g$ be the real parts of its Frobenius eigenvalues.*

(i) If g is even then

$$-2qg \leq \#X(\mathbb{F}_{q^2}) - (q^2+1) + \frac{1}{g}\left(\#X(\mathbb{F}_q) - (q+1)\right)^2 \tag{9}$$

and equality holds if and only if $g/2$ values α_j equal $-\sqrt{q}$ whereas the others equal $\sqrt{q}$.

(ii) If g is odd then

$$-2q\left(g - \frac{2}{g}\right) \leq \#X(\mathbb{F}_{q^2}) - (q^2+1) + \frac{1}{g}\left(\#X(\mathbb{F}_q) - (q+1)\right)^2 \tag{10}$$

and equality holds if and only if $(g-1)/2$ or $(g+1)/2$ values α_j equal $-\sqrt{q}$ whereas the others equal $\sqrt{q}$.

Remark 3.5. The necessary conditions on the α_j's established in the previous proposition help us to find, with the help of the LMFDB database ([19]), curves of genus 2 which reach the lower bound. Let us give some examples when g is even. For instance the (Diophantine-stable) hyperelliptic curve X of equation $y^2 = x^5 + 4x$ defined over $\mathbb{F}_5$ satisfies $\#X(\mathbb{F}_q) = \#X(\mathbb{F}_{q^2}) = 6$ and we can check the equality $\#X(\mathbb{F}_{q^2}) - (q^2+1) + \frac{1}{g}\left(\#X(\mathbb{F}_q) - (q+1)\right)^2 = -2qg$. We have also found examples when $q = 7$, $q = 8$ or $q = 11$. For the case $q = 9$ the necessary conditions lead us to $N_1 = 10$, $N_2 = 46$, and to an isogeny class which does not contain any Jacobian according to the LMFDB database[2]. For the case $q = 13$ one can find (at least) two hyperelliptic curves of genus 2 with equations $y^2 = x^5 + 12x$ and $y^2 = x^6 + 2x^3 + 8$ such that $\alpha_1 = -\sqrt{13}$ and $\alpha_2 = \sqrt{13}$. We thus have $N_1 = 14$, $N_2 = 118$ and the lower bound is reached.

Remark 3.6. We are indebted to Christophe Ritzenthaler for pointing out the following example of a non-elliptic curve of odd genus which reaches the lower bound. We take $q = 49$ and we consider a generator a of the multiplicative group $\mathbb{F}_q^*$. Thus the curve of equation $x^4 + a^{43}x^3y + a^{36}x^3z + a^{27}x^2y^2 + a^{10}x^2yz + a^{31}x^2z^2 + a^9xy^3 + a^{47}xy^2z + a^6xyz^2 + a^{19}xz^3 + a^9y^4 + 6y^3z + a^{46}y^2z^2 + a^{22}yz^3 + 6z^4 = 0$ has genus 3 and is such that $N_1 = 2108$ whereas $N_2 = 36$, and so we can check $\#X(\mathbb{F}_{q^2}) - (q^2+1) + \frac{1}{g}\left(\#X(\mathbb{F}_q) - (q+1)\right)^2 = -2q(g - \frac{2}{g})$. The curve is obtained by twisting the curve of equation $x^4 + y^4 + z^4 = 0$ so that it only changes one elliptic factor.

3.2. **Weil polynomials and Diophantine maximal curves.** In this subsection we relate the topic of DM-curves with the question of determining whether a polynomial $f(T) = 1 + a_1T + a_2T^2 + \cdots + a_2q^{g-2}T^{2g-2} + a_1q^{g-1}T^{2g-1} + q^gT^{2g} \in \mathbb{Z}[T]$ can be the reciprocal polynomial of the characteristic polynomial of a dimension g abelian variety over $\mathbb{F}_q$. We provide necessary conditions in terms of the two first coefficients a_1 and a_2 and in terms of δ_{DM}.

Recall that a q-Weil number is an algebraic integer such that its image under every embedding has absolute value $\sqrt{q}$ and that a monic polynomial of $\mathbb{Z}[T]$ is called a q-Weil polynomial if all its roots are q-Weil numbers. The Honda-Tate theorem (see [30]) establishes a bijection between the simple abelian varieties over $\mathbb{F}_q$ up to isogeny and the q-Weil numbers up to conjugation so that a well-known necessary condition for f is to be a q-Weil polynomial.

[2] https://www.lmfdb.org/Variety/Abelian/Fq/2/9/a_as

When $g = 2$ Rück has proved in Theorem 1.1. of [25] that a necessary condition is that $a_2 \leq \frac{a_1^2}{4} + 2q$. In the same direction, Haloui has proved in [8] that for $g = 3$ a necessary condition is that $a_2 \leq \frac{a_1^2}{3} + 3q$. Haloui and Singh have also proved in [9] that if $g = 4$ a necessary condition is given by $a_2 \leq \frac{3}{8}a_1^2 + 4q$. We will propose a generalization of this series of necessary condition for any g.

We start expliciting relations between the coefficients a_1, a_2 and the defect δ_{DM} in the case where $f(T)$ is the L-polynomial of a curve X.

Proposition 3.7. *Let X be a curve defined over $\mathbb{F}_q$ of genus $g \geq 2$ and with DM-defect δ_{DM}. For any $j \in \{1, \dots, g\}$ we also note α_j for the real part of the inverse root ω_j of the L-polynomial $L_X(T) = 1 + a_1T + a_2T^2 + \cdots + a_2q^{g-2}T^{2g-2} + a_1q^{g-1}T^{2g-1} + q^gT^{2g}$. We have*

$$\sum_{j=1}^{g} \alpha_j^2 = \frac{a_1^2 + \delta_{DM}}{4g}. \tag{11}$$

and we deduce

$$a_2 \leq \frac{a_1^2(g-1)}{2g} + gq \tag{12}$$

with equality if and only if X is a DM-curve.

Proof. For the first point we express equality (i) in Theorem 3.2 in terms of the coefficient a_1 which satisfies $a_1 = -2\sum_{j=1}^{g} \alpha_j$.

For the second point we start from equality (ii) in Theorem 3.2. We use $a_1 = -2\sigma_1$ and $a_2 = gq + 4\sigma_2$ to get $2ga_2 + \delta_{DM} = (g-1)a_1^2 + 2g^2q$ and thus $a_2 = a_1^2(g-1)/2g + gq - \delta_{DM}/2g$. But δ_{DM} is non-negative and equals zero if and only if the curve is Diophantine-maximal. □

Remark 3.8. Let us point out a geometric interpretation in the euclidean space $\mathbb{R}^g$ if we associate to a a curve X the point P_X of coordinates $(\alpha_j)_{1\leq j\leq g}$. Thanks to the Riemann Hypothesis we know that this point belongs to the closed ball $\overline{B}_\infty(0, \sqrt{q})$. Now we fix the value of a_1. A way to translate Proposition (3.7) is to say that the point P_X belongs to the affine plane $\mathcal{P}$ of equation $\sum_{j=1}^{g} x_j = -\frac{a_1}{2}$ as well as to the sphere $\mathcal{S}$ of equation $\sum_{j=1}^{g} x_j^2 = \frac{a_1^2+\delta}{4g}$. Since the radius $r := \sqrt{\frac{a_1^2+\delta}{4g}}$ of $\mathcal{S}$ is greater than or equal to the distance $d := \frac{|a_1|}{2\sqrt{g}}$ from the origin to the plane $\mathcal{P}$, we deduce that the intersection of $\mathcal{S}$ and $\mathcal{P}$ is nonempty. Moreover, $d = r$ if and only if $\delta_{DM} = 0$. In other words, X is a DM-curve if and only if $\mathcal{P}$ is tangent to $\mathcal{S}$.

Now, we just assume that f is a q-Weil polynomial. If we still denote by α_j the real parts of its complex roots ω_j and by δ_{DM} the variance of the α_j's, the equality (11) remains valid. But the variance stays non-negative and we thus obtain the following generalization of the results of Rück, Haloui and Haloui and Singh.

Theorem 3.9. *If $T^{2g} + a_1 T^{2g-1} + a_2 T^{2g-2} + \cdots + a_2 q^{g-2} T^2 + a_1 q^{g-1} T + q^g$ is a q-Weil polynomial then*

$$a_2 \leq \frac{a_1^2(g-1)}{2g} + gq.$$

4. Ihara-maximal curves

This section is devoted to the study of curves whose number of rational points reaches the Ihara bound (2), which will be called Ihara-maximal curves.

4.1. A characterization of Ihara-maximal curves.

As stated in Proposition 4.1 below, the Ihara-maximal curves will appear as the curves which are both Diophantine-maximal and Diophantine-stable.

Proposition 4.1. *Let X be a curve of genus $g \geq 1$ defined over $\mathbb{F}_q$. The following assertions are equivalent.*

(i) X is a Ihara-maximal curve.
(ii) X is both a DM-curve and a DS-curve .
(iii) $Z_X(T) = \frac{(1-2\alpha T+qT^2)^g}{(1-T)(1-qT)}$ where $\alpha = \frac{1}{4} - \frac{\sqrt{(8q+1)g^2+(4q^2-4q)g}}{4g}$.

Proof. For the first implication $(i) \Rightarrow (ii)$, it is sufficient to notice that the original proof of Ihara ([16]) rests on three inequalities, namely a quadratic inequality, the arithmetic inequality $N_1 \leq N_2$ and the Cauchy-Schwarz inequality $g\sum_{j=1}^{g} \alpha_j^2 \geq \left(\sum_{j=1}^{g} \alpha_j\right)^2$, where the α_j's are the real parts of the Frobenius eigenvalues of X. When X is Ihara-maximal, they all become equalities. But the equality $N_1 = N_2$ defines the Diophantine-stability. And the Cauchy-Schwarz inequality becomes an equality if and only if the α_j's are all equal, which characterizes DM-curves by Proposition 2.1.

To prove $(ii) \Rightarrow (iii)$ one can notice that $N_2-(q^2+1) = 2gq - \frac{1}{g}\big(N_1-(q+1)\big)^2$. Hence $N_1 = N_2$ implies $N_1 = q+1+\frac{1}{2}\left(\sqrt{(8q+1)g^2+(4q^2-4q)g} - g\right)$. Thus one can use Proposition 2.1 to obtain the claimed form for the zeta function of X.

Finally, if (iii) is verified we are again in the context of Proposition 2.1 and then the knowledge of the value $\alpha = \frac{q+1-\sharp X(\mathbb{F}_q)}{2g}$ enables to determine N_1 and to conclude that X is Ihara-maximal. □

Remark 4.2. As a straightforward application we identify a family of Ihara-maximal curves. Indeed, for $t \geq 1$ and $q = 2^{2t+1}$ let us consider the Deligne-Lusztig curve of Suzuki type (called Suzuki curve from now), associated to the Suzuki group $Sz(q)$, that is the curve defined over $\mathbb{F}_q$ as the non-singular model S of the plane curve given by the equation $y^q - y = x^{q_0}(x^q - x)$ where $q_0 = 2^t$. It is well-known (see Proposition 4.3 of [10] or [11] or [12]) that this curve has genus $g = \frac{\sqrt{q}(q-1)}{\sqrt{2}}$, satisfies $L_S(T) = (1 + \sqrt{2}\sqrt{q}T + qT^2)^g$ and has $q^2 + 1$ rational points over $\mathbb{F}_q$ and over $\mathbb{F}_{q^2}$.

So this curve is Diophantine-stable, and Proposition (2.1) ensures the Diophantine-maximality. A Suzuki curve is thus Ihara-maximal.

4.2. **An analog of a theorem of Rück and Stichtenoth.** Ihara has proved in [16] that a Weil-maximal curve X defined over $\mathbb{F}_q$ has a genus less than or equal to $\frac{\sqrt{q}(\sqrt{q}-1)}{2}$. Indeed, the Ihara bound $N_2^* := q + 1 + (\sqrt{(8q+1)g^2 + 4qg(q-1)} - g)/2$ becomes sharper than the Weil-bound $N_1^* := q + 1 + 2g\sqrt{q}$ for $g > g_2 := \frac{\sqrt{q}(\sqrt{q}-1)}{2}$. Hallouin and Perret have proposed an even sharper bound N_3^* (see [7] and [2]) valid when $g \geq g_3 := \frac{\sqrt{q}(q-1)}{\sqrt{2}}$, which implies that the genus of a Ihara-maximal curve is less than or equal to g_3.

And in this direction they provide (Theorem 14 of ([7])) an increasing sequence $(g_n)_{n \geq 1}$ of integers and a sequence of upper bounds $N_n^*(g)$ such that $N_n^*(g)$ is a valid bound for $\#X(\mathbb{F}_q)$ when the genus g of the curve is greater than or equal to g_n. The bounds are proven to be sharper and sharper (see point 18 of Theorem 14 in ([7])) and the following expression for g_n is established: $g_n = \sqrt{q}^{n+1} \sum_{k=1}^{n} \frac{1}{\sqrt{q}^k} \cos\left(\frac{k\pi}{n+1}\right)$.

Let us now come back to the case of the Weil bound. Rück and Stichtenoth have characterized the Weil-maximal curves of maximal genus the following way.

Theorem 4.3. ***(Rück and Stichtenoth, [26])*** *We suppose that q is a square and we consider a curve X defined over $\mathbb{F}_q$. Suppose that X has genus $g = \frac{\sqrt{q}(\sqrt{q}-1)}{2}$. Then X is Weil-maximal if and only if X is $\mathbb{F}_q$-isomorphic to the Hermitian curve whose equation is $x^{\sqrt{q}+1} + y^{\sqrt{q}+1} + z^{\sqrt{q}+1} = 0$.*

We notice that it is possible to obtain an analogue of this result for Ihara-maximal curves by reformulating a maximality theorem of Fuhrmann and Torres (Theorem 2 in [6]) for Suzuki curves. This theorem asserts that if $t \geq 1$, $q = 2^{2t+1}$ and $q_0 = 2^t$ then any curve of genus $g = q_0(q-1)$ and such that $\#X(\mathbb{F}_q) = q^2 + 1$ is isomorphic to the Suzuki curve. It is thus sufficient to verify (with a tedious but straightforward computation) that in this setting q^2+1 equals the Ihara-bound to obtain the following analogue of the Rück and Stichtenoth theorem.

Theorem 4.4. ***(Reformulation of Theorem 2 in [6])*** *We consider $t \geq 1$ and $q = 2^{2t+1}$. Let X be a curve defined over $\mathbb{F}_q$. Suppose that X has genus $g = \frac{\sqrt{q}(q-1)}{\sqrt{2}}$. Then X is Ihara-maximal if and only if X is $\mathbb{F}_q$-isomorphic to the Suzuki curve S which is the non-singular model of the curve of equation $y^q - y = x^{q_0}(x^q - x)$ where $q_0 = 2^t$.*

4.3. **Determination of Ihara-maximal curves for small values of g or q.**

Proposition 4.5. *We give in the Table 1 the complete list, up to isomorphism, of Ihara-maximal curves defined over $\mathbb{F}_q$ of genus $g \leq 18$, except for the case $g = 7$. In this case, we know that $q = 7$ and that there exists at least one Ihara-maximal curve, but we do not know if there is unicity.*

g	q	$N_1 = N_2$	$L_X(T)$	Ihara-maximal curve	Weil-max
	2	5	$1+2T+2T^2$	$y^2+y=x^3+x$	
1	3	7	$1+3T+3T^2$	$y^2=x^3+2x+1$	
	4	9	$1+4T+4T^2$	$y^2+y=x^3$	×
	4	14	$(1+3T+4T^2)^3$	$x^4+x^2y+xy^3+x+y^2=0$	
3	4	14	$(1+3T+4T^2)^3$	$x^4+x^2y^2+y^4+x^2y+xy^2+$ $+x^2+xy+y^2+1=0$	
	9	28	$(1+3T)^6$	$x^4+y^4+z^4=0$	×
6	16	65	$(1+8T+16T^2)^6$	$x^5+y^5+z^5=0$	×
7	7	36	$(1+4T+7T^2)^7$	There exists at least one curve: the fibre product $y_1^3=5(x+2)(x+5)/x$ $y_2^3=3x^2(x+5)/(x+3)$	
10	25	126	$(1+10T+25T^2)^{10}$	$x^6+y^6+z^6=0$	×
14	8	65	$(1+4T+8T^2)^{14}$	$y^8-y=x^2(x^8-x)$	

TABLE 1. List (up to isomorphism) of all Ihara-maximal curves of genus $g \leq 18$ except for $g = 7$.

We also provide in each case the number $N_1 = N_2$ of rational points over $\mathbb{F}_q$ (and $\mathbb{F}_{q^2}$), the L-polynomial of the curve, an equation of an affine model of the curve and we indicate with a × in the sixth column whether the curve is Weil-maximal.

Proof. With the help of a Python program we first list all the couples (g, q) for which $(q - \sqrt{q})/2 \leq g$ and for which $(8q+1)g^2 + 4qg(q-1)$ is a square. We also list the corresponding values of $N_1 (= N_2)$.

Some of these couples can be discarded because the Ihara bound is greater than a known upper bound of $N_q(g)$. We sum up in Table 2 these discarded couples with the helpful references (obtained for the most part thanks to ManyPoints [32]).

g	4	6	8	8	8	10	10	15	16	16	18	18
q	8	9	11	11	19	5	16	25	4	13	29	41
Ihara bound	29	40	56	29	88	36	87	161	45	102	204	270
Up. bound on $N_q(g)$	25	38	55	88	84	33	86	160	38	101	203	258
Reference	[27]	[14]	[18]	[18]	[29]	[14]	[14]	[14]	[29]	[18]	[14]	[28]

TABLE 2. List of discarded cases in proof of Proposition 4.5

Let us now treat the remaining cases with the help of Proposition 4.1 which ensures that a curve is Ihara-maximal if and only if it is both Diophantine-maximal and Diophantine-stable. The case of curves of genus one is easily handled as such a curve is always Diophantine-maximal (see Remark 2.2) and as Bars, Lario and Vrioni provide in Proposition 3.1 in [4] the only isomorphism classes of Diophantine-stable curves.

When $g = 3$ and $q = 4$, we should have $N_1 = N_2 = 14$. In [4] the authors indicate the two only isomorphism classes of Diophantine-stable curves for the extension $\mathbb{F}_{16}/\mathbb{F}_4$.

When $(g, q) = (3, 9), (6, 16)$ or $(10, 25)$ we notice that $g = \sqrt{q}(\sqrt{q} - 1)/2$ and that the candidate curve must be Weil-maximal. So by the Theorem of Rück and Stichtenoth quoted in Subsection 4.2 we know that we deal with an Hermitian curve.

If $g = 14$ and $q = 8$, it is remarkable that $g = \sqrt{q}(q-1)/\sqrt{2}$, so Theorem 4.4 ensures that in this case a Ihara-maximal curve is a Suzuki curve.

Finally, Özbudak, Temür and Yayla provide in [24] a fibre product of Kummer extensions which is an example of Ihara-maximal curve of genus 7 defined over $\mathbb{F}_7$ and such that $N_1 = 36$. □

Remark 4.6. If we refer to the discussion of Subsection 4.2 and to Theorem 14 in [7], we know that a curve of genus $g > g_3$ cannot be Ihara-maximal. It enables us to proceed in the same way to obtain the list of Ihara-maximal curves defined over $\mathbb{F}_q$ for $q \leq 13$.

This time we list all the couples (g, q) for which $(q - \sqrt{q})/2 \leq g \leq \sqrt{q}(q-1)/\sqrt{2}$, $q \leq 13$ and for which $(8q+1)g^2 + 4qg(q-1)$ is a square. In comparison with the proof of Proposition 4.5 the only additional case is that of the couple $(g, q) = (25, 13)$ which would lead to a curve with 144 rational points. But it is known that in this context $N_1 \leq 142$ (see [18]).

Thus the complete list, up to isomorphism, of Ihara-maximal curves defined over $\mathbb{F}_q$ for $q \leq 13$ of genus $g \geq 1$, except for the case $q = 7$, is given by the eight curves of Table 1 corresponding to the couples $(g, q) \in \{(1, 2), (1, 3), (1, 4), (3, 4), (3, 9), (7, 7), (14, 8)\}$.

Remark 4.7. For low values of g, the first case we do not know how to treat is the one of a possible curve of genus $g = 19$ defined over $\mathbb{F}_{19}$ and such that $N_1 = N_2 = 153$. For low values of q, we are reduced to study the existence of curves of genus 24 defined over $\mathbb{F}_{16}$ with $N_1 = N_2 = 161$.

5. Abelian surfaces isogenous to Jacobians of DM-curves

We focus in this section on abelian surfaces which are isogenous to Jacobians of genus-2 DM-curves. Recall that by Proposition 2.1, the L-polynomial of such a curve reads $(1 + aT + qT^2)^2$, and so the characteristic polynomial of the Jacobian of such a curve expresses $(T^2 + aT + q)^2$. The next result will answer the natural question: among the polynomials $(T^2 + aT + q)^2$, which ones do correspond to the characteristic polynomial of the Jacobian of a DM-curve?

The proof will be based upon successive works which aim to describe the characteristic polynomials which can be associated to abelian and Jacobian surfaces. Maisner and Nart have characterized in [20] when a q-Weil polynomial of degree 4 corresponds to an abelian surface defined over $\mathbb{F}_q$ and when this abelian surface is simple. Furthermore, Howe, Nart and Ritzenthaler have determined in [15] when simple abelian surfaces are isogenous to a Jacobian and also when the isogeny class of a square of an ordinary elliptic curve contains a Jacobian. Finally, we will make use of the characterization of abelian surfaces whose isogeny class contains the product of two supersingular elliptic curves given in characteristic 2 by Maisner and Nart ([21]), in characteristic 3 by Howe ([13]) and in greater characteristic by Howe, Nart and Ritzenthaler ([15]). When the abelian surface is not simple, we use the characterization of Waterhouse (see [33]) of a polynomial of degree 2 arising as the characteristic polynomial of an elliptic curve. Putting all of this together, one obtains the following characterization of DM-curves of genus 2 related to the structure of their Jacobians.

Theorem 5.1. *Let $q = p^n$ be a power of a prime p. We consider a polynomial $f(T) = (T^2 + aT + q)^2 \in \mathbb{Z}[T]$ (which amounts to saying that $a \in \mathbb{Z}$). Then $f(T)$ is the characteristic polynomial of a Jacobian of a DM-curve defined over $\mathbb{F}_q$ if and only if one of the following conditions holds.*

1) Simple abelian surface case

(1.1) n is even and $p \equiv 1 \pmod 4$ and $a = 0$.
(1.2) n is even and $p \equiv 1 \pmod 3$ and $a = \pm\sqrt{p^n}$.

In these cases, $f(T)$ is the characteristic polynomial of a simple abelian surface defined over $\mathbb{F}_q$ which is supersingular and isogenous to the Jacobian of a DM-curve.

2) Split ordinary case

$|a| \leq 2\sqrt{q}$, $(a,p) = 1$ *and* $a^2 - 4q \notin \{-3, -4, -7\}$.

In this case, $f(T)$ is the characteristic polynomial of an abelian surface defined over $\mathbb{F}_q$ which is isogenous to $E \times E$ where E is an ordinary elliptic curve and its isogeny class contains the Jacobian of a DM-curve.

3) Split supersingular case

(3.1) $p = 2$ and $n > 1$:
- *(i) n odd and $a = 0$.*
- *(ii) n odd and $a = \pm\sqrt{2p^n}$.*
- *(iii) n even and $a = 0$.*
- *(iv) n even and $a = \pm p^{\frac{n}{2}}$.*
- *(v) $n \geq 4$ even and $a = \pm 2p^{\frac{n}{2}}$.*

(3.2) $p = 3$:
- *(i) $n \geq 3$ odd and $a = 0$.*
- *(ii) n even and a verifies one of the following conditions:*

$$(a = 0) \quad \text{or} \quad \left(a = \pm p^{\frac{n}{2}}\right) \quad \text{or} \quad \left(a = \pm 2p^{\frac{n}{2}} \quad \text{and} \quad n \geq 4\right).$$

(3.3) $p > 3$:
- *(i) n even and $a = \pm 2p^{\frac{n}{2}}$.*
- *(ii) n even, $p \not\equiv 1 \pmod 3$ and $a = \pm p^{\frac{n}{2}}$.*
- *(iii) n odd and $a = 0$.*
- *(iv) n even, $p \not\equiv 1 \pmod 4$ and $a = 0$.*

In all these cases, $f(T)$ is the characteristic polynomial of an abelian surface defined over $\mathbb{F}_q$ which is isogenous to $E \times E$ where E is a supersingular elliptic curve and its isogeny class contains the Jacobian of a DM-curve.

Proof. Throughough the proof we will consider a *square* polynomial $f(T) = (T^2 + aT + q)^2 = T^4 + (2a)T^3 + (a^2 + 2q)T^2 + (2a)qT + q^2 \in \mathbb{Z}[T]$. By Rück's result (see [25]) we know that f is a q-Weil polynomial if and only if $|a| \leq 2\sqrt{q}$ and $4|a|\sqrt{q} \leq a^2 + 4q$.

(1) Let us first consider the case of simple abelian surfaces. Theorem 2.9 in [20] provides necessary and sufficient conditions for a q-Weil polynomial $f(T) = T^4 + a_1T^3 + a_2T^2 + qa_1T + q^2 \in \mathbb{Z}[T]$ to be the characteristic polynomial of a simple abelian surface defined over $\mathbb{F}_q$.

This theorem classifies such surfaces in four families called mixed (M), ordinary (O) and supersingular (SS1) and (SS2). This classification involves the integer $\Delta = a_1^2 - 4a_2 + 8q$, but when $f(T) = (T^2 + aT + q)^2$ we have $\Delta = 4a^2 - 4(a^2 + 2q) + 8q = 0$. In our case Δ is a square and so the cases (M) and (O) are discarded. It is not possible to fulfill the conditions of the cases (SS1) or (SS2), except when $a = 0$, n is even and $p \equiv 1 \pmod 4$ or when $a = \pm\sqrt{q}$, n is even and $p \equiv 1 \pmod 3$. For these values we check that $f(T) = (T^2 + aT + q)^2$ does correspond to a Weil polynomial thanks to Rück's result.

Moreover, Howe, Nart and Ritzenthaler have given in Theorem 1.2 of [15] necessary conditions on a Weil polynomial to be the characteristic polynomial of a simple abelian surface defined over $\mathbb{F}_q$ which is not isogenous to a Jacobian. Since $a_2 = a^2 + 2q$ the condition $a_2 < 0$ of Table 1.2 of [15] is never satisfied. We conclude that in the considered cases $f(T)$ is indeed the characteristic polynomial of a simple abelian surface A isogeneous to a Jacobian. This concludes the case (1) concerning simple abelian surfaces.

(2) If $(a, q) = 1$ then by Deuring and Waterhouse the polynomial $T^2 + aT + q$ is the characteristic polynomial of an ordinary elliptic curve E and thus $(T^2 + aT + q)^2$ is the characteristic polynomial of an abelian surface isogenous to $E \times E$. Moreover, by Theorem 2.3 of [15], $E \times E$ is isogeneous to the Jacobian of a curve X if and only if $a^2 - 4q$ is neither -3 nor -4 nor -7. In this case, we know that $L_X(T) = (T^2 + aT + q)^2$ and so X is indeed a DM-curve by Proposition 2.1.

(3) It remains to consider abelian surfaces which are isogenous to the product of two supersingular elliptic curves.

(3.1) We first treat the characteristic 2 case. Maisner and Nart have given in Table 1 (respectively Table 2) in [21] the list of the 6 (respectively 15) isogeny classes of abelian surfaces in characteristic 2 that contain the product of two supersingular elliptic curves over $\mathbb{F}_{2^n}$ when n is odd (respectively even), together with the numbers of $\mathbb{F}_{2^n}$-isomorphism classes of supersingular curves of genus 2 whose Jacobian lies in each isogeny class. Theirs tables involve the couples $(a_1, a_2) = (2a, a^2 + 2q)$.

The cases in Table 1 and Table 3 for which the condition $a_2 = \frac{a_1^2}{4} + 2q$ holds are the following:

$$\begin{cases} (a_1, a_2) &=& (0, 2q) & \text{for odd } n > 1 & (\text{so } a = 0), \\ (a_1, a_2) &=& (\pm 2\sqrt{2q}, 4q) & \text{for odd } n > 1 & (\text{so } a = \pm\sqrt{2q}), \\ (a_1, a_2) &=& (0, 2q) & \text{for even } n & (\text{so } a = 0), \\ (a_1, a_2) &=& (\pm 2\sqrt{q}, 3q) & \text{for even } n & (\text{so } a = \pm\sqrt{q}), \text{ and} \\ (a_1, a_2) &=& (\pm 4\sqrt{q}, 6q) & \text{for even } n > 2 & (\text{so } a = \pm 2\sqrt{q}). \end{cases}$$

So when n and a fulfill one of these conditions we know that there exists an elliptic curve E such that $E \times E$ is isogeneous to the Jacobian of a curve and admits $f(T) = (T^2 + aT + q)^2$ as characteristic polynomial. We manage to prove that this elliptic curve is supersingular with Deuring-Waterhouse theorem ([33]).

(3.2) We now deal with the characteristic 3 case. In this context, Howe has determined all the polynomials that occur as characteristic polynomials of abelian surfaces and whose isogeny class contains the Jacobian of a curve of genus 2 (see Theorem 1.1 in [13]). We can identify among them the only polynomials which are square polynomials $f(T) = (T^2 + aT + q)^2$:

$$\begin{cases} f(T) = (T^2 + q)^2 & \text{for odd } n \geq 3 & (\text{so } a = 0), \\ f(T) = (T^2 + q)^2 & \text{for even } n & (\text{so } a = 0), \\ f(T) = (T^2 \pm \sqrt{q}T + q)^2 & \text{for even } n & (\text{so } a = \pm\sqrt{q}), \\ f(T) = (T^2 \pm 2\sqrt{q}T + q)^2 & \text{for even } n \geq 4 & (\text{so } a = \pm 2\sqrt{q}). \end{cases}$$

What is left is to show that in any case the corresponding abelian surface is not simple (for this it is sufficient to check that none of the condition of the case (S) is satisfied) and that $T^2 + aT + q$ is indeed the characteristic polynomial of a supersingular elliptic curve (for this we check $|a| \leq 2\sqrt{q}$ and one of the conditions (2), (3), (4) or (5) of Deuring-Waterhouse theorem (Theorem 4.1 in [33]).

(3.3) The case of characteristic $p > 3$ will be a conclusion of works undertaken by Howe, Nart and Ritzenthaler. In this setting, Theorem 2.4 of [15] asserts that there is a Jacobian isogeneous to the product of two supersingular elliptic curves defined over $\mathbb{F}_q$ if and only if the squares of the traces of the Frobenius of the curves are equal. One can deduce that if E is a supersingular elliptic curve then the isogeny class of $E \times E$ always contains the Jacobian of a DM-curve. Once again, Theorem 4.1 of [33] gives the conditions on a we have to verify to be sure that E is a supersingular elliptic curve. □

Recall that any curve of genus 1 is a DM-curve. As a consequence, there exists a DM-curve whatever the base field. What about the question for non-elliptic curves? The previous theorem brings the following answer.

Proposition 5.2. *Over any finite field there exists a non-elliptic DM-curve.*

More precisely, there exists a DM-curve of genus 2 defined over $\mathbb{F}_q$ *if and only if* $q > 2$*, and there exists a DM-curve of genus 3 over* $\mathbb{F}_2$*.*

Proof. Maisner, Nart and Howe have provided in [20] a complete description of the curves of genus 2 defined over $\mathbb{F}_2$ up to $\mathbb{F}_2$-isomorphism and quadratic twist. None of the couples (a_1, a_2) given in Table 2 in [20] satisfies the condition $a_2 = a_2/4 + 2q$, so there is no DM-curve of genus 2 over $\mathbb{F}_2$.

Now we assume that $q > 2$, and we first consider the case of characteristic 2. When $q = 4$, Table 7 of [20] shows (first row and last column) the existence of a curve such that $a_1 = 0$ and $a_2 = 8 = \frac{a_1^2}{4} + 2q$ so the existence of a DM-curve of genus 2 over $\mathbb{F}_4$ is established.

If $q = 2^n$ with $n \geq 3$, the choice $a = 3$ gives $|a| \leq 2\sqrt{q}$ and fulfills the conditions $(a, p) = 1$ and $a^2 - 4q \notin \{-3, -4, -7\}$ of point (2) of Theorem 5.1. Thus the polynomial $(T^2 + 3T + q)^2$ is the characteristic polynomial of an abelian surface (isogenous to $E \times E$ where E is an ordinary elliptic curve) whose isogeny class contains the Jacobian of a DM-curve of genus 2 defined over $\mathbb{F}_q$.

Finally in the case of characteristic greater than or equal to 3 one can take $a = 2$, so once again the conditions of point (2) of Theorem 5.1 are fulfilled, and so there exists a DM-curve of genus 2 defined over $\mathbb{F}_q$.

For the remaining case $q = 2$, the LMFDB database ([19]) provides a genus-3 DM-curve, namely[3] the pointless curve defined by the equation $x^4 + x^2y^2 + x^2yz + x^2z^2 + xy^2z + xyz^2 + y^4 + y^2z^2 + z^4 = 0$ which admits 14 points over $\mathbb{F}_4$. □

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