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Every triangle-free induced subgraph of the triangular lattice is $(5m, 2m)$-choosable

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ABSTRACT

A graph G is (a, b)-choosable if for any color list of size a associated with each vertex, one can choose a subset of b colors such that adjacent vertices are colored with disjoint color sets. This paper proves that for any integer $m \geq 1$, every finite triangle-free induced subgraph of the triangular lattice is $(5m, 2m)$-choosable.

1. Introduction

Let $G = (V(G), E(G))$ be a graph where $V(G)$ is the set of vertices and $E(G)$ is the set of edges, and let a, b, n and e be integers.

Given a list assignment L of G *i.e.* a map $L : V(G) \to \mathcal{P}(\mathbb{N})$ and a weight function w of G *i.e.* a map $w : V(G) \to \mathbb{N}$, an (L, w)-*coloring* c of G is a list assignment of the weighted graph G such that for all $v \in V(G)$,

$$c(v) \subset L(v) \quad \text{and} \quad |c(v)| = w(v),$$

and for all $vv' \in E(G)$,

$$c(v) \cap c(v') = \emptyset.$$

We say that G is (L, w)-*colorable* if there exists an (L, w)-coloring c of G. An (L, b)-coloring c of G is an (L, w)-coloring such that for all $v \in V(G)$, we have $w(v) = b$. A a-list assignment L of G is a list assignment of G such that for all $v \in V(G)$, we have $|L(v)| = a$. The graph G is said to be (a, b)-*choosable* if for any a-list assignment L of G, there exists an (L, b)-coloring c of G. If the graph is (a, b)-choosable for the a-list assignment L such that $L(v) = L(v')$ for all vertices v, v', then G is (a, b)-colorable.

The concept of choosability of a graph has been introduced by Vizing [16], and independently by Erdős, Rubin and Taylor [4]. It contains of course the colorability as a particular case. Since its introduction, choosability has been extensively studied (see for example [1,2,14,15,6,17] and more recently [7,9]). Even for the original (unweighted) version, the problem proves to be difficult, and is NP-complete for very restricted graph classes.

Every graph that is (a, b)-colorable is also trivially (am, bm)-colorable for any integer $m \geq 1$. Erdős, Rubin and Taylor [4] conjectured the following:

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Conjecture 1 (*[4]*). *The* (a, b)*-choosability of a graph* G *implies* (am, bm)*-choosability for all* $m \in \mathbb{N},\ m \geq 1$.

In relation with this question, Gutner and Tarsi [7] have recently exhibited graphs G that are (a, b)-choosable but not (c, d)-choosable, with $\frac{c}{d} > \frac{a}{b} \geq 3$.

List multicoloring problems on graphs can be used to model channel assignment problems in wireless systems. Sets of radio frequencies are to be assigned to transmitters such that adjacent transmitters are assigned disjoint sets of frequencies. Often these transmitters are laid out like vertices of a triangular lattice in a plane. This problem corresponds to the problem of multicoloring an induced subgraph of a triangular lattice with integer demands associated with each vertex. Since more than a decade, multicoloring of subgraphs of the triangular lattice has been the subject of many papers (see e.g. [12,8,10,11]). Mc Diarmid and Reed [12] have made the following conjecture when the subgraph is induced and contains no triangle:

Conjecture 2 (*[12]*). *Every triangle-free induced subgraph of the triangular lattice is* $\left(\left\lceil \frac{9b}{4} \right\rceil, b\right)$*-colorable.*

The ratio $\frac{9b}{4}$ is best possible since the cycle of length 9 is a triangle-free induced subgraph of the triangular lattice and is not (a, b)-colorable for any a such that $a/b < 9/4$. Some progress has been made regarding this conjecture with Havet [8] proving the $(5, 2)$-colorability and $(7, 3)$-colorability and Sudeep and Vishwanathan [13] (with a simpler proof) the $(14, 6)$-colorability of any triangle-free induced subgraph of the triangular lattice.

The main result is Theorem 14 of Section 3 which shows the $(5m, 2m)$-choosability of triangle-free induced subgraphs of the triangular lattice. The method is similar with that of Havet [8], that uses precoloring extensions and decomposition into induced path (called handles). However, we need here some results on the choosability of a weighted path. We find convenient to work on a type of list assignment of the path that we call *waterfall* list assignment. This is the subject of the next section. We think that the choosability results for paths presented in the next section can be of interest to derive other choosability results on triangular lattices or other graphs.

2. Waterfall list assignments on the path

We first define the similarity of two list assignments with respect to a weighted graph:

Definition 1. Let (G, w) be a weighted graph. Two list assignments L and L' are said to be *similar* if this assertion is true:

G is (L, w)-colorable $\Leftrightarrow$ G is (L', w)-colorable.

The *path* P_{n+1} of length n is the graph with vertex set $V = \{v_0, v_1, \ldots, v_n\}$ and edge set $E = \bigcup_{i=0}^{n-1}\{v_iv_{i+1}\}$. To simplify the notation, $L(i)$ denotes $L(v_i)$, $c(i)$ denotes $c(v_i)$ and $w(i)$ denotes $w(v_i)$.

By analogy with the flow of water in waterfalls, we define a waterfall list assignment as follows:

Definition 2. A *waterfall* list assignment L of a path P_{n+1} of length n is a list assignment L such that for all $i, j \in \{0, \ldots, n\}$ with $|i - j| \geq 2$, we have $L(i) \cap L(j) = \emptyset$.

Notice that another similar definition of a waterfall list assignment is that any color is present only on one list or on two lists of consecutive vertices. The list assignment L^c on the bottom right of Fig. 1 is a waterfall list assignment.

Definition 3. For a weighted path (P_{n+1}, w),

- A list assignment L is *good* if $|L(i)| \geq w(i) + w(i + 1)$ for any i, $1 \leq i \leq n - 1$.
- The *amplitude* $A(i, j)(L)$ (or $A(i, j)$) of a list assignment L is $A(i, j)(L) = \cup_{k=i}^{j} L(k)$.

As the next result shows, the condition of being a good list assignment is sufficient to allows the transformation of the list assignment into a similar waterfall list assignment. Notice that if the condition $|L(i)| \geq w(i) + w(i + 1)$ would be also required for the vertex v_0 in a good list assignment L (and if $|L(n)| \geq w(n)$), then an (L, w)-coloring could be obtained easily by greedily coloring the vertices from v_n to v_0. However we will show, using Hall's condition, that the path P_{n+1} remains (L, w)-colorable for good waterfall lists L with some additional properties (Corollary 7, Propositions 8 and 9).

Proposition 4. *For any good list assignment* L *of* P_{n+1}*, there exists a similar waterfall list assignment* L^c *with* $|L^c(i)| = |L(i)|$ *for all* $i \in \{0, \ldots, n\}$.

Proof. We are going to transform a good list assignment L of P_{n+1} into a waterfall list assignment L^c and we will prove that L^c is similar with L.

First, remark that if y is a new color ($y \notin A(0, n)(L)$), then if a color $x \in L(i - 1)$ but $x \notin L(i)$ for some i with $1 \leq i \leq n - 1$, then for any $j > i$, one can change the color x by y in the list $L(j)$, resulting in a similar list assignment. With this remark in hand, we can assume that L is such that any color x appears on the lists of consecutive vertices $i_x, \ldots, j_x$.

Now, by permuting the colors if necessary, we can assume that if $x < y$ then $i_x < i_y$ or $i_x = i_y$ and $j_x \leq j_y$.

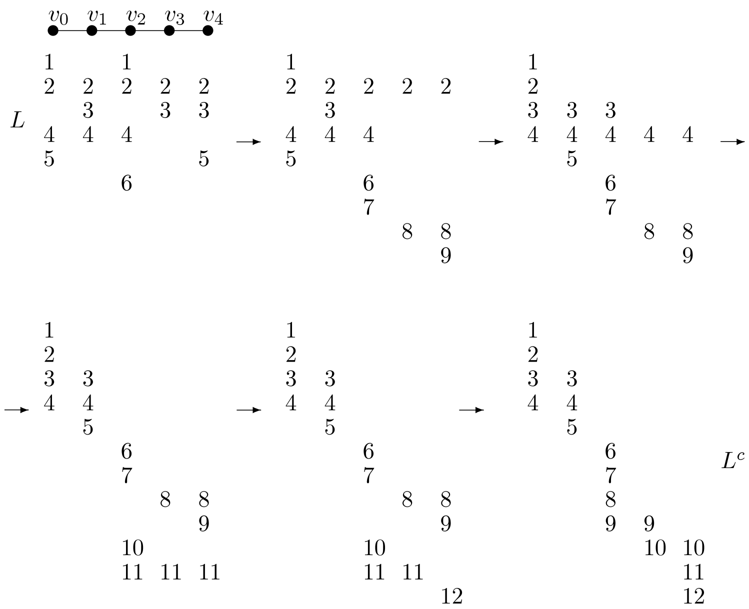

Fig. 1. Example of the transformation of a list assignment L of P_5 into a similar waterfall list assignment L^c as in proof of Proposition 4.

Repeat the following transformation:

1. Take the minimum color x for which $j_x \geq i_x + 2$, i.e. the color x is present on at least three vertices $i_x, i_x + 1, i_x + 2, \ldots, j_x$;
2. Replace color x by a new color y in lists $L(i_x + 2), \ldots, L(j_x)$;

until the obtained list assignment is a waterfall list assignment (obviously, the number of iterations is always finite).

Fig. 1 illustrates these steps, starting from a list assignment L of the path P_5. The second list assignment is after changing the non-contiguous colors (1, 3, and 5) along the path; the third one is after reordering the colors; the fourth one is after applying transformation on colors 3 and 4; the fifth one is after applying transformation on color 11 (this is already a waterfall list assignment) and the sixth one is after reordering the colors (optional).

Now, in order to show that this transformation preserves the choosability of the list assignment, we prove that the list assignment L_s obtained from the list assignment L after s iterations of the above transformation is similar with the list assignment L_{s-1} (setting $L_0 = L$), for $s \geq 1$.

If c is an (L_{s-1}, w)-coloring of P_{n+1} then the coloring c' obtained from c by changing the color x by the color y in the color set $c(k)$ of each vertex $k \geq i_x + 2$ (containing x) is an (L_s, w)-coloring since y is a new color.

Conversely, if c' is an (L_s, w)-coloring of P_{n+1}, we consider two cases:

Case 1: $x \notin c'(i_x + 1)$ or $y \notin c'(i_x + 2)$. In this case, the coloring c obtained from c' by changing the color y by the color x in the color set $c'(k)$ of each vertex $k \geq i_x + 2$ (containing y) is an (L_{s-1}, w)-coloring.

Case 2: $x \in c'(i_x + 1)$ and $y \in c'(i_x + 2)$. We have to consider two subcases:

- Subcase 1: $L_s(i_x + 1) \not\subset (c'(i_x) \cup c'(i_x + 1) \cup c'(i_x + 2))$. There exists $z \in L_s(i_x + 1) \setminus (c'(i_x) \cup c'(i_x + 1) \cup c'(i_x + 2))$ and the coloring c obtained from c' by changing the color x by the color z in $c'(i_x + 1)$ and replacing the color y by the color x in the color set $c'(k)$ of each vertex $k \geq i_x + 2$ (containing y) is an (L_{s-1}, w)-coloring.
- Subcase 2: $L_s(i_x + 1) \subset (c'(i_x) \cup c'(i_x + 1) \cup c'(i_x + 2))$. We have

$$|L_s(i_x + 1)| = \left|\big((c'(i_x)) \cup c'(i_x + 1) \cup c'(i_x + 2)\big) \cap L_s(i_x + 1)\right|.$$

As c' is an (L_s, w)-coloring of P_{n+1}, we have

$$|L_s(i_x + 1)| = |c'(i_x + 2) \cap L_s(i_x + 1)| + |c'(i_x + 1) \cap L_s(i_x + 1)| + \left|\big(c'(i_x) \backslash c'(i_x + 2)\big) \cap L_s(i_x + 1)\right|,$$

$$|L_s(i_x + 1)| - w(i_x + 1) - |c'(i_x + 2) \cap L_s(i_x + 1)| = \left|\big(c'(i_x) \backslash c'(i_x + 2)\big) \cap L_s(i_x + 1)\right|.$$

Since $y \in c'(i_x + 2)$ and $y \notin L_s(i_x + 1)$, we obtain that

$$|c'(i_x + 2) \cap L_s(i_x + 1)| \leq w(i_x + 2) - 1,$$

hence

$$(|L_s(i_x + 1)| - w(i_x + 1) - w(i_x + 2)) + 1 \leq \left|\big(c'(i_x) \backslash c'(i_x + 2)\big) \cap L_s(i_x + 1)\right|.$$

But, by hypothesis, L is a good list assignment, hence so are list assignments L_ℓ, $\ell \geq 0$. Thus $|L_{s-1}(i_x + 1)| = |L_s(i_x + 1)| \geq w(i_x + 1) + w(i_x + 2)$ and

$$1 \leq \left|\big(c'(i_x) \backslash c'(i_x + 2)\big) \cap L_s(i_x + 1)\right|.$$

Consequently, there exists $z \in \left(c'(i_x)\backslash c'(i_x+2)\right) \cap L_s(i_x+1)$. The coloring c is then constructed from c' by changing the color x by the color z in $c'(i_x+1)$, the color z by the color x in $c'(i_x)$ and the color y by the color x in the set $c'(k)$ of each vertex $k \geq i_x+2$ (containing y). □

Cropper et al. [3] consider Philip Hall's theorem on systems of distinct representatives and its improvement by Halmos and Vaughan as statements about the existence of proper list colorings or list multicolorings of complete graphs. The necessary and sufficient condition in these theorems is generalized in the new setting as "Hall's condition":

$$\forall H \subset G, \sum_{k \in C} \alpha(H, L, k) \geq \sum_{v \in V(H)} w(v),$$

where $C = \bigcup_{v \in V(H)} L(v)$ and $\alpha(H, L, k)$ is the independence number[1] of the subgraph of H induced by the vertices containing k in their color list. Notice that H can be restricted to be a connected induced subgraph of G.

It is easily seen that Hall's condition is necessary for a graph to be (L, w)-colorable. Cropper et al. [3] showed that the condition is also sufficient for some graphs, including paths:

Theorem 5 ([3]). *For the following graphs, Hall's condition is sufficient to ensure an (L, w)-coloring:*

(a) *cliques;*
(b) *two cliques joined by a cut-vertex;*
(c) *paths;*
(d) *a triangle with a path of length two added at one of its vertices;*
(e) *a triangle with an edge added at two of its three vertices.*

This result is very nice, however, it is often hard to compute the left part of Hall's condition, even for paths. However, as the next result shows, Hall's condition is very easy to check when restricted to waterfall list assignments.

Theorem 6. *Let L^c be a waterfall list assignment of a weighted path (P_{n+1}, w). Then P_{n+1} is (L^c, w)-colorable if and only if:*

$$\forall i, j \in \{0, \dots, n\}, \quad \left| \bigcup_{k=i}^{j} L^c(k) \right| \geq \sum_{k=i}^{j} w(k).$$

Proof. "if" part: Recall that $A(i,j) = \cup_{k=i}^{j} L^c(k)$. For $i, j \in \{0, \dots, n\}$, let $P_{i,j}$ be the subpath of P_{n+1} induced by the vertices $i, \dots, j$. By Theorem 5, it is sufficient to show that

$$\forall i, j \in \{0, \dots, n\}, \quad \sum_{x \in A(i,j)} \alpha(P_{i,j}, L^c, x) \geq \sum_{k=i}^{j} w(k).$$

Since L^c is a waterfall list assignment, then for each color $x \in A(i,j)$, $\alpha(P_{i,j}, L^c, x) = 1$ and thus $\sum_{x \in A(i,j)} \alpha(P_{i,j}, L^c, x) = |A(i,j)| = \left| \bigcup_{k=i}^{j} L^c(k) \right|$.

"only if" part: If c is an (L^c, w)-coloring of P_{n+1} then

$$\forall i, j \in \{0, \dots, n\} : \bigcup_{k=i}^{j} L^c(k) \supset \bigcup_{k=i}^{j} c(k).$$

Since L^c is a waterfall list assignment, it is easily seen that $\left| \bigcup_{k=i}^{j} c(k) \right| = \sum_{k=i}^{j} w(k)$. Therefore, $\forall i, j \in \{0, \dots, n\}$: $\left| \bigcup_{k=i}^{j} L^c(k) \right| \geq \sum_{k=i}^{j} w(k)$. □

Theorem 6 has the following corollary when the list assignment is a good waterfall list assignment and $|L(n)| \geq w(n)$.

Corollary 7. *Let L^c be a waterfall list assignment of a weighted path (P_{n+1}, w) such that for any i, $1 \leq i \leq n-1$, $|L^c(i)| \geq w(i) + w(i+1)$ and $|L^c(n)| \geq w(n)$. Then P_{n+1} is (L^c, w)-colorable if and only if*

$$\forall j \in \{0, \dots, n\}, \quad \left| \bigcup_{k=0}^{j} L^c(k) \right| \geq \sum_{k=0}^{j} w(k).$$

[1] The independence number of a graph is the size of the largest set of isolated vertices.

Proof. Under the hypothesis, if P_{n+1} is (L^c, w)-colorable, then Theorem 6 proves in particular the result.

Conversely, since L^c is a waterfall list assignment of P_{n+1}, we have:

$$\forall i, j \in \{1, \dots, n\}, \quad |A(i,j)| = \left|\bigcup_{k=i}^{j} L^c(k)\right| \geq \left|\bigcup_{\substack{k=i \\ k-i \text{ even}}}^{j} L^c(k)\right| = \sum_{\substack{k=i \\ k-i \text{ even}}}^{j} |L^c(k)|.$$

Since L^c is a good list assignment of P_{n+1} (for simplicity, we set $w(n+1) = 0$):

$$\forall i, j \in \{1, \dots, n\}, \quad \sum_{\substack{k=i \\ k-i \text{ even}}}^{j} |L^c(k)| \geq \sum_{\substack{k=i \\ k-i \text{ even}}}^{j} (w(k) + w(k+1)) \geq \sum_{k=i}^{j} w(k),$$

and then we obtain for all $i, j \in \{1, \dots, n\}$, $|A(i,j)| \geq \sum_{k=i}^{j} w(k)$. Since for all $j \in \{0, \dots, n\}$, $|A(0,j)| \geq \sum_{k=0}^{j} w(k)$, Theorem 6 concludes the proof. $\square$

Another interesting result holds for list assignments L such that $|L(0)| = |L(n)| = b$, and for all $i \in \{1, \dots, n-1\}$, $|L(i)| = a$. The function Even is defined for any real x by: $\text{Even}(x)$ is the smallest even integer p such that $p \geq x$.

Proposition 8. *Let L be a list assignment of P_{n+1} such that $|L(0)| = |L(n)| = b$, and $|L(i)| = a = 2b+e$ for all $i \in \{1, \dots, n-1\}$ (with $e > 0$).*

$$\text{If } n \geq \text{Even}\left(\frac{2b}{e}\right) \quad \text{then } P_{n+1} \text{ is } (L, b)\text{-colorable}.$$

Proof. The hypothesis implies that L is a good list assignment of P_{n+1}, hence by Proposition 4, there exists a waterfall list assignment L^c similar to L. So we get:

$$\forall i \in \{1, \dots, n-1\}, \quad |L^c(i)| \geq 2b = w(i) + w(i+1)$$

and $|L^c(n)| \geq b = w(n)$. By Corollary 7 it remains to prove that:

$$\forall j \in \{0, \dots, n\}, \quad |A(0,j)| \geq \sum_{k=0}^{j} w(k) = (j+1)b.$$

Case 1: $j = 0$. By hypothesis, we have $|A(0,0)| = |L^c(0)| \geq b$.

Case 2: $j \in \{1, \dots, n-1\}$. Since L^c is a waterfall list assignment of P_{n+1} we obtain that:
if j is even

$$|A(0,j)| \geq \sum_{\substack{k=0 \\ k \text{ even}}}^{j} |L^c(k)| = b + \sum_{\substack{k=2 \\ k \text{ even}}}^{j} 2b = b + \frac{j}{2} 2b = (j+1)b,$$

and if j is odd

$$|A(0,j)| \geq \sum_{\substack{k=0 \\ k \text{ odd}}}^{j} |L^c(k)| = \sum_{\substack{k=1 \\ k \text{ odd}}}^{j} 2b = \frac{j+1}{2} 2b = (j+1)b.$$

Hence for all $j \in \{0, \dots, n-1\}$, $|A(0,j)| \geq (j+1)b$.

Case 3: $j = n$. Since $n \geq \text{Even}\left(\frac{2b}{e}\right)$ by hypothesis, and

$$|A(0,n)| \geq \sum_{\substack{k=0 \\ k \text{ odd}}}^{n} |L^c(k)| = \begin{cases} a\dfrac{n}{2}, & \text{if } n \text{ is even;} \\ b + a\dfrac{n-1}{2}, & \text{otherwise;} \end{cases}$$

we deduce that $|A(0,n)| \geq (n+1)b$, which concludes the proof. $\square$

Also, we will need the following result with more restrictions on the lists of colors on the two last vertices of the path.

Proposition 9. *Let n, a, b, e be positive integers such that $a = 2b + e$ and $e < b$. Let L be a list assignment of a path P_{n+1} such that $|L(0)| = b$, and for any $i \in \{1, \dots, n-2\}$: $|L(i)| = a$, $|L(n-1)| = |L(n)| = b + e$, and $|A(n-1, n)(L)| \geq 2b$.*

$$\text{If } n = \text{Even}\left(\frac{2b}{e}\right) \quad \text{then } P_{n+1} \text{ is } (L, b)\text{-colorable}.$$

Proof. Since

$$|A(n-1,n)| = |L(n-1)| + |L(n)\backslash L(n-1)| = (b+e) + |L(n)\backslash L(n-1)| \geq 2b,$$

we obtain that $|L(n)\backslash L(n-1)| \geq b-e$. Let D be a set such that $D \subset L(n)\backslash L(n-1)$ and $|D| = b-e$.

Let L' be the new list assignment constructed with L such that $L'(i) = L(i)$ if $i \in \{0,\dots,n-1\}$ and $L'(n) = L(n)\backslash D$ and let w' be a new weight function defined by $w'(i) = b$ if $i \in \{0,\dots,n-1\}$ and $w'(n) = e$.

We are going to prove that if P_{n+1} is (L', w')-colorable then P_{n+1} is (L, b)-colorable. Indeed, if c' is an (L', w')-coloring of P_{n+1}, then we construct c such that:

$$c(i) = \begin{cases} c'(i), & \text{if } i \in \{0,\dots,n-1\}; \\ c'(n) \cup D, & \text{otherwise.} \end{cases}$$

Since $D \cap L(n-1) = \emptyset$, we have $c(n-1) \cap c(n) = \emptyset$ and then c is an (L, b)-coloring of P_{n+1}.

Now, this new list assignment L' is a good list assignment of P_{n+1} and $|L'(n)| \geq w'(n)$. Proposition 4 shows that there exists a waterfall list assignment L^c similar to L' such that for all k we have $|L^c(k)| = |L'(k)|$.

Thanks to Corollary 7, it remains to check that:

$$\forall j \in \{0,\dots,n\} : |A(0,j)(L^c)| \geq \sum_{k=0}^{j} w'(k).$$

Case 1: $j \in \{0,\dots,n-2\}$. Since the list assignment is a waterfall list assignment, we have:

$$|A(0,j)(L^c)| \geq \begin{cases} \displaystyle\sum_{\substack{k=0 \\ k \text{ even}}}^{j} |L^c(k)| = b + a\frac{j}{2}, & \text{if } j \text{ is even;} \\ \displaystyle\sum_{\substack{k=0 \\ k \text{ odd}}}^{j} |L^c(k)| = a\frac{j+1}{2}, & \text{otherwise;} \end{cases}$$

and the weight function satisfies $\sum_{k=0}^{j} w'(k) = (j+1)b$. Hence, we deduce that

$$|A(0,j)(L^c)| \geq \sum_{k=0}^{j} w'(k).$$

Case 2: $j = n-1$. Since the list assignment is a waterfall list assignment, we have:

$$|A(0,n-1)(L^c)| \geq \sum_{\substack{k=0 \\ k \text{ odd}}}^{n-1} |L^c(k)| = (b+e) + a\frac{n-2}{2},$$

and $\sum_{k=0}^{n-1} w'(k) = nb$. Then $b + e + a\frac{n-2}{2} \geq nb$ if and only if $\frac{en}{2} \geq b$, which is true by hypothesis since $n = \text{Even}\left(\frac{2b}{e}\right)$, thus

$$|A(0,n-1)(L^c)| \geq \sum_{k=0}^{n-1} w'(k).$$

Case 3: $j = n$. Since the list assignment is a waterfall list assignment, we have:

$$|A(0,n)(L^c)| \geq \sum_{\substack{k=0 \\ k \text{ even}}}^{n} |L^c(k)| = b + 2e + a\frac{n-2}{2},$$

and $\sum_{k=0}^{n} w'(k) = nb+e$. Then $b+2e+a\frac{n-2}{2} \geq nb+e$ if and only if $\frac{en}{2} \geq b$, which is true by hypothesis since $n = \text{Even}\left(\frac{2b}{e}\right)$, thus

$$|A(0,n)(L^c)| \geq \sum_{k=0}^{n} w'(k). \quad \square$$

3. Choosability of the triangular lattice

Let $\mathcal{R}$ be a finite triangle-free induced subgraph of the triangular lattice. Recall that the triangular lattice is embedded in an Euclidean space and the vertices are all integer linear combinations $x\vec{p} + y\vec{q}$ of the two vectors $\vec{p} = (1,0)$ and $\vec{q} = \left(1/2, \sqrt{3}/2\right)$. Hence, we may identify the vertices with the pairs (x, y) of integers and any vertex (x, y) of $\mathcal{R}$ has at

Fig. 2. Left node and right node.

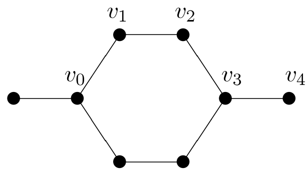

Fig. 3. Cutting handle of length 3.

most six neighbors: its neighbor on the left $(x-1, y)$, its neighbor on the right $(x+1, y)$, its neighbor on the top left $(x-1, y+1)$, its neighbor on the top right $(x, y+1)$, its neighbor on the bottom left $(x, y-1)$ and its neighbor on the bottom right $(x+1, y-1)$.

Following the terminology used in [8], we define nodes and handles of $\mathcal{R}$.

Definition 10. The nodes of $\mathcal{R}$ are the vertices of degree 3. There are two kinds of nodes: the left nodes whose neighbors are the neighbors on the left, on the top right, and on the bottom right; and the right nodes whose neighbors are the neighbors on the right, on the top left and on the bottom left (see Fig. 2).

Definition 11. A *cutting node* of $\mathcal{R}$ is a left node (x, y) such that for any node (x', y'), we have $y \geq y'$, and for any left node (x', y), we have $x' \leq x$. A *handle* H of $\mathcal{R}$ is a subpath of $\mathcal{R}$ such that its extremal vertices are nodes and its internal vertices have degree 2 in $\mathcal{R}$. The set of the internal vertices of a handle H is denoted by $\dot{H}$. A *cutting handle* of $\mathcal{R}$ is a handle H such that one of its extremal vertices is the cutting node (x, y) and $(x, y+1)$ is its neighbor in H.

Note that the end-vertices of a handle may be the same vertex (i.e. the handle is an induced cycle of $\mathcal{R}$).

We have the following easy results:

Lemma 12. *If $\mathcal{R}$ has minimum degree* 2 *and at least a node, then it has a cutting handle.*

Proof. First, we can suppose that $\mathcal{R}$ contains a cutting node v_0 (by considering the mirror graph if there is no left node among the nodes with highest ordinate). Second, as the vertices are of degree 2 or 3, the number of nodes is even and there are at least two nodes. Then the induced path starting at v_0, crossing $v_1 = (x, y+1)$ and continuing along the induced path of $\mathcal{R}$ (which has minimum degree 2) until a node v_n is reached is a cutting handle. $\square$

Lemma 13. *Let H be a cutting handle of $\mathcal{R}$ such that $V(H) = \{v_0, \ldots, v_n\}$. If the length n of H is less than or equal to* 3*, then $n = 3$ and v_3 has a neighbor $v_4 \neq v_2$ of degree less than or equal to* 2*.*

Proof. Let $v_0 = (x, y)$ be the cutting node of H. Then, by definition, $v_1 = (x, y+1)$.

If $n = 1$ then v_1 is a right node with higher ordinate than v_0, which contradicts the hypothesis that v_0 is a cutting node.

If $n \geq 2$ then v_2 can be neither $(x+1, y)$ nor $(x-1, y+1)$ since otherwise v_0, v_1, v_2 would form a triangle in $\mathcal{R}$. Hence if $n = 2$, then v_2 is a node with a higher ordinate than v_0, which contradicts the hypothesis.

Then $n = 3$ and $v_3 = (x+2, y)$ has the same ordinate than v_0 and is a right node. We are thus in the configuration depicted in Fig. 3. The vertex $v_4 = (x+3, y)$ cannot be a node since otherwise it would be a left node on the right from v_0, which, again, contradicts the hypothesis. Hence v_4 has degree one or two. $\square$

Theorem 14. *For any integer $m \geq 1$, every finite triangle-free induced subgraph of the triangular lattice is $(5m, 2m)$-choosable.*

Proof. Set $a = 5m$, $b = 2m$ and $e = m$. Let G be a minimal (with respect to the number of vertices) counter-example. It is easily seen that G is connected and has no vertex of degree 0 or 1. Then G has only vertices of degree 2 or 3. Suppose G has only vertices of degree 4 Also, G cannot be reduced to a simple cycle since it has girth[2] at least 6 and since cycles of even length are $(2m, m)$-choosable and cycles of odd length $\ell > 3$ are $(5m, 2m)$-choosable [18]. Therefore G has at least one node and, by Lemma 12, G has a cutting handle H. By hypothesis, for any $5m$-list assignment L of G, there exists an $(L, 2m)$-coloring c of $G - \dot{H}$. If H is of length $n \geq 4 = \mathrm{Even}\left(\frac{4m}{m}\right)$, by Proposition 8, c can be extended to an $(L, 2m)$-coloring

2 The length of a shortest (simple) cycle in the graph.

of G, contradicting the hypothesis. Otherwise, by Lemma 13, H has length 3 and v_3 has a neighbor $v_4 \neq v_2$ of degree 2 (let v_2' be the other neighbor of v_3 and let v_5 be the other neighbor of v_4). Hence we have $|L(v_4) \setminus c(v_5)| \geq 5m - 2m = 3m = b + e$, and $|L(v_3) \setminus c(v_2')| = 5m - 2m = b + e$. Moreover, $|L(v_3) \setminus c(v_2') \cup L(v_4) \setminus c(v_5)| \geq |c(v_3) \cup c(v_4)| = 2b$ since c is an $(L, 2m)$-coloring of $G - \dot{H}$. Then, $H \cup \{v_4\}$ satisfies the conditions of Proposition 9 and the coloring c restricted to $G - (\dot{H} \cup \{v_3, v_4\})$ can be extended to an $(L, 2m)$-coloring of G, contradicting the hypothesis. □

Notice that the above proof can be easily translated to show the more general result that for any integers a, b such that $a/b \geq 5/2$, every finite triangle-free induced subgraph of the triangular lattice is (a, b)-choosable. As mentioned in the Introduction, this extension is not true for all graphs.

Also, using similar arguments with some additional results on the list-colorability of the path [5], allows us to show that triangle-free induced subgraphs of the triangular lattice are $(7, 3)$-colorable, thus giving another proof of Havet's result and for the $(7, 3)$-choosability of such graphs, Proposition 8 permits us to treat handles of length at least 6 and Proposition 9 to treat a handle of length 5 with an end-vertex that has a neighbor of degree 2.

Moreover, the method defined in this paper can serve as a starting tool in order to prove Conjecture 2. Proceeding as in the proof of Theorem 14 but with $a = 9$ and $b = 4$ (and $e = a - 2b = 1$), allows us to prove that a minimal counter-example to the conjecture does not contain handles of length $n \geq 8$ (and also no handles of length 7 with an end-vertex that has a neighbor of degree 2). By using more complex structures than handles, some additional properties of a minimal counter-example were found by Godin [5]. However, many configurations remain to be investigated in order to prove the conjecture.

Remark 15. As noticed by a referee, for $a = 5m$, $b = 2m$ and $e = m$, the coloring of a handle of length $n \geq 4$ can be done directly, without using Proposition 8: color the vertices one by one from one end of the path until there remain only the three vertices v_1, v_2, v_3 with lists $|L(v_1) \setminus c(v_0)| = |L(v_3) \setminus c(v4)| = 3m$ and $|L(v_2)| = 5m$. Color v_2 by m colors from $L(v_2) \setminus (L(v_1) \setminus c(v_0))$ and m further colors from $L(v_2) \setminus (L(v_3) \setminus c(v4))$. However, we choose to keep Proposition 8 since it can be of interest to prove other coloring results. For instance if $a = 9m$ and $b = 4m$ (and $e = a - 2b = m$), finding directly a coloring of a handle of length 8 seems not so easy.

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