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On the characteristic polynomials of the Frobenius endomorphism for projective curves over finite fields

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Abstract

We give a formula for the number of rational points of projective algebraic curves defined over a finite field, and a bound "à la Weil" for connected ones. More precisely, we give the characteristic polynomials of the Frobenius endomorphism on the étale ℓ-adic cohomology groups of the curve. Finally, as an analogue of Artin's holomorphy conjecture, we prove that, if $Y \longrightarrow X$ is a finite flat morphism between two varieties over a finite field, then the characteristic polynomial of the Frobenius morphism on $H^i_c(X, \mathbf{Q}_\ell)$ divides that of $H^i_c(Y, \mathbf{Q}_\ell)$ for any i. We are then enable to give an estimate for the number of rational points in a flat covering of curves.

MSC: 11G20; 14G10; 14G15

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1. Introduction

Absolutely reducible projective curves arise naturally in different ways in Arithmetic and Geometry. For example when we reduce, modulo a prime, a

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projective curve defined over a number field, or when we consider intersections of projective varieties.

We are interested in this paper in the number of rational points of such a curve X defined over a finite field k. It is convenient to introduce the zeta function of X, denoted by $Z_{k,X}(T)$ or simply $Z_X(T)$, as

$$Z_X(T) = \exp\left(\sum_{n=1}^{\infty} \sharp X(k_n)\frac{T^n}{n}\right),$$

where k_n denotes the finite field extension of degree n of k. Let q be the order of k, and F be the endomorphism on the ℓ-adic étale cohomology groups $H^i_{\mathrm{et}}(\bar{X}, \mathbf{Q}_\ell)$ of X (for some prime ℓ different from the characteristic of k and $\bar{X}$ a geometric model of X) induced by the endomorphism $x \longmapsto x^q$. The Grothendieck–Lefschetz formula expresses this zeta function as a rational fraction in terms of the (reciprocal) characteristic polynomials of F on the H^i's

$$Z_X(T) = \frac{\det(I - TF \mid H^1_{\mathrm{et}}(\bar{X}, \mathbf{Q}_\ell))}{\det(I - TF \mid H^0_{\mathrm{et}}(\bar{X}, \mathbf{Q}_\ell)) \det(I - TF | H^2_{\mathrm{et}}(\bar{X}, \mathbf{Q}_\ell))}.$$

In other words, the number of k_n-rational points of X equals

$$\sharp X(k_n) = \sum \alpha^n_{2,j} - \sum \alpha^n_{1,j} + \sum \alpha^n_{0,j},$$

where the $\alpha_{i,j}$'s are the eigenvalues of F on $H^i_{\mathrm{et}}(\bar{X}, \mathbf{Q}_\ell)$. The aim of this paper is to determine them. Consider for instance a k-irreducible projective curve X having two absolutely irreducible components X_1 and X_2 defined over k_2, conjugated under $\mathrm{Gal}(k_2/k)$. It is easily seen that $X_1 \cap X_2$ is defined over k, and an elementary counting argument shows that

$$\sharp X(k_n) = \begin{cases} \sharp X_1(k_n) + \sharp X_2(k_n) - \sharp X_1 \cap X_2(k_n) & \text{if } n \text{ is even,} \\ \sharp X_1 \cap X_2(k_n) & \text{if } n \text{ is odd.} \end{cases}$$

It is not clear what could be these numbers $\alpha_{i,j}$ (whose existence follows from the above Grothendieck–Lefschetz formula) summing-up these two-case formulae into a closed one (see Example 2). This will be done in the general case.

In the general reducible case, if $X = X_1 \cup \cdots \cup X_r$ is a decomposition of X into its k-irreducible components, it is enticing to compute $\sharp X(k_n)$ using the well-known inclusion–exclusion formula in terms of the jth intersections $X_{i_1} \cap \cdots \cap X_{i_j}$. In fact, this approach is not effective. Indeed, we obtain the eigenvalues $\alpha_{i,j}$'s unfortunately only up to roots of unity (Theorem 1). However, this is sufficient to deduce a Weil inequality (Corollary 3).

We use in the next section the cohomological approach to determine, without any indetermination, the eigenvalues of the Frobenius on X in terms of the eigenvalues of the Frobenius of the normalizations of the absolutely irreducible components of $\bar{X}$, the (finite) set of singular points of these absolutely irreducible components, and the (finite) set of intersection points of these components (Theorems 9–11). In view of these results, we point out that the contributions of these finite sets are very easy to handle, as shown by Lemma 8. Moreover, the multiple intersections between the absolutely irreducible components do not appear in the results (see Example 3), which is nice, both for theoretical and computational approaches.

Finally, we consider in the final section the behaviour of the eigenvalues of the Frobenius in a covering $Y \longrightarrow X$ of d-dimensional ($d \geqslant 1$) non-proper varieties. In analogy with a conjecture of Artin, we prove a divisibility result for such finite flat morphisms (Corollary 13). Together with the results of the preceding section, this enables us to derive some upper bound for the number of points if X and Y are projective curves in a surjective flat morphism (Theorem 14), including the known results as special cases.

Because the cohomology groups of a disjoint union of varieties is the direct sum as F-modules of those of its connected components, the characteristic polynomial of the Frobenius is the product of those on each component. Hence, we will restrict ourselves to connected curves.

Let us fix some notations. If V is a scheme over a field k, we denote by $|V|$ the set of closed points of V, by $k(P)$ the residue field of a point $P \in |V|$, and by $d_P = [k(P) : k]$ the degree of P over k. In this paper, k will always be the finite field with q elements and $\bar{k}$ an algebraic closure of k. The normalization map is denoted by $v_V : \tilde{V} \longrightarrow V$ and we denote by $\bar{V} = V \times_k \bar{k}$ the extension of V to $\bar{k}$. We set π_V for the arithmetic genus of V, g_V for its geometric genus and $\Delta_V = \sharp(\tilde{V}(\bar{k}) - V(\bar{k}))$.

For simplicity, we denote by $H^i(V)$ (respectively $H^i_c(V)$) the ith ℓ-adic étale cohomology group (resp. with compact support) $H^i_{et}(\bar{V}, \mathbf{Q}_\ell)$ (resp. $H^i_c(\bar{V}, \mathbf{Q}_\ell)$) of V. Then, we denote by

$$P_{k,H^i(V)}(T)$$

the characteristic polynomial $\det(I - TF \mid H^i(V))$ of the Frobenius endomorphism F of the variety V over the field k. We use the same notation, but with a subscript "c", when we deal with cohomology with compact support.

Some varieties $\bar{V}$ will naturally be introduced over $\bar{k}$. They will be denoted with an overline. When it will be proved that they can be defined over the finite field extension k_n of k, we will denote by V (without overline) the variety over k_n such that $\bar{V} = V \times_{k_n} \bar{k}$.

2. A counting approach

Let us remark that if $\{V_i\}$ is a finite covering of a variety V defined over k by subvarieties defined over k, then the following inclusion–exclusion formula

$$\sharp V(k_n) = \sum_{j \geqslant 1} (-1)^{j+1} \sum_{i_1 < \cdots < i_j} \sharp(V_{i_1} \cap \cdots \cap V_{i_j})(k_n)$$

gives

$$Z_{k,V}(T) = \prod_{j\geqslant 1} \prod_{i_1<\cdots<i_j} Z_{V_{i_1}\cap\cdots\cap V_{i_j}}(T)^{(-1)^{j+1}}.$$

Theorem 1. *Let X be a connected projective curve defined over k and $\bar{X} = \bar{X}_1 \cup \cdots \cup \bar{X}_{\bar{r}}$ be its decomposition into its $\bar{k}$-irreducible components. Then the number of rational points of X over k_n is of the form*

$$\sharp X(k_n) = \sum_{i=1}^{\bar{r}} \rho_i^n - \sum_{i=1}^{\bar{r}} \sum_{j=1}^{2g_{\tilde{X}_i}} \omega_{i_j}^n - \sum_{i=1}^{\Delta_X - \bar{r}} \beta_i^n$$

for some algebraic integers ρ_i of modulus q, some algebraic integers ω_{i_j} of modulus $\sqrt{q}$ and some roots of unity β_i in $\mathbf{C}$.

Proof. Let us assume to begin with that all absolutely irreducible components of X, so as all its singular points and the points above them by the normalization map, are rational over k. We set $Z = \bigcup_{i<j}(X_i \cap X_j)$ and $Z_i = Z \cap X_i$ as k-varieties (overlines for X_i's can be dropped thanks to the last paragraph of the introduction).

Then, the inclusion–exclusion formula applied to $X = \bigcup_i X_i$ gives

$$Z_{k,X}(T) = \prod_{j\geqslant 1} \prod_{i_1<\cdots<i_j} Z_{X_{i_1}\cap\cdots\cap X_{i_j}}(T)^{(-1)^{j+1}}$$

and then applied to $Z = \bigcup_i Z_i$ gives

$$Z_{k,Z}(T) = \prod_{j\geqslant 1} \prod_{i_1<\cdots<i_j} Z_{Z_{i_1}\cap\cdots\cap Z_{i_j}}(T)^{(-1)^{j+1}}.$$

Remarking that $Z_{i_1} \cap \cdots \cap Z_{i_j} = X_{i_1} \cap \cdots \cap X_{i_j}$ for $j \geqslant 2$, we obtain

$$Z_{k,X}(T) = \prod_{i=1}^{\bar{r}} Z_{k,X_i}(T) \times \frac{Z_{k,Z}(T)}{\prod_i Z_{k,Z_i}(T)}.$$

Since the X_i's are absolutely irreducible curves, we know by [1] that their zeta function are given by

$$Z_{k,X_i}(T) = \frac{P_{X_i/\widetilde{X_i}}(T) P_{k,H^1(\widetilde{X_i})}(T)}{(1-T)(1-qT)},$$

where the polynomial $P_{k,H^1(\widetilde{X_i})}(T)$ has degree $2g_{X_i}$, i.e. twice the geometric genus of X_i and has root of modulus $\sqrt{q}$ by the Riemann hypothesis and $P_{X_i/\tilde{X}_i}(T)$ is the following polynomial of degree Δ_{X_i} whose roots have modulus 1

$$P_{X_i/\widetilde{X_i}}(T) = \prod_{P\in|X|}\left(\frac{\prod_{\tilde{P}\in\nu_X^{-1}(P)}(1-T^{d_{\tilde{P}}})}{1-T^{d_P}}\right).$$

Under our rationality assumptions, we have here: $P_{X_i/\widetilde{X_i}}(T) = (1-T)^{\Delta_{X_i}}$.

Moreover, for any zero-dimensional algebraic set V defined over k all of whose closed points are rational over k, we have clearly

$$Z_V(T) = \frac{1}{(1-T)^{\sharp V(k)}}.$$

Thus $Z_{k,X}(T)$ can be written as

$$Z_{k,X}(T) = \frac{\prod_{i=1}^{\bar{r}}\ (P_{X_i/\widetilde{X_i}}(T)P_{k,H^1(\widetilde{X_i})}(T)) \times (1-T)^{(\sum_{i=1}^{\bar{r}}\ \sharp Z_i(k))-\sharp Z(k)-\bar{r}}}{(1-qT)^{\bar{r}}}.$$

Hence

$$Z_{k,X}(T) = \frac{\left(\prod_{i=1}^{\bar{r}}\ P_{k,H^1(\widetilde{X_i})}(T)\right) \times (1-T)^{\Delta_X-\bar{r}}}{(1-qT)^{\bar{r}}},$$

since

$$\Delta_X = \left(\sum_{i=1}^{\bar{r}}\ \Delta_{X_i}\right) + \left(\sum_{i=1}^{\bar{r}}\ \sharp Z_i(k)\right) - \sharp Z(k).$$

This means that

$$\sharp X(k_n) = \bar{r}q^n - \sum_{i=1}^{\bar{r}}\ \sum_{j=1}^{2g_{\tilde{X}_i}} \omega_{i_j}^n - \sum_{i=1}^{\Delta_X-\bar{r}} 1^n,$$

so that the theorem is proved in this case.

In the general case, the well-known formula

$$\prod_{\zeta^m=1} Z_{k,X}(\zeta T) = Z_{k_m,X\times_k k_m}(T^m)$$

holding for any $m\in\mathbf{N}^*$, proves that the absolute values of the zeros and poles of $Z_{k,X}(T)$ are some mth roots of the zeros and poles of $Z_{k_m,X\times_k k_m}(T)$. Hence, the general case follows from the particular one after a suitable base-field extension k_m of k, and the theorem is proved. □

Lemma 2. *Let X be a connected projective curve defined over k of arithmetic genus π_X, and $\bar{X} = \bar{X}_1 \cup \cdots \cup \bar{X}_{\bar{r}}$ be its decomposition into $\bar{k}$-irreducible projective curves $\bar{X}_i$ of geometric genus g_{X_i}. Let $\bar{c}$ be the number of absolutely connected components of $\bar{X}$. Then, we have*

$$\Delta_X \leqslant \pi_X - \sum_{i=1}^{\bar{r}} g_{\bar{X}_i} + \bar{r} - \bar{c}.$$

Proof. Since the problem is geometric, we can work on the algebraic closure $\bar{k}$ of k. If P is a closed point of $\bar{X}$, let $\mathcal{O}_{P,\bar{X}}$ be the local ring of $\bar{X}$ at P, $\mathrm{Frac}(\mathcal{O}_{P,\bar{X}})$ be the localization of $\mathcal{O}_{P,\bar{X}}$ at the multiplicative set of non-zero divisors of $\mathcal{O}_{P,\bar{X}}$ and $\overline{\mathcal{O}_{P,\bar{X}}}$ be its integral closure in $\mathrm{Frac}(\mathcal{O}_{P,\bar{X}})$. Define

$$\delta(P, \bar{X}) = \dim_{\bar{k}} \frac{\overline{\mathcal{O}_{P,\bar{X}}}}{\mathcal{O}_{P,\bar{X}}}.$$

We have the following short exact sequence:

$$0 \longrightarrow \mathcal{O}_{\bar{X}} \longrightarrow (v_{\bar{X}})_* \mathcal{O}_{\tilde{\bar{X}}} \longrightarrow (v_{\bar{X}})_* \mathcal{O}_{\tilde{\bar{X}}} / \mathcal{O}_{\bar{X}} \longrightarrow 0,$$

where $\mathcal{O}_{\bar{X}}$ is the structure sheaf of $\bar{X}$ and $(v_{\bar{X}})_* \mathcal{O}_{\tilde{\bar{X}}}$ the direct image sheaf.

Then the long exact sequence in cohomology associated implies (taking into account that $\dim H^0(\mathcal{O}_{\bar{X}}) = \bar{c}$, that $\dim 0((v_{\bar{X}})_* \mathcal{O}_{\tilde{\bar{X}}}) = \bar{r}$ and finally that $\dim H^1((v_{\bar{X}})_* \mathcal{O}_{\tilde{\bar{X}}} / \mathcal{O}_{\bar{X}}) = 0$):

$$\pi_X = \sum_{i=1}^{\bar{r}} g_{X_i} + \sum_{P \in \bar{X}} \delta(P, \bar{X}) - \bar{r} + \bar{c}.$$

We are then reduced to prove the following inequality:

$$\Delta_X \leqslant \sum_{P \in \bar{X}} \delta(P, \bar{X}).$$

If $P \in X(\bar{k})$, let $\alpha(P, \bar{X})$ be the number of closed points in $v_{\bar{X}}^{-1}(P)$. We have to prove that

$$\alpha(P, \bar{X}) - 1 \leqslant \delta(P, \bar{X}).$$

The total fraction ring of $\mathcal{O}_{P,\bar{X}}$ is isomorphic to the direct product $\bar{k}(\bar{X}_1) \times \cdots \times \bar{k}(\bar{X}_r)$ of the function fields of the irreducible components of $\bar{X}$. The integral closure $\overline{\mathcal{O}_{P,\bar{X}}}$ in it is then isomorphic to the direct product of the integral closures $\overline{\mathcal{O}_{P,\bar{X}_i}}$ of the domains $\mathcal{O}_{P,\bar{X}_i} \subset \bar{k}(\bar{X}_i)$. But each $\overline{\mathcal{O}_{P,\bar{X}_i}}$ is a semi-local ring

$$\overline{\mathcal{O}_{P,\bar{X}_i}} = \bigcap_{P \leftarrow \tilde{P}_{i_j} \in \widetilde{\bar{X}}_i} \mathcal{O}_{\tilde{P}_{i_j},\bar{X}_i}.$$

Let $1 \leqslant i \leqslant \bar{r}$ be fixed. We introduce the evaluation map on the points of $\widetilde{\bar{X}}_i$ lying over P:

$$\begin{array}{rccc} \phi_i: & \overline{\mathcal{O}_{P,\bar{X}_i}} & \longrightarrow & \bar{k}^{\alpha(P,\bar{X}_i)} \\ & f & \longmapsto & (f(\tilde{P}_{i_1}), \ldots, f(\tilde{P}_{i_{\alpha(P,\bar{X}_i)}})). \end{array}$$

Note that if $P \notin \bar{X}_i$ then $\mathcal{O}_{P,\bar{X}_i} = \bar{k}(\bar{X}_i) = \overline{\mathcal{O}_{P,\bar{X}_i}}$ and $\alpha(P, \bar{X}_i) = 0$, so that the map $\phi_i : \bar{k}(\bar{X}_i) \longrightarrow \bar{k}^0 = \{0\}$ is the zero map.

This is a $\bar{k}$-linear map which is surjective thanks to the weak approximation theorem for the global field $\bar{k}(\widetilde{\bar{X}}_i)$. They fit together in a surjective $\bar{k}$-linear map

$$\Phi : \overline{\mathcal{O}_{P,\bar{X}}} = \overline{\mathcal{O}_{P,\bar{X}_1}} \times \cdots \times \overline{\mathcal{O}_{P,\bar{X}_{\bar{r}}}} \longrightarrow \prod_{i=1}^{r} \bar{k}^{\alpha(P,\bar{X}_i)} = \bar{k}^{\sum \alpha(P,\bar{X}_i)} = \bar{k}^{\alpha(P,\bar{X})}$$

sending $f = (f_1, \ldots, f_{\bar{r}})$ to

$$(f_1(\tilde{P}_{1_1}), \ldots, f_1(\tilde{P}_{1_{\alpha(P,\bar{X}_1)}}), \ldots, f_{\bar{r}}(\tilde{P}_{\bar{r}_1}), \ldots, f_{\bar{r}}(\tilde{P}_{\bar{r}_{\alpha(P,\bar{X}_{\bar{r}})}})),$$

which sends $\mathcal{O}_{P,\bar{X}}$ onto the diagonal line. The inequality is then proved and so is the lemma. $\square$

Theorem 1 together with Lemma 2 admit for example the following corollary:

Corollary 3. *Let X be an absolutely connected projective curve defined over k and $X = X_1 \cup \cdots \cup X_r$ be its decomposition into k-irreducible projective curves which are absolutely irreducible. Then*

$$|\sharp X(k) - (rq + 1)| \leqslant 2 \sum_{i=1}^{r} g_{X_i} \sqrt{q} + \Delta_X - r + 1 \leqslant 2\pi_X \sqrt{q}.$$

Proof. We write the formula of Theorem 1 for the number of k-rational points with $\rho_i = q$ for any i. Then, taking modulus we get the first inequality, and the second inequality follows from Lemma 2. $\square$

Remark that we can improve as in [10] the inequalities of the preceding corollary by replacing $2\sqrt{q}$ by its integer part.

3. Frobenius on the cohomology

Let us begin by some lemmas, which will be useful later.

3.1. Two lemmas

Lemma 4. *Let X be a projective curve defined over k, and $Z \subset X$ be a non-empty zero-dimensional subvariety defined over k. Let $U = X - Z$. Then $P_{k,H^0(X)}(T)$ divides $P_{k,H^0(Z)}(T)$ and*

$$P_{k,H^1_c(U)}(T) = P_{k,H^1(X)}(T)\frac{P_{k,H^0(Z)}(T)}{P_{k,H^0(X)}(T)}.$$

Proof. Since $H^1(Z) = 0$ for a zero-dimensional scheme, and $H^0_c(U) = 0$ for a non-proper variety U, the following long exact sequence of F-modules (see [9, Remark 1.30, p. 94]):

$$\cdots \longrightarrow H^{i-1}_c(Z) \longrightarrow H^i_c(U) \longrightarrow H^i_c(X) \longrightarrow H^i_c(Z) \longrightarrow \cdots \tag{1}$$

becomes

$$0 \longrightarrow H^0(X) \longrightarrow H^0(Z) \longrightarrow H^1_c(U) \longrightarrow H^1(X) \longrightarrow 0.$$

The lemma follows taking characteristic polynomials of the Frobenius. □

Lemma 5. *Let V be an irreducible (respectively connected) k-variety, and suppose that*

$$\bar{V} = \bar{V}_1 \cup \cdots \cup \bar{V}_m$$

for some disjoint absolutely irreducible (respectively, absolutely connected) subvarieties $\bar{V}_i$ over $\bar{k}$. Then, $\bar{V}_1, \ldots, \bar{V}_m$ are defined over k_m, are conjugated under $\mathrm{Gal}(k_m/k)$, *and*

$$P_{k,H^i_c(V)}(T) = P_{k_m,H^i_c(V_1)}(T^m).$$

Proof. Let k_n be the smallest extension of k, in which each V_i, $1 \leqslant i \leqslant m$, are defined. Then $\mathrm{Gal}(k_n/k)$ acts on the set $\{V_1, \ldots, V_m\}$. The union of those V_i's in an orbit for this action is defined over k, and is irreducible (resp. connected) over k. Since V is irreducible (resp. connected) by assumption, this action is transitive, so that $n \geqslant m$. On the other side, each V_i is defined over the fixed field of k_n by the common stabilizer. By minimality of n, this stabilizer is trivial, hence $n = m$, which proves the first and the second assertions of the lemma.

Now, the disjointness of the V_i's implies that

$$H^i_c(V) = H^i_c(V_1) \oplus \cdots \oplus H^i_c(V_m)$$

as vector spaces. We just saw that, up to a labelling, F cyclically permutes $V_1, \dots, V_m$. Let $\mathscr{B}_1 = \{e_1, \dots, e_b\}$ be a basis of $H^i_c(V_1)$, so that $\mathscr{B}_k = \{F^{k-1}(e_1), \dots, F^{k-1}(e_b)\}$ is a basis of $H^i_c(V_k)$. In the basis $\mathscr{B} = \mathscr{B}_1 \cup \cdots \cup \mathscr{B}_m$ of $H^i_c(V)$, the matrix of F is

$$\mathrm{Mat}_{\mathscr{B}}(F \mid H^i_c(V)) = \begin{pmatrix} 0 & 0 & \cdots & \cdots & A \\ I & 0 & \cdots & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I & 0 \end{pmatrix}$$

for some matrix $A \in M_b(\mathbf{Q}_\ell)$. Hence,

$$\mathrm{Mat}_{\mathscr{B}}(\varphi_{q^m} \mid H^i_c(V)) = \mathrm{Mat}_{\mathscr{B}}(F \mid H^i_c(V))^m = \begin{pmatrix} A & 0 & \cdots & 0 \\ 0 & A & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A \end{pmatrix};$$

but the matrix $\mathrm{Mat}_{\mathscr{B}}(\varphi_{q^m} \mid H^i_c(V))$ also equals

$$\begin{pmatrix} \mathrm{Mat}_{\mathscr{B}_1}(\varphi_{q^m} \mid H^i_c(V_1)) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathrm{Mat}_{\mathscr{B}_m}(\varphi_{q^m} \mid H^i_c(V_m)) \end{pmatrix},$$

so that $A = \mathrm{Mat}_{\mathscr{B}_1}(F^m \mid H^i_c(V_1))$. Now, the lemma follows from the easy fact that, if $A \in \mathscr{M}_b(\mathbf{Q}_\ell)$ and I is the identity matrix in $\mathscr{M}_b(\mathbf{Q}_\ell)$ then

$$\det \begin{pmatrix} I & 0 & \cdots & 0 & -TA \\ -TI & I & \ddots & & 0 \\ 0 & -TI & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & I & 0 \\ 0 & \cdots & 0 & -TI & I \end{pmatrix} = \det(I - T^m A). \qquad \square$$

3.2. The zeroth and second cohomology groups

Proposition 6. *Let X be a connected projective algebraic curve defined over k. Let $\bar{c}$ be the number of connected components of $\bar{X}$. Then these connected components are defined over $k_{\bar{c}}$, are conjugate under* $\mathrm{Gal}(k_{\bar{c}}/k)$, *and*

$$P_{k,H^0(X)}(T) = 1 - T^{\bar{c}}.$$

Proof. This follows immediately from Lemma 5 and the fact that

$$P_{k_{\bar{c}},H^0(X_1)}(T) = 1 - T$$

for an absolutely connected component X_1 of X, defined over $k_{\bar{c}}$. $\square$

Proposition 7. *Let X be a connected projective curve defined over k, and let*

$$\bar{X} = \bar{\mathscr{X}}_1 \cup \cdots \cup \bar{\mathscr{X}}_{\bar{c}}$$

be the decomposition of $\bar{X}$ into its absolutely connected components. The $\bar{\mathscr{X}}_i$'s are defined over $k_{\bar{c}}$ and conjugated under $\mathrm{Gal}(k_{\bar{c}}/k)$. *Let*

$$\mathscr{X}_1 = X_1 \cup \cdots \cup X_r$$

be the decomposition of $\mathscr{X}_1$ into its $k_{\bar{c}}$-irreducible components and let

$$\bar{X}_i = \bar{X}_{i,1} \cup \cdots \cup \bar{X}_{i,\overline{r_i}}$$

be the decomposition of $\bar{X}_i$ into absolutely irreducible components $\bar{X}_{i,j}$, $1 \leqslant i \leqslant r, 1 \leqslant j \leqslant \overline{r_i}$.

Then, $\bar{X}_{i,1}, \ldots, \bar{X}_{i,\overline{r_i}}$ are defined over $k_{\bar{c}\cdot\overline{r_i}}$, are conjugate under $\mathrm{Gal}(k_{\bar{c}\cdot\overline{r_i}}/k)$, *and*

$$P_{k,H^2(X)}(T) = \prod_{i=1}^{r} (1 - (qT)^{\bar{c}\cdot\overline{r_i}}).$$

Proof. Lemma 5 says that the $\bar{\mathscr{X}}_i$'s are defined over $k_{\bar{c}}$ and conjugated under $\mathrm{Gal}(k_{\bar{c}}/k)$ and that

$$P_{k,H^2(X)}(T) = P_{k_{\bar{c}},H^2(\mathscr{X}_1)}(T^{\bar{c}}). \tag{2}$$

Let

$$\bar{Z}_i = \bigcup_{1 \leqslant j < k \leqslant \bar{r}_i} (\bar{X}_{i,j} \cap \bar{X}_{i,k}).$$

This variety is the extension to $\bar{k}$ of a (zero-dimensional)-variety Z_i defined over $k_{\bar{c}}$, so that $X_i - Z_i$ is defined over $k_{\bar{c}}$. Let $\bar{Z}_{i,j} = \bar{X}_{i,j} \cap \bar{Z}_i$. Lemma 5 for the disjoint decomposition

$$\bar{X}_i - \bar{Z}_i = \bigcup_{j=1}^{\bar{r}_i} (\bar{X}_{i,j} - \bar{Z}_{i,j})$$

proves that the $\bar{X}_{i,j} - \bar{Z}_{i,j}$ are defined over $k_{\bar{c}\cdot\bar{r}_i}$ and conjugated under $\mathrm{Gal}(k_{\bar{c}\cdot\bar{r}_i}/k)$. Hence, this is also the case for their completions $\bar{X}_{i,j}$.

Let Z' be the algebraic set

$$Z' = \bigcup_{i \neq j} (X_i \cap X_j)$$

and $Z'_i = Z' \cap X_i$. Then Z' and Z'_i are obviously defined over $k_{\bar{c}}$.

Now, the exact sequence (1) for closed subschemes, together with Mayer–Vietoris sequence and the fact that $H^1(Z') = 0$ for the finite subscheme of intersections points of the X_i's, imply that $H^2(\mathscr{X}_1) = H^2(\mathscr{X}_1 - Z') = \oplus_{i=1}^r H_c^2(X_i - Z'_i) = \oplus_{i=1}^r H^2(X_i)$ as a direct sum of F-modules. Thus,

$$P_{k_{\bar{c}}, H^2(\mathscr{X}_1)}(T) = \prod_{i=1}^{r} P_{k_{\bar{c}}, H_c^2(X_i)}(T).$$

The last part of the proposition follows from (2) and the fact that

$$P_{k_{\bar{c}}, H_c^2(X_i)}(T) = 1 - (q^{\bar{c}} T)^{\bar{r}_i}. \qquad \square$$

3.3. *The first cohomology group*

We will give in this section the characteristic polynomial on the first cohomology group of a connected projective curve X over k depending only on the characteristic polynomial for the smooth models of the absolutely irreducible components of X, the singular points of the absolutely irreducible components of X, and on the 0-dimensional subvariety of pairwise intersections of the irreducible components of $\bar{X}$.

This will be done by successive reductions, starting from the smooth absolutely irreducible case to the absolutely irreducible one (Theorem 9), then from the absolutely irreducible case to the irreducible one (Theorem 10), and finally from the irreducible case to the connected one (Theorem 11). Let us begin with the following trivial consequence of Lemma 5.

Lemma 8. *Let Z be a* 0*-dimensional algebraic set defined over k. Then*

$$P_{k, H^0(Z)}(T) = \prod_{P \in |Z|} (1 - T^{d_P}).$$

Theorem 9. *Let X be an absolutely irreducible projective curve defined over k, with normalization map $v_X : \tilde{X} \longrightarrow X$. Then, we have*

$$P_{k, H^1(X)}(T) = P_{k, H^1(\tilde{X})}(T) \prod_{P \in |X|} \frac{\prod_{v_X(\tilde{P}) = P} (1 - T^{d_{\tilde{P}}})}{(1 - T^{d_P})}.$$

Proof. We can assume that X is singular, because otherwise there is nothing to prove. We apply Lemma 4 to both situations $\mathrm{Sing}\,X \subset X$ and $v_X^{-1}(\mathrm{Sing}\,X) \subset \tilde{X}$. We obtain

$$\begin{aligned} P_{k,H^1(\tilde{X})}(T)\frac{P_{k,H^0(v_X^{-1}(\mathrm{Sing}\,X))}(T)}{P_{k,H^0(\tilde{X})}(T)} &= P_{k,H^1_c(\tilde{X}-v_X^{-1}(\mathrm{Sing}\,X))}(T) \\ &= P_{k,H^1_c(X-\mathrm{Sing}\,X)}(T) \\ &= P_{k,H^1(X)}(T)\frac{P_{k,H^0(\mathrm{Sing}\,X)}(T)}{P_{k,H^0(X)}(T)}, \end{aligned}$$

where the middle equality follows from the fact that the normalization map v_X is an isomorphism from $\tilde{X} - v_X^{-1}(\mathrm{Sing}\,X)$ to $X - \mathrm{Sing}\,X$. Then, Lemma 8 applied to $\mathrm{Sing}\,X$ and $v_X^{-1}(\mathrm{Sing}\,X)$ and Proposition 6 applied to X and $\tilde{X}$ gives the result. $\square$

Note that an elementary proof for Theorem 9 can be found in [1].

Theorem 10. *Let X be an irreducible projective curve defined over k with $\bar{c}$ absolutely connected components, and let*

$$\bar{X} = \bar{X}_1 \cup \cdots \cup \bar{X}_{\bar{r}}$$

be the decomposition of $\bar{X}$ into its absolutely irreducible components. Let $\bar{Z}$ be the algebraic set

$$\bar{Z} = \bigcup_{i \neq j} \bar{X}_i \cap \bar{X}_j$$

and $\bar{Z}_i = \bar{Z} \cap \bar{X}_i$. Then,

- *$\bar{Z}$ is defined over k;*
- *$\bar{Z}_i$ are defined over $k_{\bar{r}}$;*
- *$\bar{X}_i$ are defined over $k_{\bar{r}}$, and are conjugated under* $\mathrm{Gal}(k_{\bar{r}}/k)$, *and*

$$P_{k,H^1(X)}(T) = P_{k_{\bar{r}},H^1(X_1)}(T^{\bar{r}})\frac{P_{k_{\bar{r}},H^0(Z_1)}(T^{\bar{r}})/P_{k,H^0(Z)}(T)}{(1-T^{\bar{r}})/(1-T^{\bar{c}})}$$

Proof. The assertions on the field of definition follow from Lemma 5 in the same way as in the proof of Proposition 7 for the decomposition $\bar{X} - \bar{Z} = \bigcup_{i=1}^{\bar{r}} (\bar{X}_i - \bar{Z}_i)$. Lemma 4 applied to $Z \subset X$ as k-varieties implies that

$$P_{k,H^1_c(X-Z)}(T) = P_{k,H^1(X)}(T)\frac{P_{k,H^0(Z)}(T)}{P_{k,H^0(X)(T)}}.$$

Now, Lemma 4 applied to $Z_1 \subset X_1$ as $k_{\bar{r}}$-varieties says that

$$P_{k_{\bar{r}},H^1_c(X_1-Z_1)}(T) = P_{k_{\bar{r}},H^1(X_1)}(T)\frac{P_{k_{\bar{r}},H^0(Z_1)}(T)}{P_{k_{\bar{r}},H^0(X_1)}(T)}.$$

But Lemma 5 applied to the variety $U = U_1 \cup \cdots \cup U_{\bar{r}}$ where $U = X - Z$ and $U_i = X_i - Z_i$, proves that

$$P_{k,H^1_c(U)}(T) = P_{k_{\bar{r}},H^1_c(U_1)}(T^{\bar{r}}).$$

Hence the theorem follows thanks to Proposition 6 applied to X over k and to X_1 over $k_{\bar{r}}$. $\square$

Theorem 11. *Let X be a connected projective curve defined over k, and let*

$$\bar{X} = \bar{\mathscr{X}}_1 \cup \cdots \cup \bar{\mathscr{X}}_{\bar{c}}$$

be the decomposition of $\bar{X}$ into its absolutely connected components. The $\bar{\mathscr{X}}_i$'s are defined over $k_{\bar{c}}$ and conjugated under $\mathrm{Gal}(k_{\bar{c}}/k)$. *Let*

$$\mathscr{X}_1 = X_1 \cup \cdots \cup X_r$$

be the decomposition of $\mathscr{X}_1$ into its $k_{\bar{c}}$-irreducible components and let $\bar{c}_i$ be the number of absolutely connected components of X_i. Let Z be the algebraic set

$$Z = \bigcup_{i \neq j} X_i \cap X_j$$

and $Z_i = Z \cap X_i$. Then Z and Z_i are defined over $k_{\bar{c}}$, and

$$P_{k,H^1(X)}(T) = \prod_{i=1}^{r} P_{k_{\bar{c}},H^1(X_i)}(T^{\bar{c}}) \times \frac{\prod_{i=1}^{r} P_{k_{\bar{c}},H^0(Z_i)}(T^{\bar{c}})}{P_{k_{\bar{c}},H^0(Z)}(T^{\bar{c}})} \times \frac{(1 - T^{\bar{c}})}{\prod_{i=1}^{r} (1 - T^{\bar{c}.\bar{c}_i})}.$$

Proof. Lemma 5 implies that the $\bar{\mathscr{X}}_i$'s are defined over $k_{\bar{c}}$ and conjugated under $\mathrm{Gal}(k_{\bar{c}}/k)$ and that

$$P_{k,H^1(X)}(T) = P_{k_{\bar{c}},H^1(\mathscr{X}_1)}(T^{\bar{c}}).$$

Since the X_i's are, by definition, defined over $k_{\bar{c}}$, this implies obviously that Z and Z_i are also defined over $k_{\bar{c}}$. Now, Lemma 4 applied to $Z \subset \mathscr{X}_1$, and to $Z_i \subset X_i$, together with the fact that $\mathscr{X}_1 - Z$ is equal to the disjoint union of the $(X_i - Z_i)$'s and with Lemma 6 enables us to conclude. $\square$

Note that it can happen for the X_i's to be absolutely disconnected as shown by the example of the curve $\mathscr{X}_1$ in $\mathbf{P}^2$ with equation $(X^2+Z^2)(Y^3+YZ^2+Z^3)$ over a finite field k for which both factors are k-irreducible. In this case, $\bar{X}_1$ has two connected components, and $\bar{X}_2$ has three connected components.

4. Examples

Let us now look at some examples.

Example 1. Let X be the projective plane curve with equation $x^2+y^2=0$ over the field k with q elements, where $q \equiv 3 \pmod 4$. Then $\bar{X}$ is the union of two projective lines meeting at the k-rational point $[0:0:1]$, and $\bar{X} = \bar{X}_1 \cup \bar{X}_2$, where $\bar{X}_1$ is the k_2-rational projective line whose equation is $x - \imath y = 0$ ($\imath$ being a primitive root of -1), and X_2 is the $\mathrm{Gal}(k_2/k)$-conjugate of X_1.

Propositions 6, 7 and Theorem 10 give us the spectrums of the Frobenius on the étale cohomology groups:

- $\mathrm{Spec}(F \mid H^0(X)) = \{1\}$;
- $\mathrm{Spec}(F \mid H^1(X)) = \{0\}$;
- $\mathrm{Spec}(F \mid H^0(X)) = \{q, -q\}$.

Indeed, Theorem 10 with $\bar{r} = 2, Z = \{[0:0:1]\}$ and $Z_1 = Z \cap X_1$ says that

$$P_{k,H^1(X)} = \frac{(1-T^2)/(1-T)}{(1-T^2)/(1-T)} = 1.$$

The Grothendieck–Lefschetz formula then gives, for any $n \in \mathbf{N}^*$:

$$\sharp X(k_n) = q^n + (-q)^n - 0^n + 1^n$$

$$= \begin{cases} 2q^n + 1 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd,} \end{cases}$$

which is the expected value.

Example 2. More generally, let X be an irreducible and absolutely connected projective curve defined over k, having exactly two absolutely irreducible components $\bar{X}_1$ and $\bar{X}_2$ over $\bar{k}$. By Lemma 5, the $\bar{X}_i$'s are extension to $\bar{k}$ of two curves X_1 and X_2 defined over k_2, and conjugated under $\mathrm{Gal}(k_2/k) = \mathbf{Z}/2\mathbf{Z}$. Moreover, $X_1 \cap X_2$ is defined over k by Theorem 10. In particular, we have

$$X(k_n) = \begin{cases} X_1(k_n) \cup X_2(k_n) & \text{if } n \text{ is even,} \\ X_1 \cap X_2(k_n) & \text{if } n \text{ is odd.} \end{cases}$$

Hence,

$$\sharp X(k_n) = \begin{cases} \sharp X_1(k_n) + \sharp X_2(k_n) - \sharp X_1 \cap X_2(k_n) & \text{if } n \text{ is even,} \\ \sharp X_1 \cap X_2(k_n) & \text{if } n \text{ is odd.} \end{cases}$$

Let us verify that Propositions 6, 7 and Theorem 10 are in accordance with this naive counting. Indeed, Theorem 10 says, since $Z = Z_1$, that

$$\begin{aligned} P_{k,H^1(X)}(T) &= P_{k_2,H^1(X_1)}(T^2)\frac{P_{k_2,H^0(Z_1)}(T^2)/P_{k,H^0(Z)}(T)}{(1-T^2)/(1-T)} \\ &= P_{k_2,H^1(X_1)}(T^2)\frac{P_{k_2,H^0(Z)}(T^2)/P_{k,H^0(Z)}(T)}{1+T}. \end{aligned}$$

Let $\omega_1, \dots, \omega_a$ be the eigenvalues of the Frobenius $F^2 = F \circ F$ on $H^1(X_1)$ with multiplicities, and $\alpha_1, \dots, \alpha_b$ be those of the Frobenius F on $H^0(Z)$ with multiplicities. Note that X_1 being defined over k_2 and being eventually singular, some ω_i's have modulus $\sqrt{q^2} = q$ and the others have modulus $\sqrt{1} = 1$. Note also that 1 is always an eigenvalue on $H^0(Z)$, so that we can assume that $\alpha_1 = 1$. We have then

$$\sharp X_1(k_n) = q^n - \sum_{i=1}^{a} \omega_i^n + 1$$

for n even, and

$$\sharp X_1 \cap X_2(k_n) = \sum_{i=1}^{b} \alpha_i^n = 1 + \sum_{i=2}^{b} \alpha_i^n$$

for any n. The above formula for $P_{k,H^1(X)}$ implies that the eigenvalues of the Frobenius on $H^1(X)$ with multiplicities are

$$\sqrt{\omega}_1, -\sqrt{\omega}_1, \dots, \sqrt{\omega}_a, -\sqrt{\omega}_a, -\alpha_2, \dots, -\alpha_b.$$

Moreover, Propositions 6 and 7 implies that the eigenvalue of the Frobenius on $H^0(X)$ is just 1 and the eigenvalues on $H^2(X)$ are q and $-q$. Then, we have by the

Grothendieck–Lefschetz formula

$$\begin{aligned}\sharp X(k_n) &= q^n + (-q)^n - \left(\sum_{i=1}^{a} (1+(-1)^n)\sqrt{\omega_i^n} + \sum_{j=2}^{b} (-\alpha_j)^n \right) + 1 \\ &= \begin{cases} 2q^n - (2(q^n+1-\sharp X_1(k_n)) + \sharp X_1 \cap X_2(k_n) - 1) + 1 & \text{if } n \text{ is even;} \\ -(-(\sharp X_1 \cap X_2(k_n) - 1)) + 1 & \text{if } n \text{ is odd} \end{cases} \\ &= \begin{cases} \sharp X_1(k_n) + \sharp X_2(k_n) - \sharp X_1 \cap X_2(k_n) & \text{if } n \text{ is even;} \\ \sharp X_1 \cap X_2(k_n) & \text{if } n \text{ is odd,} \end{cases}\end{aligned}$$

as promised in the Introduction (note that $\sharp X_1(k_n) = \sharp X_2(k_n)$ if n is even).

Example 3. The aim of this example is to show that on the contrary to what may be thought, formulas of Theorems 10 and 11 really took into account the multiple intersections between the X_i's and not just the pairwise intersections. Indeed, let X be an absolutely connected projective curve, union of r absolutely irreducible components defined over k:

$$X = X_1 \cup \cdots \cup X_r.$$

Suppose for simplicity that all intersection points of the X_i's are also defined over k, that is to say that $Z(k) = Z(\bar{k})$. Then, Theorem 11, together with Propositions 6 and 7 and with the Grothendieck–Lefschetz formula, imply

$$\begin{aligned}\sharp X(k) &= rq - \left(\sum_{i=1}^{r} \sum_{\omega \in \mathrm{Spec}(F \mid H^1(X_i))} \omega + \sum_{i=1}^{r} \sharp Z_i - \sharp Z - (r-1) \right) + 1 \\ &= rq - \left(\sum_{i=1}^{r} (q+1-\sharp X_i(k)) + \sum_{i=1}^{r} \sharp Z_i - \sharp Z - (r-1) \right) + 1 \\ &= \sum_{i=1}^{r} \sharp X_i(k) - \left(\sum_{i=1}^{r} \sharp Z_i - \sharp Z \right).\end{aligned}$$

Since

$$\begin{aligned}\sharp Z = \sum_{i=1}^{r} \sharp Z_i & \\ - \sum_{1 \leqslant i_1 < i_2 \leqslant r} & \sharp X_{i_1}(k) \cap X_{i_2}(k) \\ + \sum_{1 \leqslant i_1 < i_2 < i_3 \leqslant r} & \sharp X_{i_1}(k) \cap X_{i_2}(k) \cap X_{i_3}(k) \\ -, \ldots, &\end{aligned}$$

we obtain the well-known inclusion–exclusion formula for $\sharp X(k)$!

5. Analogue of an Artin conjecture for algebraic varieties

For a finite extension of number fields E/F, Artin's holomorphy conjecture asserts that the quotient $\zeta_E(s)/\zeta_F(s)$ of their Dedekind zeta functions is an entire function of the complex variable s (this conjecture was proved independently by Aramata and Brauer in the Galois case (see [5] for instance).

Let $Y \longrightarrow X$ be a surjective morphism between algebraic varieties defined over k. One can ask whether the quotient $Z_Y(T)/Z_X(T)$ of their zeta function (which are rational fractions thanks to Dwork's theorem) is a polynomial in T. The Grothendieck–Lefschetz formula gives, as in Section 1 in the one dimensional case, the following form for the zeta function of an algebraic variety X defined over a finite field k:

$$Z_X(T) = \prod_{i=0}^{2\dim X} (\det(1 - TF \mid H^i_c(X)))^{(-1)^{i+1}} = \prod_{i=0}^{2\dim X} (P_{k,H^i_c(X)}(T))^{(-1)^{i+1}}.$$

Therefore, the real question becomes whether the polynomials $P_{k,H^i_c(X)}(T)$ divide the polynomials $P_{k,H^i_c(Y)}(T)$ (see [3] for a detailed discussion).

The following proposition, whose proof has been communicated to the authors by N. Katz, gives an answer to this question.

Proposition 12. *Let $f : Y \longrightarrow X$ be a finite flat morphism between varieties over k and G be a constructible $\mathbf{Q}_\ell$-sheaf on X. Then the compact cohomology group $H^i_c(\bar{X}, G)$ is a direct factor of $H^i_c(\bar{Y}, f^*(G))$ for any $i \geqslant 0$ has a F-module.*

Without hypothesis on the morphism, this turns to be false as shown by the example of the normalization map of a nodal singular curve.

This proposition was proved by Kleiman in [7] if both Y and X are smooth projective algebraic varieties, and by the authors in [3] for absolutely irreducible projective curves.

Proof. Since f is finite, we have $H^i_c(\bar{Y}, f^*(G)) = H^i_c(\bar{X}, f_*f^*(G))$. Since f is flat, there is a Trace morphism $f_*f^*(G) \longrightarrow G$, such that the composite with the natural morphism $G \longrightarrow f_*f^*(G)$ is the multiplication by $\deg(f)$ on G (see [6, Exposé XVIII, Theorem 2.9]). If we choose ℓ prime to $\deg(f)$, then $\deg(f)$ is invertible in $\mathbf{Q}_\ell$, so that G injects in $f_*f^*(G)$ and $f_*f^*(G)$ surjects in G. Hence, we get by elementary linear algebra that G is a direct factor of $f_*f^*(G)$, which gives the desired result. $\square$

When G is the constant sheaf $\mathbf{Q}_\ell$, we obtain

Corollary 13. *Let $f : Y \longrightarrow X$ be a finite flat map between varieties defined over the finite field k. Then, for any positive i, the reciprocal polynomial of the characteristic* *polynomial of the Frobenius morphism* $P_{k,H_c^i(X)}(T)$ *on* $H_c^i(X)$ *divides that of* $H_c^i(Y)$ *in the polynomial ring* $\mathbf{Z}[T]$.

Note that there is no completness or dimensional assumption on X and Y in this corollary.

Propositions 6, 7 and Theorems 9–11, together with Corollary 13, imply

Theorem 14. *Let* $f: Y \longrightarrow X$ *be a surjective flat morphism between absolutely connected projective curves defined over the finite field* k *with* q *elements, having respectively* $\bar{r}_Y$ *and* $\bar{r}_X$ $\bar{k}$*-irreducible components* $\bar{Y}_i$ *and* $\bar{X}_i$ *of geometric genus* $g_{\bar{Y}_i}$ *and* $g_{\bar{X}_i}$. *We have*

$$|\sharp Y(k) - \sharp X(k)| \leqslant (\bar{r}_Y - \bar{r}_X)q + 2\left(\sum_{i=1}^{\bar{r}_Y} g_{\bar{Y}_i} - \sum_{i=1}^{\bar{r}_X} g_{\bar{X}_i}\right)\sqrt{q} + \Delta_Y - \Delta_X - (\bar{r}_Y - \bar{r}_X).$$

For $X = \mathbf{P}^1$ and an absolutely irreducible smooth curve Y, this is nothing else than Weil's bound. In this case, the flatness hypothesis is always satisfied. Without the smoothness assumption on Y, this is the bound for singular curves proved in [1] (see also [4,8]). For absolutely irreducible curves X and Y, we recover the bound given in [2].

6. Remark

In the particular case of absolutely connected curves X defined over k for which the k-irreducible components $\bar{X}_1, \ldots, \bar{X}_{\bar{r}}$ are absolutely irreducible, we can have the following approach for $P_{k,H^1(X)}(T)$. Consider the jacobian J_X of X which is the group scheme defined as the identity component of the Picard scheme Pic_X of X. This is a semi-abelian variety: J_X is an extension of the abelian variety $J_{\tilde{X}}$ (the jacobian of the desingularization $\tilde{X}$ of X) by a smooth connected linear algebraic group L_X, and the latter can be written as the product of a unipotent group U_X by a torus T_X. We have quoted in [3], using a result of Deligne, that the polynomial $P_{k,H^1(X)}(T)$ is related to the Tate module $T_\ell(J_X)$ of the jacobian of X by

$$P_{k,H^1(X)}(T) = \det(1 - TF \mid T_\ell(J_X) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell).$$

Moreover, this polynomial can be viewed as the product

$$P_{k,H^1(X)}(T) = P_{k,H^1(\tilde{X})}(T) \times P_{X/\tilde{X}}(T), \tag{3}$$

where the last polynomial corresponds to the following weight-zero part (see [3]):

$$P_{X/\tilde{X}}(T) = \det(1 - TF \mid T_\ell(T_X) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell). \tag{4}$$

The (absolutely) irreducible components of $\tilde{X}$ are the normalizations $\tilde{\bar{X}}_i$ of the (absolutely) irreducible components $\bar{X}_i$. The following exact sequence of sheaves:

$$1 \longrightarrow \mathcal{O}_X^* \longrightarrow v_{X,*}\mathcal{O}_{\tilde{X}}^* \longrightarrow v_{X,*}\mathcal{O}_{\tilde{X}}^*/\mathcal{O}_X^* \longrightarrow 1$$

gives the following long exact sequence:

$$1 \longrightarrow H^0(X, \mathcal{O}_X^*) \longrightarrow \prod_{i=1}^{\bar{r}} H^0(\tilde{X}_i, \mathcal{O}_{\tilde{X}_i}^*) \longrightarrow \prod_{i=1}^{N}\left(\prod_{j=1}^{n_i} k(\tilde{P}_{i_j})\right)/k(P_i)$$
$$\longrightarrow J_X \longrightarrow \prod_{i=1}^{\bar{r}} J_{\tilde{X}_i} \longrightarrow 1,$$

where P_i, $i = 1, \ldots, N$ are the singular points of X and $P_{i_j}, j = 1, \ldots, n_i$ the points of $\tilde{X}$ lying above P_i and where $k(P_i)$ and $k(\tilde{P}_{i_j})$ are theirs residue field.

Thus, the kernel of $J_X \longrightarrow \prod_{i=1}^{r} J_{\tilde{X}_i}$ is a torus of rank equal to $\Delta_X - \bar{r} + 1$. This kernel is equal to the toric part T_X of the jacobian of X which gives, by (4), the weight-zero part $P_{X/\tilde{X}}(T)$ of $P_{k,H^1(X)}(T)$. So, by (3), we get

$$P_{k,H^1(X)}(T) = P_{X/\tilde{X}}(T) \times \prod_{i=1}^{\bar{r}} P_{k,H^1(\tilde{X})}(T),$$

where $P_{X/\tilde{X}}(T)$ is a polynomial of degree $\Delta_X - \bar{r} + 1$.

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