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RATIONAL POINTS ON SMOOTH SURFACES IN $\mathbb{P}^3$ OVER FINITE FIELDS

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Abstract

We improve a bound due to Voloch ['Surfaces in $\mathbb{P}^3$ over finite fields', in: *Topics in Algebraic and Noncommutative Geometry*, Contemporary Mathematics, 324 (eds. C. G. Melles, J.-P. Brasselet, G. Kennedy, K. Lauter and L. McEwan) (American Mathematical Society, Providence, RI, 2003), 219–226] on the number of rational points on smooth surfaces in $\mathbb{P}^3$ over finite fields and look at families of surfaces that achieve or nearly achieve this bound, for which we compute their exact number of rational points. These computations may have independent interest.

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1. Introduction

Let S be a smooth surface in $\mathbb{P}^3$ of degree d defined over the finite field $\mathbb{F}_p$ with p prime. We are interested in the number $\sharp S(\mathbb{F}_p)$ of rational points of S over $\mathbb{F}_p$. Deligne has proved, as a consequence of his proof of the Weil conjectures in [3], that

$$\sharp S(\mathbb{F}_p) \leq p^2 + 1 + (d^3 - 4d^2 + 6d - 2)p.$$

There are, in addition, several bounds that can be obtained by elementary methods. See [4, 5] and the references therein. An example of such a bound was proved by the second author in [10, Proposition 2], namely

$$\sharp S(\mathbb{F}_p) \leq (d-1)(p+1)^2 + p + 1,$$

which is better than Deligne's bound if $d > \sqrt{p/3}$ approximately. Moreover, if we suppose that S does not contain a line defined over $\mathbb{F}_p$, then the above bound can be improved to

$$\sharp S(\mathbb{F}_p) \leq (d-1)p^2 + (d-2)(p+1) + 1.$$

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By a different method, inspired by [8], the second author proved in [10, Theorem 2] that if $2 < d < p$ and if m denotes the number of $\mathbb{F}_p$-lines contained in S, then

$$\sharp S(\mathbb{F}_p) \leq \frac{d(d+p-1)(d+2p-2)}{6} + m(p+1) \tag{1.1}$$

and so, in particular,

$$\sharp S(\mathbb{F}_p) \leq \frac{d(d+p-1)(d+2p-2)}{6} + d(11d-24)(p+1).$$

The main purpose of this paper is to improve this last bound and to provide examples of surfaces for which we are able to determine the number of rational points together with the number of $\mathbb{F}_p$-lines contained in the surface, and that achieve or get close to this bound.

2. The main result

Let S be a smooth surface in $\mathbb{P}^3$ of degree d defined over the finite field $\mathbb{F}_p$ with p prime and $2 < d < p$, and let $L_1, \ldots, L_m$ be the $\mathbb{F}_p$-lines on S.

Suppose that the surface S is given as the set of zeros in $\mathbb{P}^3$ of a homogeneous polynomial $f(x_0, x_1, x_2, x_3)$ with coefficients in $\mathbb{F}_p$. Consider the surface S_1 given as the zero locus of the polynomial $\sum_{i=0}^{3}(\partial f/\partial x_i)x_i^p$ and the surface S_2 given by the polynomial $\sum_{i=0}^{3}\sum_{j=0}^{3}(\partial^2 f/\partial x_i x_j)x_i^p x_j^p$.

We consider, as in [10], the scheme $X := S \cap S_1 \cap S_2$. The following lemma will allow us to estimate the zero-dimensional part of X.

Lemma 2.1. *Let $\mathcal{S}$ be a smooth surface in $\mathbb{P}^3$ of degree d and $\mathcal{S}_i$ surfaces in $\mathbb{P}^3$ of degree d_i $(i = 1, 2)$. Assume that $\mathcal{S}, \mathcal{S}_1, \mathcal{S}_2$ intersect in a zero-dimensional scheme Γ and lines $L_1, \ldots, L_m$ with multiplicity one. Then,*

$$2m \leq \deg \Gamma - (dd_1d_2 - m(d + d_1 + d_2)) \leq m(m+1).$$

Proof. We work with intersection products on $\mathcal{S}$. The surfaces $\mathcal{S}_i$ cut $\mathcal{S}$ in divisors $C_i + L_1 + \cdots + L_m$ and we are interested in $\Gamma = C_1 \cdot C_2$. Letting H denote a hyperplane section of $\mathcal{S}$,

$$\deg \Gamma = (d_1H - (L_1 + \cdots + L_m)) \cdot (d_2H - (L_1 + \cdots + L_m)). \tag{2.1}$$

By classical intersection theory on surfaces,

$$H^2 = d, \quad L_i.H = 1 \quad \text{and} \quad L_i^2 = 2 - d.$$

The first follows since $\mathcal{S}$ has degree d. For the second, a line and a plane meet at one point, but if the line is contained in $\mathcal{S}$, the intersection point is also in $\mathcal{S}$. The third follows from the previous one and the adjunction formula. Indeed, if L is a nonsingular curve of genus g on a surface S and K is the canonical divisor on S, then the adjunction formula gives

$$2g - 2 = L.(L + K) = L^2 + L.K.$$

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If $L = L_i$ is a line, its genus is zero and we get

$$-2 = L_i^2 + L_i.K.$$

Moreover, the canonical divisor K on a smooth surface in $\mathbb{P}^3$ of degree d satisfies

$$K \sim (d-4)H,$$

where H is a hyperplane section. So, we obtain $-2 = L_i^2 + (d-4)L_i.H$ and, thus, $L_i^2 = 2 - d$ since $L_i.H = 1$. Hence, the right-hand side of (2.1) expands to

$$\begin{aligned} d_1 d_2 H^2 &- (d_1 + d_2)(L_1 + \cdots + L_m) \cdot H + (L_1 + \cdots + L_m)^2 \\ &= d d_1 d_2 - m(d_1 + d_2) + (L_1 + \cdots + L_m)^2. \end{aligned}$$

Finally, $L_i \cdot L_j = 0$ or 1 if $i \neq j$, depending on whether the lines intersect or not, so

$$(L_1 + \cdots + L_m)^2 = \sum_{i=1}^m L_i^2 + \sum_{i \neq j} L_i \cdot L_j \leq m(2-d) + m(m-1) = m(m+1-d)$$

and

$$(L_1 + \cdots + L_m)^2 = \sum_{i=1}^m L_i^2 + \sum_{i \neq j} L_i \cdot L_j \geq m(2-d),$$

which combines with (2.1) to prove the lemma. □

The next theorem improves the bound in (1.1).

THEOREM 2.2. *Let S be a smooth surface in $\mathbb{P}^3$ of degree d defined over the finite field $\mathbb{F}_p$ with p prime and $2 < d < p$, and suppose S is given by the zero locus of the polynomial $f(x_0, x_1, x_2, x_3)$. Let m be the number of lines defined over $\mathbb{F}_p$ on S and suppose that they have multiplicity one in $X := S \cap S_1 \cap S_2$, where S_1 and S_2 are the surfaces given respectively by the zero locus of the polynomial $\sum_{i=0}^3 (\partial f/\partial x_i) x_i^p$ and the polynomial $\sum_{i=0}^3 \sum_{j=0}^3 (\partial^2 f/\partial x_i x_j) x_i^p x_j^p$.*

Then, the number of rational points on S is bounded above by

$$\sharp S(\mathbb{F}_p) \leq \frac{d(d+p-1)(d+2p-2)}{6} - \frac{3m(d+p-1) - m(m+1)}{6} + m(p+1). \quad (2.2)$$

In particular,

$$\sharp S(\mathbb{F}_p) \leq \frac{d(d+p-1)(d+2p-2)}{6} + \frac{(11d^2 - 30d + 18)(11d^2 - 33d + 28 + 3p)}{6}.$$

PROOF. In [10], it is proved that the one-dimensional components of X are lines. Hence, the situation here is as in Lemma 2.1, with the surface S_1 of degree $d_1 = d + p - 1$ and the surface S_2 of degree $d_2 = d + 2p - 2$. Since we have supposed that the $\mathbb{F}_p$-lines $L_1, \ldots, L_m$ on S have multiplicity one in X, it follows from Lemma 2.1 that the zero-dimensional subscheme Γ of X given by its isolated rational points

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satisfies

$$\deg \Gamma \leq d(d+p-1)(d+2p-2) - m(3d+3p-3) + m(m+1).$$

Moreover, as shown in [10, proof of Theorem 2], outside of the lines contained in S, the rational points of S are isolated points of $S \cap S_1 \cap S_2$ with multiplicity at least 6. So, the rational points of S not on lines have multiplicity at least 6 in Γ and lead to the bound

$$\#S(\mathbb{F}_p) \leq \frac{d(d+p-1)(d+2p-2) - m(3d+3p-3) + m(m+1)}{6} + m(p+1),$$

which can be rewritten as

$$\#S(\mathbb{F}_p) \leq \frac{d(d+p-1)(d+2p-2) - m(3d-7-m)}{6} + \frac{m(p+1)}{2}.$$

Furthermore, Bauer and Rams in [2, Theorem 1] show that a smooth surface in $\mathbb{P}^3$ of degree $d > 2$ over a field of characteristic 0 or of characteristic $p > d$ contains at most $11d^2 - 30d + 18$ lines. So, the last bound of the theorem follows. □

REMARK 2.3. To see when Theorem 2.2 improves the bound (1.1), we set

$$G := \frac{3m(d+p-1) - m(m+1)}{6}.$$

It is easy to see that $G \geq 0$ if and only if $m \leq 3d + 3p - 4$. However, we have already seen that, as a smooth surface in $\mathbb{P}^3$ of degree d with $2 < d < p$ and defined over $\mathbb{F}_p$, S contains at most $11d^2 - 30d + 18$ lines. So, as soon as $11d^2 - 30d + 18 \leq 3d + 3p - 4$, we will have $G \geq 0$. Thus, if $p > (11d^2 - 33d + 22)/3$, then $G > 0$ and Theorem 2.2 improves the bound (1.1).

REMARK 2.4. We have tried, but not succeeded, to prove that under the other hypothesis of Theorem 2.2, the lines of $X := S \cap S_1 \cap S_2$ always occur with multiplicity one. We will prove that this is the case for Fermat surfaces in Proposition 3.1. We have also verified by computer that this is so for a number of examples. We expect that this is always the case.

REMARK 2.5. Let S be a smooth surface in $\mathbb{P}^3$ of degree d defined over a finite field $\mathbb{F}_q$ of characteristic p.

If $d = 3$, then S is a nonsingular cubic surface and it is well known that S contains at most 27 lines and one knows very precise results on its number of rational points. Indeed,

$$\sharp S(\mathbb{F}_q) = q^2 + nq + 1,$$

where $-2 \leq n \leq 7$ with $n \neq 6$ (see for example [6, Table 1]). Swinnerton-Dyer proved in [9] that $n \leq 5$ for $q = 2, 3$ or 5. Following the proof of Theorem 2.2, all points in $X := S \cap S_1 \cap S_2$ that are not on a line are rational. Indeed, an asymptotic line at a point of S that is not contained in S does not meet S again, since S has degree 3. These points occur with multiplicity 6 on X if, furthermore, they are not contained in an (irrational)

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line of S. We hoped to recover the known results on the number of rational points of cubic surfaces from these facts, but did not succeed.

If $d = 4$ and $p \neq 2, 3$, then Rams and Schütt in [7] proved that a smooth quartic surface S contains at most 64 lines and this bound is sharp. So, for $p \geq 5$, the bound (1.1) gives

$$\sharp S(\mathbb{F}_p) \leq \frac{4(p+3)(2p+2)}{6} + 64p + 64.$$

This is improved by the bound (2.2) of Theorem 2.2 as soon as $p \geq 19$, giving

$$\sharp S(\mathbb{F}_p) \leq \frac{4(p+3)(2p+2)}{6} + 32p + 661.$$

3. Surfaces without isolated rational points

Consider Fermat-type surfaces S in $\mathbb{P}^3$ defined over $\mathbb{F}_p$ with $p \equiv 1 \pmod d$ given by the equation

$$x^d + y^d - z^d - w^d = 0.$$

PROPOSITION 3.1. *If -1 is a dth power modulo p, then the surface S contains $3d^2$ lines defined over $\mathbb{F}_p$ and all the lines in the intersection $X := S \cap S_1 \cap S_2$ have multiplicity one.*

PROOF. Following [1] (which is ostensibly over the field $\mathbb{C}$ of complex numbers, but one can see that the proofs work over finite fields), one can prove that S has $3d^2$ lines defined over $\mathbb{F}_p$, namely the lines with equations

$$\begin{cases} w = \eta^i x, \\ y = \eta^k z, \end{cases} \quad \begin{cases} x = \eta^{k+i} z, \\ w = \eta^i y, \end{cases} \quad \begin{cases} x = \nu\eta^i y, \\ w = \eta^{k+i} z, \end{cases}$$

where η is a primitive dth root of unity in $\mathbb{F}_p$ and $\nu \in \mathbb{F}_p$ is such that $\nu^d = -1$.

One can prove that already for $S \cap S_1$ (and *a posteriori* for X), the multiplicity of the lines is one. We show this by verifying that S and S_1 intersect transversely outside of a finite set. We work in the affine patch $w = 1$ (and the other patches are similar). Then, S has equation $x^d + y^d - z^d = 1$ with gradient $(dx^{d-1}, dy^{d-1}, -dz^{d-1})$ and S_1 has equation $x^{d+q-1} + y^{d+q-1} - z^{d+q-1} = 1$ with gradient $((d-1)x^{d+q-2}, (d-1)y^{d+q-2}, -(d-1)z^{d+q-2})$.

The intersection $S \cap S_1$ is not transversal at points where the gradients are proportional. For this to happen, there is a constant λ for which $dx^{d-1} = \lambda(d-1)x^{d+q-2}$ and similarly for y, z. So, $x^{q-1} = d/(\lambda(d-1))$ and similarly for y, z. So, $x = \lambda^{-1/(q-1)} c_x$, $y = \lambda^{-1/(q-1)} c_y$, $z = \lambda^{-1/(q-1)} c_z$, where c_x, c_y, c_z come from the same finite set with $c^{q-1} = d/(d-1)$. However, $(x, y, z) \in S$, so $\lambda^{d/(q-1)} = c_x^d + c_y^d + c_z^d$, so λ is also on a finite set, proving that S and S_1 intersect transversely outside a finite set. □

Consider now the surface S in $\mathbb{P}^3$ defined over $\mathbb{F}_p$ given by the equation

$$x^{(p-1)/2} + y^{(p-1)/2} - z^{(p-1)/2} - w^{(p-1)/2} = 0.$$

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If we denote by f the polynomial $x^{(p-1)/2} + y^{(p-1)/2} - z^{(p-1)/2} - w^{(p-1)/2}$, then the two surfaces S_1 and S_2 given respectively by the polynomials $\sum_{i=0}^{3}(\partial f/\partial x_i)x_i^p$ and $\sum_{i=0}^{3}\sum_{j=0}^{3}(\partial^2 f/\partial x_i x_j)x_i^p x_j^p$ have respective equations

$$x^{3(p-1)/2} + y^{3(p-1)/2} - z^{3(p-1)/2} - w^{3(p-1)/2} = 0$$

and

$$x^{5(p-1)/2} + y^{5(p-1)/2} - z^{5(p-1)/2} - w^{5(p-1)/2} = 0.$$

The next proposition gives the number of rational points on S and shows that the scheme $X := S \cap S_1 \cap S_2$ does not contain isolated rational points.

Proposition 3.2. *The number of rational points of the surface S in $\mathbb{P}^3$ defined over $\mathbb{F}_p$ by $x^{(p-1)/2} + y^{(p-1)/2} - z^{(p-1)/2} - w^{(p-1)/2} = 0$ is given by*

$$\sharp S(\mathbb{F}_p) = \tfrac{3}{8}(p^3 - 3p^2 + 11p - 9).$$

This is an example of a surface for which $S \cap S_1 \cap S_2$ does not contain isolated rational points.

Proof. Let us remark that if $a \in \mathbb{F}_p$, then $a^{(p-1)/2} = 0$ if and only if $a = 0$, and it is equal to 1 if a is a quadratic residue modulo p and equal to -1 otherwise. It is well known that the number of nonzero quadratic residues, and also the number of quadratic nonresidues modulo p, is equal to $(p-1)/2$. We begin by counting the number of affine rational points on S. When none of the coordinates are zero, we have 6 choices for the positions of the $+1$, so $6((p-1)/2)^4$ solutions. If two coordinates are zero, then we have also 6 choices for the zero coordinates and then two possibilities for the choices of $+1$ or -1; thus, $6 \times 2 \times ((p-1)/2)^2$ solutions. Adding the point with all coordinates zero, we obtain the number N_{aff} of affine rational points on S, which is equal to $6((p-1)/2)^4 + 12((p-1)/2)^2 + 1$, and then $\sharp S(\mathbb{F}_p) = (N_{\mathrm{aff}} - 1)/(p-1)$, which gives the result.

Let us calculate the degree of Γ, the zero-dimensional scheme appearing in the intersection $X := S \cap S_1 \cap S_2$. By Proposition 3.1, the lines in X have multiplicity one and so we can apply Lemma 2.1 to get

$$\deg \Gamma = dd_1d_2 - m(d + d_1 + d_2) + 2m + \sum_{i \neq j} L_i \cdot L_j,$$

with $d = (p-1)/2, d_1 = 3(p-1)/2$ and $d_2 = 5(p-1)/2$. Let us scrutinise the intersections of the lines L_i. Since -1 is clearly a $(p-1)/2$th power modulo p, one can use Proposition 3.1 to assert that the lines L_i contained in X are of three types:

$$\begin{cases} x = ay, \\ z = bw, \end{cases} \quad \begin{cases} x = a'z, \\ y = b'w, \end{cases} \quad \begin{cases} x = a'w, \\ y = b'z, \end{cases}$$

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where a, b are quadratic nonresidues modulo p and a', b' are (nonzero) quadratic residues modulo p. Thus, according to Proposition 3.1, the number of lines is $m = 3((p-1)/2)^2$.

If L_1 and L_2 are two lines of same type given, for example, by equations

$$\begin{cases} x = a_1 y, \\ z = b_1 w, \end{cases} \quad \begin{cases} x = a_2 y, \\ z = b_2 w, \end{cases}$$

then $L_1.L_2 = 1$ if and only if $a_1 = a_2$ or $b_1 = b_2$. Hence, in the case of lines L_i and L_j of the same type, the contribution will be

$$\sum_{i \neq j, \text{same type}} L_i.L_j = 3\left(\frac{p-1}{2}\right)^2 \left(\left(\frac{p-1}{2} - 1\right) \times 2\right) = 3\left(\frac{p-1}{2}\right)^2 (p-3).$$

If L_1 and L_2 are two lines of different types given, for example, by equations

$$\begin{cases} x = ay, \\ z = bw, \end{cases} \quad \begin{cases} x = a'z, \\ y = b'w, \end{cases}$$

then $L_1.L_2 = 1$ if and only if $ab' = a'b$. Hence, in the case of lines L_i and L_j of different type, the contribution will be

$$\sum_{i \neq j, \text{different type}} L_i.L_j = 6\left(\frac{p-1}{2}\right)^3.$$

Finally, we obtain

$$\sum_{i \neq j} L_i.L_j = 6\left(\frac{p-1}{2}\right)^2 (p-2).$$

One can now compute $\deg \Gamma = dd_1d_2 - m(d + d_1 + d_2) + 2m + \sum_{i \neq j} L_i \cdot L_j$, which gives

$$\deg \Gamma = 15\left(\frac{p-1}{2}\right)^3 - 3\left(\frac{p-1}{2}\right)^2 \left(9\left(\frac{p-1}{2}\right)\right) + 6\left(\frac{p-1}{2}\right)^2 + 6\left(\frac{p-1}{2}\right)^2 (p-2) = 0.$$

Thus, $S \cap S_1 \cap S_2$ does not contain isolated rational points. □

4. Surfaces without rational lines

We are interested in the surface S in $\mathbb{P}^3$ defined over $\mathbb{F}_p$ by the equation

$$x^{2(p-1)/5} + y^{2(p-1)/5} + x^{(p-1)/5} y^{(p-1)/5} + z^{2(p-1)/5} + w^{2(p-1)/5} = 0$$

with $p \equiv 1 \pmod 5$.

Let us remark that if $t \in \mathbb{F}_p^*$, then $(t^{2(p-1)/5})^5 = 1$ and, thus, $t^{2(p-1)/5}$ is a 5th root of unity in $\mathbb{F}_p$. It is well known that if ζ is a primitive 5th root of unity, then $1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 = 0$. Except for certain values of p, one can show that this is the only way that

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a sum of at most five 5th roots of unity in $\mathbb{F}_p$ can be zero, that is, a sum of at most five 5th roots of unity in $\mathbb{F}_p$ can be zero only if they are distinct.

LEMMA 4.1. *Let p be a prime such that $p \equiv 1 \pmod 5$ and $p \neq 11, 41, 61$. Then, a sum of at most five 5th roots of unity in $\mathbb{F}_p$ can be zero only if this is the sum of the five distinct 5th roots of unity.*

PROOF. Let ζ_5 be a primitive 5th root of unity in $\mathbb{C}$ and $\mu_5(\mathbb{C})$ the group of 5th roots of unity in $\mathbb{C}$. In the cyclotomic number field $\mathbb{Q}(\zeta_5)$, we consider the sums $x_1 + x_2 + x_3 + x_4 + x_5$, where $x_i \in \mu_5(\mathbb{C}) \cup \{0\}$. We are looking for the primes $p \equiv 1 \pmod 5$ dividing the norm of such sums.

Consider, for example, the algebraic integer $\alpha := 1 + 1 + \zeta_5 + \zeta_5 + \zeta_5 \in \mathbb{Q}(\zeta_5)$. The norm of α is equal to

$$N_{\mathbb{Q}(\zeta_5)/\mathbb{Q}}(\alpha) = \prod_{(k,5)=1} \sigma_k(\alpha),$$

where $\sigma_k(\zeta_5) = \zeta_5^k$. Hence,

$$N_{\mathbb{Q}(\zeta_5)/\mathbb{Q}}(\alpha) = 3^4 \prod_{k=1}^{4} \Big(-\frac{2}{3} - \zeta_5^k \Big) = 3^4 \Phi_5\Big(-\frac{2}{3} \Big),$$

where $\Phi_5(X) = \prod_{k=1}^{4}(X - \zeta_5^k) = X^4 + X^3 + X^2 + X + 1$ is the 5th cyclotomic polynomial. Thus, $N_{\mathbb{Q}(\zeta_5)/\mathbb{Q}}(\alpha) = 5 \times 11$.

Since $11 \equiv 1 \pmod 5$, the prime $p = 11$ totally splits in the extension $\mathbb{Q}(\zeta_5)/\mathbb{Q}$ and the quotient of the ring of integers $\mathbb{Z}[\zeta_5]$ by the ideal generated by 11 is a product of fields

$$\mathbb{Z}[\zeta_5]/11\mathbb{Z}[\zeta_5] \simeq \mathbb{F}_{11}^4.$$

Then, we look at the image of the algebraic integer $2 + 3\zeta_5$ by the reduction morphisms which send ζ_5 to one of the primitive 5th roots of unity in $\mathbb{F}_{11}$ which are $\{3, 9, 5, 4\}$. We find that $2 + 3 \times 3 \equiv 0 \pmod{11}$, that is, $1 + 1 + 3 + 3 + 3$ is a sum of five nondistinct 5th roots of unity in $\mathbb{F}_{11}$ which is equal to zero.

In the same way, we show that the algebraic integer $1 + 4\zeta_5 \in \mathbb{Z}[\zeta_5]$ has norm equal to 5×41, and the reduction morphism which sends ζ_5 to 10, which is a primitive 5th root of unity in $\mathbb{F}_{41}$, gives $1 + 10 + 10 + 10 + 10 = 0$, a sum of five nondistinct 5th root of unity in $\mathbb{F}_{41}$ which is equal to zero.

Again, the algebraic integers $1 + 3\zeta_5$ and $3 + \zeta_5$ have norms equal to 61 and provide the following sums of four 5th roots of unity in $\mathbb{F}_{61}$ which equal zero: $20 + 20 + 20 + 1 = 0$ and $1 + 1 + 1 + 58 = 0$.

Finally, if $\alpha = x_1 + x_2 + x_3 + x_4 + x_5$, where $x_i \in \mu_5(\mathbb{C}) \cup \{0\}$, then $N_{\mathbb{Q}(\zeta_5)/\mathbb{Q}}(\alpha)$ is a product of at most five complex numbers of modulus at most 5 and, thus, the norm of α is bounded above by $5^4 = 625$. One can show that, aside from 11, 41 and 61, no other prime number $\equiv 1 \pmod 5$ and less than 625 appears as a factor of the norm of such an element α. Thus, if $p \neq 11, 41, 61$, then the only way to have a sum of at most

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five 5th roots of unity in $\mathbb{F}_p$ to be zero, up to permutations, is $1+\zeta+\zeta^2+\zeta^3+\zeta^4=0$, where ζ is a primitive 5th root of unity in $\mathbb{F}_p$. $\square$

PROPOSITION 4.2. *The surface S in $\mathbb{P}^3$ defined over $\mathbb{F}_p$ by the equation*

$$x^{2(p-1)/5}+y^{2(p-1)/5}+x^{(p-1)/5}y^{(p-1)/5}+z^{2(p-1)/5}+w^{2(p-1)/5}=0,$$

with $p\equiv 1 \pmod 5$, does not contain lines defined over $\mathbb{F}_p$.

If $p\neq 11,41,61$, then

$$\sharp S(\mathbb{F}_p)=8\Big(\frac{p-1}{5}\Big)^3.$$

Moreover, if $p=11$, we have $\sharp S(\mathbb{F}_p)=18((p-1)/5)^3=144$; if $p=41$, we have $\sharp S(\mathbb{F}_p)=10((p-1)/5)^3=5120$ and if $p=61$, we have $\sharp S(\mathbb{F}_p)=8((p-1)/5)^3+2((p-1)/5)^2=14112$.

PROOF. Let us prove the first assertion. Let H be the hyperplane given by the equation $x=0$. The intersection $S\cap H$ does not contain any rational point since this would provide a sum of at most three 5th roots of unity equal to zero and this cannot happen by Lemma 4.1. We deduce that S does not contain a line defined over $\mathbb{F}_p$ (because such a line would intersect the plane H at a rational point).

We now have to count the number of points $(x:y:z:w)\in\mathbb{P}^3(\mathbb{F}_p)$ such that $x^{2(p-1)/5}+y^{2(p-1)/5}+x^{(p-1)/5}y^{(p-1)/5}+z^{2(p-1)/5}+w^{2(p-1)/5}=0$. If $p\neq 11,41,61$, by Lemma 4.1, this is necessarily a sum of five distinct 5th roots of unity. Consider the morphism

$$\begin{array}{rccc}\psi: & \mathbb{F}_p^* & \longrightarrow & \mathbb{F}_p^* \\ & x & \longmapsto & x^{2(p-1)/5}.\end{array}$$

The kernel has order $(p-1)/5$ and the image is the group of 5th roots of unity $\mu_5(\mathbb{F}_p)$ in $\mathbb{F}_p^*$. For the choice of x, any element of $\mathbb{F}_p^*$ works, so we have $p-1$ choices. Then, for y, one can choose $p-1-(p-1)/5=4(p-1)/5$ different elements and, finally, $x^{(p-1)/5}y^{(p-1)/5}$ is completely determined. There remain $2(p-1)/5$ choices for z and, then, $(p-1)/5$ choices for w. Dividing $8(p-1)((p-1)/5)^3$ by $p-1$ to find projective points, we get the result.

For $p=11,41$ and 61, a direct computation gives the result. Extra points come from sums of less than five 5th roots of unity which are equal to zero, or from sums of five nondistinct 5th roots of unity which are also equal to zero.

For example, $1+1+3+3+3=1+5+5=1+1+1+4+4=1+1+9=1+9+4+4+4=1+3+9+9=1+1+4+5=3+3+5=1+3+3+4=1+5+9+9+9=0$ in $\mathbb{F}_{11}$. $\square$

REMARK 4.3. Consider the surfaces in $\mathbb{P}^3$ given by the equation

$$x^{2d}+x^dy^d+y^{2d}+z^{2d}+w^{2d}=0$$

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and defined over a field where they have lines

$$\begin{cases} y = \omega x, \\ z = \lambda w, \end{cases} \quad \text{with } \omega^{2d} + \omega^d + 1 = 0 \text{ and } \lambda^{2d} + 1 = 0. \tag{4.1}$$

One can show that all the lines in the intersection $X := S \cap S_1 \cap S_2$ have multiplicity one.

If $d = (p-1)/5$, we find again the surfaces defined over $\mathbb{F}_p$ with $p \equiv 1 \pmod 5$ of the previous proposition, but the lines given in (4.1) are not defined over $\mathbb{F}_p$.

The result about the multiplicity of the lines in $S \cap S_1 \cap S_2$ is false in general since it is false for the surface $zw = x^2$ and the line $y = z = 0$ (but this surface is singular).

REMARK 4.4. The bound (1.1) gives $\sharp S(\mathbb{F}_p) \leq 28((p-1)/5)^3$.

5. Other examples

In this section, we collect two more examples for which our techniques can be applied to compute a lower bound on the number of their rational points.

EXAMPLE 5.1. Define

$$G(u_0, u_1, u_2, u_3) = u_0^2 + u_0u_1 + u_1^2 + u_0u_2 + u_2^2 + u_2u_3 + u_3^2$$

and

$$F(x, y, z, w) = G(x^{(p-1)/7}, y^{(p-1)/7}, z^{(p-1)/7}, w^{(p-1)/7}).$$

If ζ is a primitive 7th root of unity, then

$$G(\zeta, \zeta^3, \zeta^{-1}, \zeta^{-3}) = \zeta^2 + \zeta^4 + \zeta^6 + 1 + \zeta^{-2} + \zeta^{-4} + \zeta^{-6} = 0$$

and

$$G(\zeta^3, \zeta, \zeta^{-3}, \zeta^{-1}) = 0.$$

We checked numerically that there are no points on the surface when $u_2 = 0, u_3 = 1$ for $p < 300, p \neq 29$, so there are no lines on the surface either for those values of p. When $p = 29$, there are points with $u_2 = 0, u_3 = 1$ from $N_{\mathbb{Q}(\zeta_7)/\mathbb{Q}}(\zeta_7^2 + \zeta_7 + 2) = 29$. The surface $F = 0$ has at least $12((p-1)/7)^3$ points coming from the above identities. For $p = 29, 43, 71, 239, 337$, it has more points, at least $m((p-1)/7)^3$ for $m = 18, 17, 16, 14, 14$, respectively.

EXAMPLE 5.2. Finally, define

$$G_2(u_0, u_1, u_2, u_3) = \sum_{i \leq j} u_i u_j$$

and

$$F_2(x, y, z, w) = G_2(x^{(p-1)/5}, y^{(p-1)/5}, z^{(p-1)/5}, w^{(p-1)/5}).$$

If u_1, u_2, u_3 are distinct and not equal to 1 in the group μ_5 of 5th roots of unity, then $G_2(1, u_1, u_2, u_3) = 0$. This gives rise to $24((p-1)/5)^3$ points in $F_2 = 0$. If $p = 31$, there

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are extra points in the surface, coming from $u_1 = u_2 = u_3$ not equal to 1 in μ_5, totalling $28((p-1)/5)^3$ points. This example, when $p = 31$, is given in [10].

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