MAXIMUM NUMBER OF RATIONAL POINTS ON HYPERSURFACES IN WEIGHTED PROJECTIVE SPACES OVER FINITE FIELDS

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Abstract. An upper bound for the maximum number of rational points on a hypersurface in a projective space over a finite field has been conjectured by Tsfasman and proved by Serre in 1989. The analogue question for hypersurfaces on weighted projective spaces has been considered by Castryck, Ghorpade, Lachaud, O'Sullivan, Ram and the first author in 2017. A conjecture has been proposed there under the assumption that the first weight is equal to one and proved in the particular case of the dimension 2. We prove here the conjecture in any dimension provided the second weight is also equal to one.

Dedicated to our friend Sudhir Ghorpade for his 60th birthday[1].

1. Introduction

Let $\mathbb{F}_q$ be the finite field with q elements and $\mathbb{P}^n(\mathbb{F}_q)$ be the set of rational points over $\mathbb{F}_q$ of the projective space of dimension $n \geq 1$. Let us set $p_n := q^n + \cdots + q + 1$ for $n \geq 0$ and $p_n := 0$ for $n < 0$. We have clearly $\sharp\mathbb{P}^n(\mathbb{F}_q) = p_n$.

Answering a conjecture that Tsfasman made at the "Journées Arithmétiques de Luminy" in 1989, Serre proved in [11] (and independently Sørensen proved later in [12]) that if F is a nonzero homogeneous polynomial in $\mathbb{F}_q[X_0, \ldots, X_n]$ of degree $d \geq 1$, then the number of rational points over $\mathbb{F}_q$ of the hypersurface $V(F)$ in $\mathbb{P}^n$ defined by F satisfies the so-called Serre bound:

$$\sharp V(F)(\mathbb{F}_q) \leq dq^{n-1} + p_{n-2}.$$

If $d \geq q+1$ then $dq^{n-1} + p_{n-2} \geq p_n = \sharp\mathbb{P}^n(\mathbb{F}_q)$ and the hypersurface defined by the degree d homogeneous polynomial $X_0^{d-q-1}(X_0^q X_1 - X_0 X_1^q)$ has p_n rational points. Thus the Serre bound holds trivially and is reached for hypersurfaces of degree greater than or equal to $q + 1$.

Furthermore, the Serre bound is reached for hypersurfaces of degree less than or equal to q. Indeed, if $d \leq q$ then the number of rational points on

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the hypersurface given by the polynomial

$$F = \prod_{i=1}^{d} (\alpha_i X_0 - \beta_i X_1),$$

where $(\alpha_1 : \beta_1), \ldots, (\alpha_d, \beta_d)$ are distincts elements of $\mathbb{P}^1(\mathbb{F}_q)$, attains the Serre bound. Note that Serre proved that the bound is reached for $d \leq q$ if and only if F is of the above form, that is $V(F)$ is the union of d hyperplanes containing a linear variety of codimension 2.

In 1997, Tsfasman and Boguslavsky in [5] have considered the analogue question for a system of r polynomial equations. They propose a conjecture for the maximum number of points in $\mathbb{P}^n(\mathbb{F}_q)$ of the projective set given by the common zeros of r linearly independent homogeneous polynomials of degree d in $\mathbb{F}_q[X_0, \ldots, X_n]$. The Tsfasman-Boguslavsky conjecture for $r = 1$ is nothing else but the Serre bound. Boguslavsky succeded to prove in [5] the case $r = 2$. In 2015, Datta and Ghorpade proved in [6] that the Tsfasman-Boguslavsky conjecture is true if $d = 2$ and $r \leq n+1$ but is false in general if $d = 2$ and $r \geq n + 2$. Moreover, in 2017 they proved in [7] that the Tsfasman-Boguslavsky conjecture is true for any positive integer d, provided $r \leq n+1$. The case for r beyond $n+1$ is specifically considered one year later by Beelen, Datta and Ghorpade in [2] and they conjectured in 2022 in [3] a general formula when $d < q$ that they were able to prove in some cases

We are interested here in a generalization in another direction, namely the question of Tsfasman and Serre in the context of weighted projective spaces $\mathbb{P}(a_0, \ldots, a_n)$, i.e. the study, for any homogeneous polynomial F in $\mathbb{F}_q[X_0, \ldots, X_n]$ of degree d (with respect to the weights $a_0, \ldots, a_n$), of the maximum number of rational points on the hypersurface $V(F)$ in $\mathbb{P}(a_0, \ldots, a_n)$. In [1], the following quantity has been introduced:

$$e_q(d; a_0, a_1, a_2, \ldots, a_n) := \max_F \sharp V(F)(\mathbb{F}_q)$$

where the maximum ranges over the set of homogeneous polynomials F in $\mathbb{F}_q[X_0, \ldots, X_n]$ of weighted degree d.

It has been conjectured in 2017 in [1] that:

Conjecture 1.1. *If* $a_0 = 1$ *and* $\mathrm{lcm}(a_1, a_2, \ldots, a_n) | d$*, and if we order the weights such that* $a_1 \leq a_2 \leq \ldots \leq a_n$ *then*

$$e_q(d; 1, a_1, a_2, \ldots, a_n) = \min\{p_n, \frac{d}{a_1} q^{n-1} + p_{n-2}\}.$$

In the case of the projective line $\mathbb{P}(a_0, a_1)$, it has been shown in [1] that $e_q(d; a_0, a_1) = \min\{p_1, d/a\}$ where $a = \mathrm{lcm}(a_0, a_1)$, so the conjecture holds in this case. Moreover, the conjecture has been proved in [1] for projective planes $\mathbb{P}(1, a_1, a_2)$ with a_1 and a_2 coprime and $a_1 < a_2$: if $F \in \mathbb{F}_q[X_0, X_1, X_2]$ is a nonzero weighted homogenous polynomial of degree $d \leq a_1(q+1)$ which is a multiple of $a_1 a_2$ then $\sharp V(F)(\mathbb{F}_q) \leq \frac{d}{a_1} q + 1$. The proof follows the one given by Serre with a new notion of lines represented by either a homogenized linear bivariate equation, or the line at infinity.

Our purpose here is to prove Conjecture 1.1 in any dimension n provided $a_1 = 1$.

We recall in Section 2 the basic facts about weighted projective spaces and a lower bound for $e_q(d; a_0, \ldots, a_n)$. Then we study in Section 3 some morphisms between weighted projective spaces and we establish a relation between the numbers of zeros of a polynomial and its pullback. Section 4 is devoted to the proof of an upper bound for the number of rational points on an hypersurface in a weighted projective space. Finally we state and prove the main result in Section 5.

2. A lower bound for the number of rational points

2.1. **Weighted projective spaces.** Let $a_0, \ldots, a_n$ be positive integers and S be the polynomial ring $\mathbb{F}_q[X_0, \ldots, X_n]$ graded by $\deg(X_i) = a_i$. The weighted projective space $\mathbb{P}(a_0, \ldots, a_n)$ over $\mathbb{F}_q$ is the scheme

$$\mathbb{P}(a_0, \ldots, a_n) = \operatorname{Proj} S,$$

and can be seen as the geometric quotient

$$\mathbb{A}^{n+1}_{\mathbb{F}_q} \setminus \{0\} / \mathbb{G}_{m,\mathbb{F}_q}$$

of the punctured affine space $\mathbb{A}^{n+1}_{\mathbb{F}_q} \setminus \{0\}$ over $\mathbb{F}_q$ under the action of the multiplicative group $\mathbb{G}_{m,\mathbb{F}_q}$ over $\mathbb{F}_q$ given for any nonzero λ in an algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$ by

$$\lambda.(x_0, \ldots, x_n) = (\lambda^{a_0} x_0, \ldots, \lambda^{a_n} x_n).$$

If the a_i's are all equal to 1, then we recover the usual (or straight) projective space: $\mathbb{P}(1, \ldots, 1) = \mathbb{P}^n$.

The corresponding equivalent class is denoted by $[x_0 : \cdots : x_n]$ without any reference to the corresponding weights $a_0, \ldots, a_n$ and is called a weighted projective point. We say that the point is $\mathbb{F}_q$-rational if $[x_0 : \cdots : x_n] = [x_0^q : \cdots : x_n^q]$. Every $\mathbb{F}_q$-rational point of a weighted projective space over $\mathbb{F}_q$ has at least one representative in $\mathbb{F}_q^{n+1} \setminus \{(0, \ldots, 0)\}$. This result has been quoted in [10] but without a complete proof. Due to a lack of proof writing, we provide the following one over any field k which has been communicated to the authors by Laurent Moret-Bailly.

Proposition 2.1. *Let k be a field and $\underline{a} = (a_0, a_1, \cdots, a_n)$ be a sequence of $n+1$ nonzero integers. Then each k-rational point $x \in \mathbb{P}(a_0, \ldots, a_n)$ has a representative $x = [x_0 : x_1 : \cdots : x_n]$ with $x_i \in k$ for any $0 \leq i \leq n$.*

Proof. [Communicated by Laurent Moret-Bailly] Given a geometric point $x = [x_i; 0 \leq i \leq n] \in \mathbb{P}(a_0, \ldots, a_n)$, we denote by $|x| := \{i \in \{0, \cdots, n\}, x_i \neq 0\}$ the support of x. Then the whole projective space is partitioned into

$$\mathbb{P}(a_0, \ldots, a_n) = \bigcup_{\emptyset \neq I \subset \{0, \cdots, n\}} W_{\underline{a}}^I,$$

where $W_{\underline{a}}^I := \{x \in \mathbb{P}(a_0, \ldots, a_n), |x| = I\}$, so that we have to prove that for any nonempty subset I of $\{0, \cdots, n\}$, any k-rational point in $W_{\underline{a}}^I$ admits a k-rational representative. For this purpose, consider the puncturing regular map defined over k

$$\begin{array}{ccc} W_{\underline{a}}^I & \longrightarrow & \mathbb{P}(a_i, i \in I) \\ [x_i; 0 \leq i \leq n] & \mapsto & [x_i; i \in I] \end{array}$$

into a weighted projective space of dimension $\sharp I - 1$. This map is injective and, in case $\sharp I \geq 2$, is an isomorphism onto the dense torus

$$T_{(a_i, i \in I)} := \{[x_i; i \in I] \in \mathbb{P}(a_i, i \in I); \forall i \in I, x_i \neq 0\}$$

of $\mathbb{P}(a_i, i \in I)$. Now if d_I denotes the gcd of the $a_i, i \in I$ and $b_i := \frac{a_i}{d_I}$ for all $i \in I$, then we have first that the $b_i, i \in I$ are coprime, second that $\mathbb{P}(a_i, i \in I)$ is k-isomorphic to $\mathbb{P}(b_i, i \in I)$ (see the lemma in section 1.1. of [9]). Hence, $W_{\underline{a}}^I$ is k-isomorphic to the dense torus $T_{(b_i, i \in I)}$ of $\mathbb{P}(b_i, i \in I)$ and we are reduced to prove the proposition only for x in the dense torus of a weighted projective space $\mathbb{P}(b_i, i \in I)$ whose weights $(b_i, i \in I)$ are coprime.

To do this, let $(u_i; i \in I) \in \mathbb{Z}^I$ such that $\sum_{i \in I} u_i b_i = 1$, and consider the subset

$$V_I = \{(x_i; i \in I) \in \mathbb{A}_{\mathbb{F}_q}^I \setminus \{0_I\}; \quad \prod_{i \in I} x_i^{u_i} = 1\}$$

of the affine space of dimension $\sharp I$. It is then easily checked that for any $x = [x_i, i \in I]$ in the dense torus of $\mathbb{P}(b_i, i \in I)$, its only representative $(\lambda^{b_i} x_i, i \in I)$ lying on V_I, for $\lambda \in \overline{k}$, is the one for $\lambda = \prod_{i \in I} x_i^{-u_i}$. This proves that $W_{\underline{a}}^I$, the dense torus $T_{(a_i, i \in I)}$ and the affine subvariety V^I are k-isomorphic, and we are done in case $\sharp I \geq 2$.

In case $\sharp I = 1$, the weighted projective space with only one weight $a \in \mathbb{N}^*$ is $\mathbb{P}(a) = \mathbb{A}_k^* / \mathbb{G}_{m,k}$ for the action $\lambda.x = \lambda^a x$, so that for any $x \in \overline{k}^*$, we have $[x] = [1]$ (take λ be any a-th rooth of x in $\overline{k}^*$), so that any point in $\mathbb{P}(a)$ is k-rational which concludes the proof. □

Furthermore, Laurent Moret-Bailly has communicated to us the following more general scheme theoretic statement.

Proposition 2.2. *(Moret-Bailly) Let* $S = \oplus_{n\geq 0} S_n$ *be a positively graded ring. Let* $X = \mathrm{Proj}(S)$, $C = \mathrm{Spec}(S) \setminus \mathrm{Spec}(S_0)$ *be the punctured cone, and* $\rho : C \longrightarrow X$ *be the natural projection. Then, for any* $x : \mathrm{Spec}(k) \longrightarrow X$*, the reduced fiber* $(C \times_{\rho,X,x} \mathrm{Spec}(k))_{red}$ *is isomorphic to* $\mathrm{Spec}(k[t,t^{-1}])$*.*

In particular, for any field k*, the map* $C(k) \longrightarrow X(k)$ *induced by* ρ *is surjective.*

Proof. Let $x : \mathrm{Spec}(k) \longrightarrow X$ be a rational point of X over k. There exists some $f \in S_d$ with $d > 0$, such that the image of x is contained in the affine open subset $D^+(f) := \mathrm{Spec}\, S_{(f)} \subset X$, the spectrum of the localization at (f). Taking the fiber product from the morphisms ρ and x, we get the diagram

$$\begin{array}{ccc} \mathrm{Spec}(S[\frac{1}{f}] \otimes_{S_{(f)}} k) = \mathrm{Spec}(S[\frac{1}{f}]) \times_{\mathrm{Spec}((S_{(f)})} \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(k) \\ \downarrow & \square & \downarrow x \\ \mathrm{Spec}(S[\frac{1}{f}]) = \rho^{-1}(D^+(f)) & \xrightarrow{\rho} & D^+(f) = \mathrm{Spec}(S_{(f)}) \\ \downarrow \cap & & \downarrow \cap \\ C & \xrightarrow{\rho} & X \end{array}$$

with $C \times_X \mathrm{Spec}(k) = \mathrm{Spec}(S[\frac{1}{f}]) \times_{\mathrm{Spec}((S_{(f)})} \mathrm{Spec}\ (k)$. We conclude using the following Lemma 2.3 for the graded algebra $B = S[\frac{1}{f}] \otimes_{S_{(f)}} k$, whose degree zero homogeneous part is a field. Indeed, the k-rational point $x : \mathrm{Spec}(k) \longrightarrow D^+(f) := \mathrm{Spec}\, S_{(f)} \subset X$ corresponds to a morphism of rings $x^\sharp : S_{(f)} \to k$, whose kernel $\mathcal{M}$ is a maximal ideal of $S_{(f)}$. From the isomorphism induced by $x^\sharp : S_{(f)}/\mathcal{M}S_{(f)} \simeq k$, we deduce the isomorphism of graded rings

$$B = S[\frac{1}{f}] \otimes_{S_{(f)}} \left(S_{(f)}/\mathcal{M}S_{(f)}\right) \simeq S[\frac{1}{f}]/\mathcal{M}S[\frac{1}{f}],$$

whose degree zero homogeneous part is $S_{(f)}/\mathcal{M}S_{(f)}$, which is isomorphic to k hence is a field. $\square$

Lemma 2.3. *(Moret-Bailly) Let* $B = \bigoplus_{n\in\mathbb{Z}} B_n$ *be a* $\mathbb{Z}$*-graded ring. Assume that* B_d *contains an element* f *invertible in* B*, for some* $d > 0$*.*

(1) Then, the morphism of $\mathbb{Z}$*-graded rings*

$$\begin{array}{cccccc} \phi_f : & B_0[t,t^{-1}] & \longrightarrow & B^{(d)} & = & \bigoplus_{n\in d\mathbb{Z}} B_n \\ & t & \mapsto & f & & \end{array}$$

is an isomorphism.

(2) If moreover B_0 is a field and d is minimal for the properties $d > 0$ and $B_d \cap B^* \neq \emptyset$, then the composite map

$$B_0[t, t^{-1}] \xrightarrow{\phi_f} B^{(d)} \hookrightarrow B \to B_{red}$$

is a graded ring isomorphism.

Proof. Let $m \in \mathbb{Z}$. Since $f \in B_d \cap B^*$, the restriction of ϕ_f to the homogeneous part of some degree $m \in \mathbb{Z}$

$$\begin{array}{ccc} B_0 t^m & \longrightarrow & B_{dm} \\ b_0 t^m & \mapsto & b_0 f^m \end{array}$$

is an isomorphism of $\mathbb{Z}$-modules, from which the first item follows.

For the second item, we begin by proving that for any $e \notin d\mathbb{Z}$ and $g \in B_e$, we have $g^d = 0_B$. Considering the Euclidean division $e = dq + r$ with $0 < r < d$ of e by d, we have that $gf^{-q} \in B_{e-dq} = B_r$ with $r > 0$, so by minimality of d we deduce that $gf^{-q} \notin B^*$. Since $f \in B^*$, it follows that $g \notin B^*$, and then that $g^d f^e \notin B^*$. But $g^d f^e \in B_{ed-de} = B_0$ which is a field, so $g^d f^e = 0_B$, hence $g^d = 0_B$.

Now, let $\mathfrak{N}$ be the nilradical of B and let $\pi : B \to B_{\text{red}} = B/\mathfrak{N}$ be the canonical morphism. We have to prove, thanks to the first item, that the graded ring morphism

$$\pi_{|_{B^{(d)}}} : B^{(d)} \hookrightarrow B \xrightarrow{\pi} B/\mathfrak{N}$$

is an isomorphism.

The morphism π is onto from $B = B^{(d)} \oplus \left(\bigoplus_{e \notin d\mathbb{Z}} B_e\right)$ to $B/\mathfrak{N}$ and sends the right part $\bigoplus_{e \notin d\mathbb{Z}} B_e$ to 0_B by the previous paragraph, so π remains onto from the first part $B^{(d)}$.

Now let $h \in B^{(d)} \cap \text{Ker}(\pi)$ and $b_{0,m} f^m$ be the homogeneous part of some degree dm with $b_{0,m} \in B_0$. Since π is a graded ring morphism, we have $b_{0,m} f^m \in \text{Ker}(\pi) = \mathfrak{N}$. From $f \in B^*$ we deduce that $b_{0,m} \in \mathfrak{N} \cap B_0$ is a nilpotent element in the field B_0, hence is equal to zero. We conclude that $\pi_{|_{B^{(d)}}}$ is an isomorphism. □

Consider a rational point of a weighted projective space over a finite field k with q elements. Starting from a rational representative whose existence follows from Proposition 2.1, one can prove (see Lemma 7 in [10]) that it has exactly $q - 1$ representatives in $k^{n+1} \setminus \{0\}$. In particular we have $\sharp\mathbb{P}(a_0, \ldots, a_n)(\mathbb{F}_q) = p_n$.

For many more details about weighted projective spaces, one can consult the article of Beltrametti and Robbiano (see [4]) for a theory over an algebraically closed field of characteristic 0, the article of Dolgachev (see

[9]) for a theory over a field of characteristic prime to all the a_i's, and the Appendix of [1] for a survey of the different points of view.

2.2. **A lower bound.** Let F be a homogeneous polynomial in S of degree d, so that

$$F(\lambda^{a_0}X_0,\dots,\lambda^{a_n}X_n)=\lambda^d F(X_0,\dots,X_n) \ \text{ for all } \ \lambda\in\overline{\mathbb{F}}_q^*$$

and let $V(F)$ be the hypersurface defined by F in $\mathbb{P}(a_0,\dots,a_n)$.

We define, as in the introduction, the quantity:

$$e_q(d;a_0,\dots,a_n):=\max_{F\in S_d\setminus\{0\}}\sharp V(F)(\mathbb{F}_q)$$

where S_d stands for the space of weighted homogeneous polynomials in S of weighted degree d. Remark that the previous quantity is only defined for $d\in a_0\mathbb{N}+\cdots+a_n\mathbb{N}$.

Consider now the polynomial

$$F=\prod_{i=1}^{d/a_{rs}}(\alpha_i X_r^{a_{rs}/a_r}-\beta_i X_s^{a_{rs}/a_s})$$

where $r,s\in\{0,\dots,n\}$ are distincts indices, $a_{rs}=\mathrm{lcm}(a_r,a_s)$, d is a multiple of a_{rs} satisfying $d\le a_{rs}(q+1)$ and the (α_i,β_i)'s are distinct elements of $\mathbb{P}^1(\mathbb{F}_q)$. It has been proved in [1] that $\sharp V(F)(\mathbb{F}_q)=(d/a_{rs})q^{n-1}+p_{n-2}$. So, if $a:=\min\{\mathrm{lcm}(a_r,a_s),0\le r<s\le n\}$ and $a\mid d$, then it implies that

$$e_q(d;a_0,\dots,a_n)\ge\min\{p_n,\frac{d}{a}q^{n-1}+p_{n-2}\}.$$

3. Some morphisms between weighted projective spaces

3.1. **The morphisms π_i.** For $i=0,\dots,n$, we consider the following morphims π_i :

$$\begin{array}{cccc}\pi_i & : & \mathbb{P}(a_0,\dots,a_{i-1},1,a_{i+1},\dots,a_n) & \longrightarrow & \mathbb{P}(a_0,\dots,a_n)\\ & & [x_0:\cdots:x_n] & \longmapsto & [x_0:\cdots:x_i^{a_i}:\cdots:x_n].\end{array}$$

Our purpose in this Section is to study the behaviour of the rational points with respect to these morphisms. For this purpose, let us fix some generator δ of the multiplicative group $\mathbb{F}_q^*$.

For any given $i\in\{0,\dots,n\}$, set $r_i=(a_i,q-1)$ the gcd of a_i and $q-1$ and consider the map φ_{a_i}:

$$\begin{array}{cccc}\varphi_{a_i} & : & \mathbb{F}_q^* & \longrightarrow & \mathbb{F}_q^*\\ & & z & \longmapsto & z^{a_i}.\end{array}$$

Recall that the map φ_{a_i} is a group homomorphism with kernel $\mathrm{Ker}(\varphi_{a_i})=<\delta^{\frac{q-1}{r_i}}>=:\mu_{a_i}$, the subgroup of $\mathbb{F}_q^*$ of a_i-th roots of unity in $\mathbb{F}_q^*$ which has order r_i, and with image $\mathrm{Im}(\varphi_{a_i}) =< \delta^{a_i} >=: \Delta^{a_i}$, the subgroup of $\mathbb{F}_q^*$ of a_i-th powers which has order $\frac{q-1}{r_i}$.

Let $\mathcal{P}$ be the whole set of rational points over $\mathbb{F}_q$ of $\mathbb{P}(a_0, \ldots, a_n)$. We have a partition $\mathcal{P} = \mathcal{R}_i \cup \mathcal{T}_i \cup \mathcal{I}_i$ with respect to the i-th coordinate, where

$$\mathcal{R}_i := \{[y_0 : \cdots : y_n] \in \mathbb{P}(a_0, \ldots, a_n)(\mathbb{F}_q) \mid y_i = 0\} \cup \{\mathcal{O}_i\},$$

$$\mathcal{T}_i := \{[y_0 : \cdots : y_n] \in \mathbb{P}(a_0, \ldots, a_n)(\mathbb{F}_q) \mid y_i = 1\} \setminus \{\mathcal{O}_i\},$$

$$\mathcal{I}_i := \{[y_0 : \cdots : y_n] \in \mathbb{P}(a_0, \ldots, a_n)(\mathbb{F}_q) \mid y_i \in \mathbb{F}_q^* \setminus \Delta^{a_i}\}$$

and $\mathcal{O}_i := [0 : \cdots : 0 : 1 : 0 : \cdots : 0]$ is the point where 1 appears at the index i.

Let us scrutinize more narrowly the sets $\mathcal{I}_i$ and $\mathcal{T}_i$. In order to do this, consider, for $j \in \{1, \ldots, q-1\}$, the sets $\mathcal{Z}_i(j)$ defined by

$$\mathcal{Z}_i(j) := \{[y_0 : \cdots : y_n] \in \mathbb{P}(a_0, \ldots, a_n)(\mathbb{F}_q) \mid y_i = \delta^j\}.$$

Lemma 3.1. *We have:*

(i) $\mathcal{Z}_i(j_1) = \mathcal{Z}_i(j_2)$ *if* $j_1 \equiv j_2 \pmod{r_i}$.

(ii) $\mathcal{Z}_i(r_i) = \mathcal{T}_i$.

(iii) $\mathcal{I}_i = \emptyset$ *if* $r_i = 1$ *and*

$$\mathcal{I}_i = \mathcal{Z}_i(1) \cup \ldots \cup \mathcal{Z}_i(r_i - 1)$$

otherwise.

Proof. We begin by proving that $\delta^{r_i} = \lambda^{a_i}$ for some $\lambda \in \mathbb{F}_q^*$, which will be used in the proof of the three items. Indeed, there exist by Bézout Theorem some integers u, v such that $r_i = ua_i + v(q-1)$, so that $\delta^{r_i} = (\delta^u)^{a_i} \times (\delta^{q-1})^v = \lambda^{a_i}$ for $\lambda = \delta^u$.

Suppose now that $j_2 = j_1 + mr_i$ for some integer m and consider some $[y_0 : \cdots : y_n] \in \mathcal{Z}_i(j_2)$. By writing $\delta^{r_i} = \lambda^{a_i}$, it is easily checked from $\delta^{j_2} = (\delta^{r_i})^m \times \delta^{j_1} = (\lambda^m)^{a_i} \times \delta^{j_1}$ that $[y_0 : \cdots : y_{i-1} : \delta^{j_2}, y_{i+1} : \cdots , y_n] = [(\lambda^{-m})^{a_0} y_0 : \cdots : (\lambda^{-m})^{a_{i-1}} y_{i-1} : \delta^{j_1} : (\lambda^{-m})^{a_{i+1}} y_{i+1} : \cdots : (\lambda^{-m})^{a_n} y_n]$ which lies in $\mathcal{Z}_i(j_1)$, so that $\mathcal{Z}_i(j_2) \subset \mathcal{Z}_i(j_1)$. The reverse inclusion follows similarly.

The second item can be proved likewise by writing $\delta^{r_i} = \lambda^{a_i}$, since then $[y_0 : \cdots : y_{i-1} : \delta^{r_i} : y_{i+1} : \cdots : y_n] = [(\lambda^{-1})^{a_0} y_0 : \cdots : (\lambda^{-1})^{a_{i-1}} y_{i-1} : 1 : (\lambda^{-1})^{a_{i+1}} y_{i+1} : \cdots : (\lambda^{-1})^{a_n} y_n]$.

Finally, the set $\mathcal{I}_i$ contains of course the union $\mathcal{Z}_i(1) \cup \ldots \cup \mathcal{Z}_i(r_i - 1)$. Conversely, given some $P = [y_0 : \cdots : y_{i-1} : \delta^h : y_{i+1} : \cdots : y_n] \in \mathcal{I}_i$ with $1 \leq h \leq q-1$ and h not divisible by a_i, then writing the Euclidean division of h by r_i gives the existence of integers m and j such that $h = r_i m + j$ with $0 \leq j \leq r_i - 1$. Thus, writing $\delta^h = (\delta^{r_i})^m \times \delta^j = (\lambda^m)^{a_i} \times \delta^j$, we get $[y_0 : \cdots : y_{i-1} : \delta^h : y_{i+1} : \cdots : y_n] = [(\lambda^{-m})^{a_0} y_0 : \cdots : (\lambda^{-m})^{a_{i-1}} y_{i-1} : \delta^j : (\lambda^{-m})^{a_{i+1}} y_{i+1} : \cdots : (\lambda^{-m})^{a_n} y_n]$, so that $P \in \mathcal{Z}_i(j)$ for this $j \in \{1, \cdots, r_i - 1\}$ which concludes the proof.

□

The following proposition describes the number of pre-images of points by the morphism π_i according to the set of the partition that they belong to.

Proposition 3.2. *Let P be a rational point of $\mathbb{P}(a_0, \ldots, a_n)$.*

(i) If $P \in \mathcal{R}_i$ then P has exactly one pre-image rational over $\mathbb{F}_q$ by π_i.

(ii) If $P \in \mathcal{T}_i$ then P has exactly r_i pre-images rational over $\mathbb{F}_q$ by π_i.

(iii) If $P \in \mathcal{I}_i$ then P has no pre-image rational over $\mathbb{F}_q$ by π_i.

Proof. (i) The point $\mathcal{O}_i := [0 : \cdots : 0 : 1 : 0 : \cdots : 0] \in \mathbb{P}(a_0, \ldots, a_n)$ has only one pre-image by π_i, namely the point $[0 : \cdots : 0 : 1 : 0 : \cdots : 0] \in \mathbb{P}(a_0, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_n)$. Moreover, the point $[y_0 : \cdots : y_{i-1} : 0 : y_{i+1} : \cdots : y_n]$ has only one pre-image by π_i, that is the point $[y_0 : \cdots : y_{i-1} : 0 : y_{i+1} : \cdots : y_n]$.

(ii) The point $[y_0 : \cdots : y_{i-1} : 1 : y_{i+1} : \cdots : y_n]$ has r_i pre-images by π_i, which are precisely the points $[y_0 : \cdots : y_{i-1} : \delta^{\frac{(q-1)k}{r_i}} : y_{i+1} : \cdots : y_n]$ for $k = 1, \ldots, r_i$ (the elements $\delta^{\frac{(q-1)k}{r_i}}$ are the a_i-th roots of unity in $\mathbb{F}_q^*$ i.e. the elements of the group μ_{a_i}).

(iii) The points $[y_0 : \cdots : y_n]$ with $y_i \notin \Delta^{a_i}$ have no rational pre-image by π_i since y_i is not a a_i-th power in $\mathbb{F}_q^*$. □

3.2. **Number of zeros of the pullback.** Let F be a homogeneous polynomial in $\mathbb{F}_q[X_0, \ldots, X_n]$ of $(a_0, \ldots, a_n)$-weighted degree $d \leq q + 1$, i.e.

$$F(\lambda^{a_0} X_0, \ldots, \lambda^{a_n} X_n) = \lambda^d F(X_0, \ldots, X_n)$$

for any $\lambda \in \overline{\mathbb{F}}_q^*$. Let

$$\pi_i^* F(X_0, \ldots, X_n) := (F \circ \pi_i)(X_0, \ldots, X_n) = F(X_0, \ldots, X_i^{a_i}, \ldots, X_n)$$

be the pullback of F, an homogeneous polynomial of $(a_0, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_n)$-weighted degree d. We consider the hypersurface $V_{\mathbb{P}(a_0, \ldots, a_n)}(F)$ of zeros of F in $\mathbb{P}(a_0, \ldots, a_n)$ whose number of rational points over $\mathbb{F}_q$ is denoted by $N(F)$. We also consider the hypersurface $V_{\mathbb{P}(a_0, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_n)}(\pi^* F)$ of zeros of $\pi^* F$ in $\mathbb{P}(a_0, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_n)$ whose number of rational points over $\mathbb{F}_q$ is denoted by $N(\pi_i^* F)$.

Let us set:

$$A(F) := \sharp(V_{\mathbb{P}(a_0,\ldots,a_n)}(F) \cap \mathcal{A})$$

for $\mathcal{A} \in \{\mathcal{R}_i, \mathcal{T}_i, \mathcal{I}_i, \mathcal{Z}_i(j)\}$. So, $N(F)$ denotes the number of rational points of $V_{\mathbb{P}(a_0,\ldots,a_n)}(F)$ and $R_i(F), T_i(F), I_i(F)$ and $Z_i(j)(F)$ denote the number of those rational points lying on $\mathcal{R}_i, \mathcal{T}_i, \mathcal{I}_i$ and $\mathcal{Z}_i(j)$ respectively.

Proposition 3.3. *We have :*

(i)

$$N(F) = R_i(F) + T_i(F) + I_i(F).$$

(ii)

$$N(\pi_i^* F) = r_i T_i(F) + R_i(F).$$

(iii) Consider the automorphism $\sigma_i : [y_0 : \cdots : y_n] \longmapsto [y_0 : \cdots : y_{i-1} : \delta y_i : y_{i+1} : \cdots : y_n]$ *of* $\mathbb{P}(a_0, \ldots, a_n)$*. If* $r_i := (a_i, q-1) \neq 1$ *then:*
(a) for $j = 1, \ldots, r_i - 1$*, we have* $T_i(F \circ \sigma_i^j) = Z_i(j)(F)$*,*
(b) for $j = r_i - 1$*, we have* $T_i(F \circ \sigma_i^j) = T_i(F)$
(c) and $R_i(F) = R_i(F \circ \sigma_i^j)$ *for* $1 \leq j \leq r_i - 1$*.*

Proof. The first equality comes from the partition $\mathcal{P} = \mathcal{R}_i \cup \mathcal{T}_i \cup \mathcal{I}_i$.

The second one from Proposition 3.2 and the fact that if P is a rational point over $\mathbb{F}_q$ of $V_{\mathbb{P}(a_0,\ldots,a_{i-1},1,a_{i+1},\ldots,a_n)}(\pi^* F)$ then $\pi_i(P)$ is a point of $V_{\mathbb{P}(a_0,\ldots,a_n)}(F)$ which is rational over $\mathbb{F}_q$.

The third one follows from the fact that the automorphism σ_i sends $\mathcal{T}_i$ to $\mathcal{Z}_i(1)$ and $\mathcal{Z}_i(j)$ to $\mathcal{Z}_i(j+1)$ for $1 \leq j \leq r_i - 1$, and by Lemma 3.1 sends $\mathcal{Z}_i(r_i - 1)$ to $\mathcal{T}_i$, and leaves $\mathcal{R}_i$ stable. □

Now we are enable to prove a relation on the numbers of points between two floors.

Proposition 3.4. *Let* F *be a homogeneous polynomial in* $\mathbb{F}_q[X_0, \ldots, X_n]$ *with respect to the weights* $(a_0, a_1, \ldots, a_n)$*. For* $i \in \{0, \ldots, n\}$*, let*

$$\begin{array}{cccc} \pi_i : & \mathbb{P}(a_0, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_n) & \longrightarrow & \mathbb{P}(a_0, \ldots, a_n) \\ & [x_0 : \cdots : x_n] & \longmapsto & [x_0 : \cdots : x_i^{a_i} : \cdots : x_n] \end{array}$$

and $\pi_i^* F(X_0, \ldots, X_n) := (F \circ \pi_i)(X_0, \ldots, X_n) = F(X_0, \ldots, X_i^{a_i}, \ldots, , X_n)$ *be the pullback of* F*.*

Let also δ *be a primitive element of* $\mathbb{F}_q^*$*, and* $\sigma_i : [y_0 : \cdots : y_n] \longmapsto [y_0 : \cdots : y_{i-1} : \delta y_i : y_{i+1} : \cdots : y_n]$ *inside* $\mathbb{P}(a_0, \ldots, a_n)$*. Denote by* $r_i = (a_i, q-1)$ *the gcd of* a_i *with* $q - 1$*.*

Then, the number $N(F)$ of rational points over $\mathbb{F}_q$ of the hypersurface of the weighted projective space $\mathbb{P}(a_0, a_1, \ldots, a_n)$ defined by F satisfies

$$N(F) \leq \frac{1}{r_i} \sum_{j=0}^{r_i-1} N(\pi_i^*(F \circ \sigma_i^j)).$$

Proof. If $r_i = 1$, then the set I_i is empty and by (i) and (ii) of Proposition 3.3, we have $N(F) = R_i(F) + T_i(F) = N(\pi_i^* F)$ which gives the result.

Suppose now that $r_i \neq 1$. By (i) of Proposition 3.3, we have:

$$r_i N(F) = (r_i T_i(F) + R_i(F)) + (r_i I_i(F) + (r_i - 1)R_i(F)).$$

On one hand, we have by (ii) of Proposition 3.3 that $r_i T_i(F) + R_i(F) = N(\pi_i^* F)$ and on the other hand, by Lemma 3.1, we can write $I_i(F) \leq \sum_{j=1}^{r_i-1} Z_i(j)(F)$. Thus, we have:

$$\begin{aligned} r_i I_i(F) + (r_i - 1)R_i(F) &\leq r_i \left(\sum_{j=1}^{r_i-1} Z_i(j)(F) \right) + (r_i - 1)R_i(F) \\ &= \sum_{j=1}^{r_i-1} \left(r_i Z_i(j)(F) + R_i(F) \right). \end{aligned}$$

Moreover, by Proposition 3.3 (iii), we have:

$$r_i Z_i(j)(F) + R_i(F) = r_i T_i(F \circ \sigma_i^j) + R_i(F \circ \sigma_i^j)$$

and we obtain with Proposition 3.3 (ii):

$$r_i Z_i(j)(F) + R_i(F) = N(\pi_i^*(F \circ \sigma_i^j)).$$

Thus we deduce that:

$$r_i I_i(F) + (r_i - 1)R_i(F) = \sum_{j=1}^{r_i-1} N(\pi_i^*(F \circ \sigma_i^j))$$

and we obtain the desired formula.

□

Remark 3.5. Note that under the additional assumption that $(a_i, a_j) = 1$ for any $1 \leq i \neq j \leq n$, we have equality in the above Proposition 3.4. This comes from the fact that, under this assumption, the sets $\mathcal{Z}_i(j)$ for $1 \leq j \leq r_i - 1$ form a partition of $\mathcal{I}_i$, hence both inequalities in the above proof are equalities. It remains to show that the sets $\mathcal{Z}_i(j)$ for $1 \leq j \leq r_i - 1$ are pairwise disjoint. Indeed, suppose that there is some common point with $\mathbb{F}_q$-coordinates

$$[y_0 : \cdots : y_{i-1} : \delta^{j_1} : y_{i+1} : \cdots : y_n] = [y'_0 : \cdots : y'_{i-1} : \delta^{j_2} : y'_{i+1} : \cdots : y'_n]$$

inside $\mathcal{Z}_i(j_1) \cap \mathcal{Z}_i(j_2)$, with say $1 \leq j_1 \leq j_2 \leq r_i - 1$. Since this point does not lie in $\mathcal{R}_i$, there is at least one position $k \neq i$, such that $y_k \neq 0 \neq y'_k$. Since they are equal, there is some $\lambda \in \overline{\mathbb{F}}_q^*$ such that

$$(y'_0, \cdots, y'_{i-1}, \delta^{j_2}, y'_{i+1}, \cdots, y'_n) = (\lambda^{a_0} y_0, \cdots, \lambda^{a_{i-1}} y_{i-1}, \lambda^{a_i} \delta^{j_1}, \lambda^{a_{i+1}} y_{i+1}, \cdots, \lambda^{a_n} y_n).$$

Looking at the k-th and the i-th position, we get $y'_k = \lambda^{a_k} y_k$ and $\delta^{j_2} = \lambda^{a_i} \delta^{j_1}$. It follows first that $\lambda^{a_k} = \frac{y'_k}{y_k} \in \mathbb{F}_q^*$, second that $\lambda^{a_i} = \delta^{j_2 - j_1}$. But from a Bézout relation $ua_k + va_i = 1$, we deduce that

$$\lambda = (\lambda^{a_k})^u \times (\lambda^{a_i})^v = (\frac{y'_k}{y_k})^u \times (\delta^{j_2 - j_1})^v \in \mathbb{F}_q^*.$$

Hence, we have $\lambda = \delta^m$ for some $m \in \mathbb{N}$, so that $\delta^{j_2 - j_1} = \lambda^{a_i} = \delta^{m a_i}$. It follows that $j_2 - j_1 \equiv m a_i \pmod{q-1}$. Since $r_i = (a_i, q-1)$ divides both a_i and $q - 1$, it divides $j_2 - j_1 \in \{0, \cdots, r_i - 1\}$, hence $j_1 = j_2$ and we are done.

4. An upper bound for the number of rational points

We prove in this section that an hypersurface in a weighted projective space cannot have more rational points than in a standard projective space. The proof is based on an unscrewing and uses Proposition 3.4.

$$\begin{array}{c}
\mathbb{P}(1,1,1,\ldots,1) = \mathbb{P}^n \\
\pi_n \big\downarrow \\
\vdots \\
\pi_1 \big\downarrow \\
\mathbb{P}(1, a_1, a_2 \ldots, a_n) \\
\pi_0 \big\downarrow \\
\mathbb{P}(a_0, a_1, a_2, \ldots, a_n)
\end{array}$$

Figure 1. Screwing of weighted projective spaces

Theorem 4.1. *Let F be a homogeneous polynomial in $\mathbb{F}_q[X_0, \ldots, X_n]$ of $(a_0, a_1, \ldots, a_n)$-weighted degree $d \leq q + 1$. Then the number $N(F)$ of rational points over $\mathbb{F}_q$ of the hypersurface of the weighted projective space $\mathbb{P}(a_0, a_1, \ldots, a_n)$ given by the set of zeros of F satisfies:*

$$N(F) \leq dq^{n-1} + p_{n-2}.$$

Proof. Let F be a homogeneous polynomial in $\mathbb{F}_q[X_0, \ldots, X_n]$ of $(a_0, a_1, \ldots, a_n)$-weighted degree d. We consider the successive pullbacks $\pi_0^*(F \circ \sigma_0^{j_0})$ with

$j_0 \in \{0, \dots, r_0 - 1\}$, and $\pi_1^*(\pi_0^*(F \circ \sigma_0^{j_0}) \circ \sigma_1^{j_1})$ with $j_1 \in \{0, \dots, r_1 - 1\}$, and so on, of F.

By Proposition 3.4, considering the morphism

$$\begin{array}{rcccl} \pi_0 & : & \mathbb{P}(1, a_1, \dots, a_n) & \longrightarrow & \mathbb{P}(a_0, a_1, \dots, a_n) \\ & & [x_0 : x_1 : \dots : x_n] & \longmapsto & [x_0^{a_0} : x_1 : \dots : x_n] \end{array}$$

we have:

$$N(F) \leq \frac{1}{r_0} \sum_{j_0=0}^{r_0-1} N(F_0(j_0))$$

where $F_0(j_0) = \pi_0^*(F \circ \sigma_0^{j_0})$. Then, considering the morphism

$$\begin{array}{rcccl} \pi_1 & : & \mathbb{P}(1, 1, a_2 \dots, a_n) & \longrightarrow & \mathbb{P}(1, a_1, \dots, a_n) \\ & & [x_0 : x_1 : x_2 : \dots : x_n] & \longmapsto & [x_0 : x_1^{a_1} : x_2 : \dots : x_n] \end{array}$$

we have for $0 \leq j_0 \leq r_0 - 1$:

$$N(F_0(j_0)) \leq \frac{1}{r_1} \sum_{j_1=0}^{r_1-1} N(F_1(j_1))$$

where $F_1(j_1) = \pi_1^*(F_0(j_0) \circ \sigma_1^{j_1})$.

Thus:

$$N(F) \leq \frac{1}{r_0 r_1} \sum_{j_0=0}^{r_0-1} \sum_{j_1=0}^{r_1-1} N(F_1(j_1)).$$

Continuing this process, we obtain

$$N(F) \leq \frac{1}{r_0 \dots r_n} \sum_{j_0=0}^{r_0-1} \dots \sum_{j_n=0}^{r_n-1} N(F_n(j_n)).$$

The last polynomials are homogeneous polynomials of degree d in the standard n-dimensional projective space $\mathbb{P}^n = \mathbb{P}(1, \dots, 1)$. Then we apply the Serre bound

$$N(F) \leq \frac{1}{r_0 \dots r_n} r_0 \dots r_n (dq^{n-1} + p_{n-2}) = dq^{n-1} + p_{n-2}$$

and we get the result. □

5. The main result

We are now enable to state and prove Conjecture 1.1 provided $a_1 = 1$ (it was already assumed in the conjecture that $a_0 = 1$).

Theorem 5.1. *For any degree d and for any nonnegative integers $a_2, \dots, a_n$, we have:*

$$e_q(d; 1, 1, a_2, \dots, a_n) = \min\{p_n, dq^{n-1} + p_{n-2}\}.$$

In other words, Conjecture 1.1 is true for any $(a_1, a_2, \dots, a_n)$ with $a_1 = 1$ and without any assumption on the degree d.

Proof. As seen in Subsection 2.1, a hypersurface of $\mathbb{P}(1,1,a_2\dots,a_n)$ has obviously a number of rational points less than or equal to p_n and the hypersurface defined by the homogeneous polynomial $X_0^{d-q-1}(X_0^qX_1-X_0X_1^q)$ of degree $d\geq q+1$ has p_n rational points (the degree is equal to d since we have supposed that the weights of X_0 and X_1 are equal to 1 in the graded ring $\mathbb{F}_q[X_0,\dots,X_n]$). Now if $d\leq q+1$, by Theorem 4.1 we have $e_q(d;1,1,a_2,\dots,a_n)\leq\min\{p_n,dq^{n-1}+p_{n-2}\}$ and the bound is met using the following degree d homogeneous polynomial:

$$F=\prod_{i=1}^{d}(\alpha_iX_0-\beta_iX_1)$$

where $(\alpha_1:\beta_1),\dots,(\alpha_d:\beta_d)$ are distinct elements of $\mathbb{P}^1(\mathbb{F}_q)$. □

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References

[1] Y. Aubry, W. Castryck, S. Ghorpade, G. Lachaud, M. O'Sullivan and S. Ram, Hypersurfaces in weighted projective spaces over finite fields with applications to coding theory, Algebraic geometry for coding theory and cryptography, 25–61, Assoc. Women Math. Ser., 9, Springer, Cham, 2017.
[2] P. Beelen, M. Datta and S. Ghorpade, Maximum number of common zeros of homogeneous polynomials over finite fields, Proc. Amer. Math. Soc. 146 (2018), no. 4, 1451–1468.
[3] P. Beelen, M. Datta and S. Ghorpade, A combinatorial approach to the number of solutions of homogeneous polynomial equations over finite fields, Moscow Math. Journal Vol. 22, Number 4, October-December 2022, Pages 565–593.
[4] M. Beltrametti and L. Robbiano, Introduction to the theory of weighted projective spaces, Expo. Math. 4 (1986), 111–162.
[5] M. Boguslavsky, On the number of solutions of polynomial systems, Finite Fields Appl. 3 (1997), no. 4, 287–299.
[6] M. Datta and S. Ghorpade, On a conjecture of Tsfasman and an inequality of Serre for the number of points of hypersurfaces over finite fields, Moscow Math. Journal Vol. 15, Number 4, October-December 2015, Pages 715–725.
[7] M. Datta and S. Ghorpade, Number of solutions of systems of homogeneous polynomial equations over finite fields, Proc. Amer. Math. Soc. 145 (2017), no. 2, 525–541.
[8] C. Delorme, Espaces projectifs anisotropes, Bull. Soc. Math. France. 103 (1975), no. 2, 203–223.
[9] I. Dolgachev, Weighted projective varieties, Group Actions and Vector Fields (Vancouver, B.C., 1981), (J. B. Carell, ed.), Lecture Notes in Mathematics, vol. 956, Springer, Berlin, 1982, pp. 34–71.

[10] M. Perret, On the number of points of some varieties over finite fields, Bull. London Math. Soc., 35 (2003), no. 3, 309–320.
[11] J. -P. Serre, Lettre à M. Tsfasman, Journées Arithmétiques, 1989, (Luminy, 1989), Astérisque, vol. 198-200, Société Mathématique de France, Paris, 1991, pp. 351–353.
[12] A. B. Sørensen, Projective Reed-Muler codes, IEEE Trans. Inform. Theory 37 (1991), no. 6, 1567–1576.

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