# MAXIMUM NUMBER OF RATIONAL POINTS ON HYPERSURFACES IN WEIGHTED PROJECTIVE SPACES OVER FINITE FIELDS

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Abstract. An upper bound for the maximum number of rational points on an hypersurface in a projective space over a finite field has been conjectured by Tsfasman and proved by Serre in 1989. The analogue question for hypersurfaces on weighted projective spaces has been considered by Castryck, Ghorpade, Lachaud, O'Sullivan, Ram and the first author in 2017. A conjecture has been proposed there and proved in the particular case of the dimension 2. We prove here the conjecture in any dimension provided the second weight is also equal to one.

*Dedicated to Sudhir Ghorpade for his $60^{th}$ birthday*[1].

## 1. Introduction

Let $\mathbb{F}_q$ be the finite field with $q$ elements and $\mathbb{P}^n(\mathbb{F}_q)$ be the set of rational points over $\mathbb{F}_q$ of the projective space of dimension $n \geq 1$. Let us set $p_n := q^n + \cdots + q + 1$ for $n \geq 0$ and $p_n := 0$ for $n < 0$. We have clearly $\sharp\mathbb{P}^n(\mathbb{F}_q) = p_n$.

Answering to a conjecture of Tsfasman made at the "Journées Arithmétiques de Luminy" in 1989, Serre proved in [11] (and independently Sørensen proved later in [12]) that if $F$ is a nonzero homogeneous polynomial in $\mathbb{F}_q[X_0, \ldots, X_n]$ of degree $d \geq 1$, then the number of rational points over $\mathbb{F}_q$ of the hypersurface $V(F)$ in $\mathbb{P}^n$ defined by $F$ satisfies the so-called Serre bound:

$$\sharp V(F)(\mathbb{F}_q) \leq dq^{n-1} + p_{n-2}.$$

Note that $dq^{n-1} + p_{n-2} \geq p_n = \sharp\mathbb{P}^n(\mathbb{F}_q)$ if $d \geq q + 1$ and thus the Serre bound is trivial in this case.

Moreover the Serre bound is reached: if $d \geq q + 1$ then the polynomial $X_0^{d-q-1}(X_0^q X_1 - X_0 X_1^q)$ is homogeneous of degree $d$ and the corresponding hypersurface has $p_m$ rational points, and if $d \leq q$ then the number of rational

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points on the hypersurface given by the polynomial

$$F = \prod_{i=1}^{d} (\alpha_i X_0 - \beta_i X_1),$$

where $(\alpha_1 : \beta_1), \ldots, (\alpha_d, \beta_d)$ are distincts elements of $\mathbb{P}^1(\mathbb{F}_q)$, attains the Serre bound. Note that Serre proved that the bound is reached for $d \leq q$ if and only if $F$ is of the above form, that is $V(F)$ is the union of $d$ hyperplanes containing a linear variety of codimension 2.

In 1997, Tsfasman and Boguslavsky in [5] have considered the analogue question for a system of $r$ polynomial equations. They propose a conjecture for the maximum number of points in $\mathbb{P}^n(\mathbb{F}_q)$ of the projective set given by the common zeros of $r$ linearly independent homogeneous polynomials of degree $d$ in $\mathbb{F}_q[X_0, \ldots, X_n]$. The Tsfasman-Boguslavsky conjecture for $r = 1$ is nothing else but the Serre bound. Boguslavsky succeded to prove in [5] the case $r = 2$. In 2015, Datta and Ghorpade proved in [6] that the Tsfasman-Boguslavsky conjecture is true if $d = 2$ and $r \leq n + 1$ but is false in general if $d = 2$ and $r \geq m + 2$. Moreover, in 2017 they proved in [7] that the Tsfasman-Boguslavsky conjecture is true for any positive integer $d$, provided $r \leq n + 1$. The case for $r$ beyond $n + 1$ is specifically considered one year later by Beelen, Datta and Ghorpade in [2] and they conjectured in 2022 in [3], and proved it in some cases, a general formula when $d < q$.

We are interested here in a generalization in another direction, namely the question of Tsfasman and Serre in the context of weighted projective spaces $\mathbb{P}(a_0, \ldots, a_n)$, i.e. the study, for any homogeneous polynomial $F$ in $\mathbb{F}_q[X_0, \ldots, X_n]$ of degree $d$ (with respect to the weights $a_0, \ldots, a_n$), of the maximum number of rational points on the hypersurface $V(F)$ in $\mathbb{P}(a_0, \ldots, a_n)$. In [1], the following quantity has been introduced:

$$e_q(d; a_0, a_1, a_2, \ldots, a_n) := \max_F \sharp V(F)(\mathbb{F}_q)$$

where the maximum ranges over the set of homogeneous polynomials $F$ in $\mathbb{F}_q[X_0, \ldots, X_n]$ of weighted degree $d$.

It has been conjectured in 2017 in [1] that:

**Conjecture 1.1.** *If $a_0 = 1$ and $\operatorname{lcm}(a_1, a_2, \ldots, a_n)|d$, and if we order the weights such that $a_1 \leq a_2 \leq \ldots \leq a_n$ then*

$$e_q(d; 1, a_1, a_2, \ldots, a_n) = \min\{p_n, \frac{d}{a_1} q^{n-1} + p_{n-2}\}.$$

In the case of the projective line $\mathbb{P}(a_0, a_1)$, it has been shown in [1] that $e_q(d; a_0, a_1) = \min\{p_1, d/a\}$ where $a = \operatorname{lcm}(a_0, a_1)$, so the conjecture holds in this case. Moreover, the conjecture has been proved in [1] for projective planes $\mathbb{P}(1, a_1, a_2)$ with $a_1$ and $a_2$ coprime and $a_1 < a_2$: if $F \in \mathbb{F}_q[X_0, X_1, X_2]$ is a nonzero weighted homogenous polynomial of degree $d \leq a_1(q+1)$ which is a multiple of $a_1 a_2$ then $\sharp V(F)(\mathbb{F}_q) \leq \frac{d}{a_1}q + 1$. The proof goes by imitating the proof of Serre by introducing a new notion of line, namely homogenized linear bivariate equations or the line at infinity.

Our purpose here is to prove Conjecture 1.1 in any dimension $n$ provided $a_1 = 1$.

We recall in Section 2 the basic facts about weighted projective spaces and a lower bound for $e_q(d; a_0, \ldots, a_n)$. Then we study in Section 3 some morphisms between weighted projective spaces and we establish a relation between the numbers of zeros of a polynomial and its pullback. Section 4 is devoted to the proof of an upper bound for the number of rational points on an hypersurface in a weighted projective space. Finally we state and prove the main result in Section 5.

## 2. A lower bound for the number of rational points

2.1. **Weighted projective spaces.** Let $a_0, \ldots, a_n$ be positive integers coprime with the characteristic of $\mathbb{F}_q$ and $S$ be the polynomial ring $\mathbb{F}_q[X_0, \ldots, X_n]$ graded by $\deg(X_i) = a_i$. The weighted projective space $\mathbb{P}(a_0, \ldots, a_n)$ is defined by

$$\mathbb{P}(a_0, \ldots, a_n) = \operatorname{Proj} S,$$

and can be seen as the geometric quotient

$$\mathbb{A}^{n+1}_{\mathbb{F}_q} \setminus \{0\} / \mathbb{G}_{m,\mathbb{F}_q}$$

of the punctured affine space $\mathbb{A}^{n+1}_{\mathbb{F}_q} \setminus \{0\}$ over $\mathbb{F}_q$ under the action of the multiplicative group $\mathbb{G}_{m,\mathbb{F}_q}$ over $\mathbb{F}_q$ given by

$$\lambda.(x_0, \ldots, x_n) = (\lambda^{a_0} x_0, \ldots, \lambda^{a_n} x_n).$$

If the $a_i$'s are all equal to 1, then we recover the usual (or straight) projective space: $\mathbb{P}(1, \ldots, 1) = \mathbb{P}^n$.

The corresponding equivalent class is denoted by $[x_0 : \cdots : x_n]$ without any reference to the corresponding weights $a_0, \ldots, a_n$ and is called a weighted projective point. We say that the point is $\mathbb{F}_q$-rational if $[x_0 : \cdots : x_n] = [x_0^q : \cdots : x_n^q]$ . It can be shown (see [10]) that every $\mathbb{F}_q$-rational point has at least one representative (in fact exactly $q-1$) in $\mathbb{F}_q^{n+1} \setminus \{(0, \ldots, 0)\}$. In particular we have $\sharp\mathbb{P}(a_0, \ldots, a_n)(\mathbb{F}_q) = p_n$.

For many more details about weighted projective spaces, one can consult the article of Beltrametti and Robbiano (see [4]) for a theory over an algebraically closed field of characteristic 0, the article of Dolgachev (see [9]) for a theory over a field of characteristic prime to all the $a_i$'s, and the Appendix of [1] for a survey of the different points of view.

2.2. **A lower bound.** Let $F$ be a homogeneous polynomial in $S$ of degree $d$, so that

$$F(\lambda^{a_0}X_0,\dots,\lambda^{a_n}X_n)=\lambda^d F(X_0,\dots,X_n)\ \text{ for all }\ \lambda\in\overline{\mathbb{F}}_q^*$$

where $\overline{\mathbb{F}}_q$ is an algebraic closure of $\mathbb{F}_q$, and let $V(F)$ be the hypersurface defined by $F$ in $\mathbb{P}(a_0,\dots,a_n)$.

We define, as in the introduction, the quantity:

$$e_q(d;a_0,\dots,a_n):=\max_{F\in S_d\setminus\{0\}}\sharp V(F)(\mathbb{F}_q)$$

where $S_d$ stands for the space of weighted homogeneous polynomials in $S$ of weighted degree $d$. Remark that the previous quantity is only defined for $d\in a_0\mathbb{N}+\cdots+a_n\mathbb{N}$.

Consider now the polynomial

$$F=\prod_{i=1}^{d/a_{rs}}(\alpha_i X_r^{a_{rs}/a_r}-\beta_i X_s^{a_{rs}/a_s})$$

where $r,s\in\{0,\dots,n\}$ are distincts indices, $a_{rs}=\mathrm{lcm}(a_r,a_s)$, $d$ is a multiple of $a_{rs}$ satisfying $d\le a_{rs}(q+1)$ and the $(\alpha_i,\beta_i)$'s are distinct elements of $\mathbb{P}^1(\mathbb{F}_q)$. It has been proved in [1] that $\sharp V(F)(\mathbb{F}_q)=(d/a_{rs})q^{n-1}+p_{n-2}$. So, if $a:=\min\{\mathrm{lcm}(a_r,a_s),0\le r<s\le n\}$ and $a\mid d$, then it implies that

$$e_q(d;a_0,\dots,a_n)\ge\min\{p_n,\frac{d}{a}q^{n-1}+p_{n-2}\}.$$

## 3. Some morphisms between weighted projective spaces

3.1. **The morphisms $\pi_i$.** For $i=0,\dots,n$, we consider the following morphims $\pi_i$ :

$$\begin{array}{cccccc}\pi_i & : & \mathbb{P}(a_0,\dots,a_{i-1},1,a_{i+1},\dots,a_n) & \longrightarrow & \mathbb{P}(a_0,\dots,a_n)\\ & & [x_0:\cdots:x_n] & \longmapsto & [x_0:\cdots:x_i^{a_i}:\cdots:x_n].\end{array}$$

Our purpose in this Section is to study the behaviour of the rational points with respect to this morphisms. For this purpose, let us fix some generator $\delta$ of the multiplicative group $\mathbb{F}_q^*$.

For any given $i\in\{0,\dots,n\}$, set $r_i=(a_i,q-1)$ the gcd of $a_i$ and $q-1$ and consider the map $\varphi_{a_i}$:

$$\begin{array}{ccccc}\varphi_{a_i} & : & \mathbb{F}_q^* & \longrightarrow & \mathbb{F}_q^*\\ & & z & \longmapsto & z^{a_i}.\end{array}$$

Recall that the map $\varphi_{a_i}$ is a group homomorphism with kernel $\mathrm{Ker}(\varphi_{a_i}) =< \delta^{\frac{q-1}{r_i}} >=: \mu_{a_i}$, the subgroup of $\mathbb{F}_q^*$ of $a_i$-th roots of unity in $\mathbb{F}_q^*$ which has order $r_i$, and with image $\mathrm{Im}(\varphi_{a_i}) =< \delta^{a_i} >=: \Delta^{a_i}$, the subgroup of $\mathbb{F}_q^*$ of $a_i$-th powers which has order $\frac{q-1}{r_i}$.

Let $\mathcal{P}$ be the whole set of rational points over $\mathbb{F}_q$ of $\mathbb{P}(a_0, \ldots, a_n)$. We have a partition $\mathcal{P} = \mathcal{R}_i \cup \mathcal{T}_i \cup \mathcal{I}_i$ with respect to the $i$-th coordinate, where

$$\mathcal{R}_i := \{[y_0 : \cdots : y_n] \in \mathbb{P}(a_0, \ldots, a_n)(\mathbb{F}_q) \mid y_i = 0\} \cup \{\mathcal{O}_i\},$$

$$\mathcal{T}_i := \{[y_0 : \cdots : y_n] \in \mathbb{P}(a_0, \ldots, a_n)(\mathbb{F}_q) \mid y_i = 1\} \setminus \{\mathcal{O}_i\},$$

$$\mathcal{I}_i := \{[y_0 : \cdots : y_n] \in \mathbb{P}(a_0, \ldots, a_n)(\mathbb{F}_q) \mid y_i \in \mathbb{F}_q^* \setminus \Delta^{a_i}\}$$

and $\mathcal{O}_i := [0 : \cdots : 0 : 1 : 0 : \cdots : 0]$ is the point where the "1" appears only for index "i".

Let us scrutinize more narrowly the sets $\mathcal{I}_i$ and $\mathcal{T}_i$. In order to do this, consider, for $j \in \{1, \ldots, q-1\}$, the sets $\mathcal{Z}_i(j)$ defined by

$$\mathcal{Z}_i(j) := \{[y_0 : \cdots : y_n] \in \mathbb{P}(a_0, \ldots, a_n)(\mathbb{F}_q) \mid y_i = \delta^j\}.$$

**Lemma 3.1.** *We have:*

*(i)* $\mathcal{Z}_i(j_1) = \mathcal{Z}_i(j_2)$ *if* $j_1 \equiv j_2 \pmod{r_i}$.
*(ii)* $\mathcal{Z}_i(r_i) = \mathcal{T}_i$.
*(iii)* $\mathcal{I}_i = \emptyset$ *if* $r_i = 1$ *and*

$$\mathcal{I}_i = \mathcal{Z}_i(1) \cup \ldots \cup \mathcal{Z}_i(r_i - 1)$$

*otherwise.*

*Proof.* We begin by proving that $\delta^{r_i} = \lambda^{a_i}$ for some $\lambda \in \mathbb{F}_q^*$, which will be used in the proof of the three items. Indeed, there exist by Bézout Theorem some integers $u, v$ such that $r_i = ua_i + v(q-1)$, so that $\delta^{r_i} = (\delta^u)^{a_i} \times (\delta^{q-1})^v = \lambda^{a_i}$ for $\lambda = \delta^u$.

Suppose now that $j_2 = j_1 + mr_i$ for some integer $m$ and consider some $[y_0 : \cdots : y_n] \in \mathcal{Z}_i(j_2)$. By writing $\delta^{r_i} = \lambda^{a_i}$, it is easily checked from $\delta^{j_2} = (\delta^{r_i})^m \times \delta^{j_1} = (\lambda^m)^{a_i} \times \delta^{j_1}$ that $[y_0 : \cdots : y_{i-1} : \delta^{j_2}, y_{i+1} : \cdots , y_n] = [(\lambda^{-m})^{a_0} y_0 : \cdots : (\lambda^{-m})^{a_{i-1}} y_{i-1} : \delta^{j_1} : (\lambda^{-m})^{a_{i+1}} y_{i+1} : \cdots : (\lambda^{-m})^{a_n} y_n]$ which lies in $\mathcal{Z}_i(j_1)$, so that $\mathcal{Z}_i(j_2) \subset \mathcal{Z}_i(j_1)$. The reverse inclusion follows similarly.

The second item can be proved likewise by writing $\delta^{r_i} = \lambda^{a_i}$, since then $[y_0 : \cdots : y_{i-1} : \delta^{r_i} : y_{i+1} : \cdots : y_n] = [(\lambda^{-1})^{a_0} y_0 : \cdots : (\lambda^{-1})^{a_{i-1}} y_{i-1} : 1 : (\lambda^{-1})^{a_{i+1}} y_{i+1} : \cdots : (\lambda^{-1})^{a_n} y_n]$.

Finally, the set $\mathcal{I}_i$ contains of course the union $\mathcal{Z}_i(1) \cup \ldots \cup \mathcal{Z}_i(r_i - 1)$. Conversely, given some $P = [y_0 : \cdots : y_{i-1} : \delta^h : y_{i+1} : \cdots : y_n] \in \mathcal{I}_i$ with $1 \leq h \leq q-1$ not divisible by $a_i$, then writing the Euclidean division of $h$ by $r_i$ gives the existence of integers $m$ and $j$ such that $h = r_i m + j$ with $0 \leq j \leq r_i - 1$. Thus, writing $\delta^h = (\delta^{r_i})^m \times \delta^j = (\lambda^m)^{a_i} \times \delta^j$, we get $[y_0 : \cdots : y_{i-1} : \delta^h : y_{i+1} : \cdots : y_n] = [(\lambda^{-m})^{a_0} y_0 : \cdots : (\lambda^{-m})^{a_{i-1}} y_{i-1} : \delta^j : (\lambda^{-m})^{a_{i+1}} y_{i+1} : \cdots : (\lambda^{-m})^{a_n} y_n]$, so that $P \in \mathcal{Z}_i(j)$ for this $j \in \{1, \cdots, r_i - 1\}$ which concludes the proof.

□

The following proposition describes the number of pre-images of points by the morphism $\pi_i$ according to the set of the partition that they belong to.

**Proposition 3.2.** *Let $P$ be a rational point of $\mathbb{P}(a_0, \ldots, a_n)$.*

*(i) If $P \in \mathcal{R}_i$ then $P$ has exactly one pre-image rational over $\mathbb{F}_q$ by $\pi_i$.*
*(ii) If $P \in \mathcal{T}_i$ then $P$ has exactly $r_i$ pre-images rational over $\mathbb{F}_q$ by $\pi_i$.*
*(iii) If $P \in \mathcal{I}_i$ then $P$ has no pre-image rational over $\mathbb{F}_q$ by $\pi_i$.*

*Proof.* $(i)$ The point $\mathcal{O}_i := [0 : \cdots : 0 : 1 : 0 : \cdots : 0] \in \mathbb{P}(a_0, \ldots, a_n)$ has only one pre-image by $\pi_i$, namely the point $[0 : \cdots : 0 : 1 : 0 : \cdots : 0] \in \mathbb{P}(a_0, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_n)$. Moreover, the point $[y_0 : \cdots : y_{i-1} : 0 : y_{i+1} : \cdots : y_n]$ has only one pre-image by $\pi_i$, expressly the point $[y_0 : \cdots : y_{i-1} : 0 : y_{i+1} : \cdots : y_n]$.

$(ii)$ The point $[y_0 : \cdots : y_{i-1} : 1 : y_{i+1} : \cdots : y_n]$ has $r_i$ pre-images by $\pi_i$, which are precisely the points $[y_0 : \cdots : y_{i-1} : \delta^{\frac{(q-1)k}{r_i}} : y_{i+1} : \cdots : y_n]$ for $k = 1, \ldots, r_i$ (the elements $\delta^{\frac{(q-1)k}{r_i}}$ are the $a_i$-th roots of unity in $\mathbb{F}_q^*$ i.e. the elements of the group $\mu_{a_i}$).

$(iii)$ The points $[y_0 : \cdots : y_n]$ with $y_i \notin \Delta^{a_i}$ have no rational pre-image by $\pi_i$ since $y_i$ is not a $a_i$-th power in $\mathbb{F}_q^*$. □

3.2. **Number of zeros of the pullback.** Let $F$ be a homogeneous polynomial in $\mathbb{F}_q[X_0, \ldots, X_n]$ of $(a_0, \ldots, a_n)$-weighted degree $d \leq q+1$, i.e.

$$F(\lambda^{a_0} X_0, \ldots, \lambda^{a_n} X_n) = \lambda^d F(X_0, \ldots, X_n)$$

for any $\lambda \in \overline{\mathbb{F}}_q^*$. Let

$$\pi_i^* F(X_0, \ldots, X_n) := (F \circ \pi_i)(X_0, \ldots, X_n) = F(X_0, \ldots, X_i^{a_i}, \ldots, X_n)$$

be the pullback of $F$, an homogeneous polynomial of $(a_0, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_n)$-weighted degree $d$. We consider the hypersurface $V_{\mathbb{P}(a_0,\ldots,a_n)}(F)$ of zeros of $F$ in $\mathbb{P}(a_0, \ldots, a_n)$ whose number of rational points over $\mathbb{F}_q$ is denoted by

$N(F)$. We also consider the hypersurface $V_{\mathbb{P}(a_0,\ldots,a_{i-1},1,a_{i+1},\ldots,a_n)}(\pi^*F)$ of zeros of $\pi^*F$ in $\mathbb{P}(a_0,\ldots,a_{i-1},1,a_{i+1},\ldots,a_n)$ whose number of rational points over $\mathbb{F}_q$ is denoted by $N(\pi_i^*F)$.

Let us set:

$$A(F) := \sharp(V_{\mathbb{P}(a_0,\ldots,a_n)}(F) \cap \mathcal{A})$$

for $\mathcal{A} \in \{\mathcal{R}_i, \mathcal{T}_i, \mathcal{I}_i, \mathcal{Z}_i(j)\}$. So, $N(F)$ denotes the number of rational points of $V_{\mathbb{P}(a_0,\ldots,a_n)}(F)$ and $R_i(F), T_i(F), I_i(F)$ and $Z_i(j)(F)$ denote the number of those rational points lying on $\mathcal{R}_i, \mathcal{T}_i, \mathcal{I}_i$ and $\mathcal{Z}_i(j)$ respectively.

**Proposition 3.3.** *We have :*

*(i)*

$$N(F) = R_i(F) + T_i(F) + I_i(F).$$

*(ii)*

$$N(\pi_i^*F) = r_i T_i(F) + R_i(F).$$

*(iii) Consider the automorphism* $\sigma_i : [y_0 : \cdots : y_n] \longmapsto [y_0 : \cdots : y_{i-1} : \delta y_i : y_{i+1} : \cdots : y_n]$ *of* $\mathbb{P}(a_0,\ldots,a_n)$. *If* $r_i := (a_i, q-1) \neq 1$ *then:*
*(a) for* $j = 1,\ldots,r_i-1$, *we have* $T_i(F \circ \sigma_i^j) = Z_i(j)(F)$,
*(b) for* $j = r_i - 1$, *we have* $T_i(F \circ \sigma_i^j) = T_i(F)$
*(c) and* $R_i(F) = R_i(F \circ \sigma_i^j)$ *for* $1 \leq j \leq r_i - 1$.

*Proof.* The first equality comes from the partition $\mathcal{P} = \mathcal{R}_i \cup \mathcal{T}_i \cup \mathcal{I}_i$.

The second one from Proposition 3.2 and the fact that if $P$ is a rational point over $\mathbb{F}_q$ of $V_{\mathbb{P}(a_0,\ldots,a_{i-1},1,a_{i+1},\ldots,a_n)}(\pi^*F)$ then $\pi_i(P)$ is a point of $V_{\mathbb{P}(a_0,\ldots,a_n)}(F)$ which is rational over $\mathbb{F}_q$.

The third one follows from the fact that the automorphism $\sigma_i$ sends $\mathcal{T}_i$ to $\mathcal{Z}_i(1)$ and $\mathcal{Z}_i(j)$ to $\mathcal{Z}_i(j+1)$ for $1 \leq j \leq r_i - 1$, and by Lemma 3.1 sends $\mathcal{Z}_i(r_i-1)$ to $\mathcal{T}_i$, and leaves $\mathcal{R}_i$ stable. □

We are enable now to prove a relation on the numbers of points between two floors.

**Proposition 3.4.** *Let* $F$ *be a homogeneous polynomial in* $\mathbb{F}_q[X_0,\ldots,X_n]$ *with respect to the weights* $(a_0,a_1,\ldots,a_n)$. *For* $i \in \{0,\ldots,n\}$, *let*

$$\begin{array}{cccc} \pi_i : & \mathbb{P}(a_0,\ldots,a_{i-1},1,a_{i+1},\ldots,a_n) & \longrightarrow & \mathbb{P}(a_0,\ldots,a_n) \\ & [x_0 : \cdots : x_n] & \longmapsto & [x_0 : \cdots : x_i^{a_i} : \cdots : x_n] \end{array}$$

*and* $\pi_i^*F(X_0,\ldots,X_n) := (F \circ \pi_i)(X_0,\ldots,X_n) = F(X_0,\ldots,X_i^{a_i},\ldots,,X_n)$ *be the pullback of* $F$.

*Let also* $\delta$ *be a primitive element of* $\mathbb{F}_q^*$, *and* $\sigma_i : [y_0 : \cdots : y_n] \longmapsto [y_0 : \cdots : y_{i-1} : \delta y_i : y_{i+1} : \cdots : y_n]$ *inside* $\mathbb{P}(a_0,\ldots,a_n)$. *Denote by* $r_i = (a_i, q-1)$ *the gcd of* $a_i$ *with* $q-1$.

*Then, the number $N(F)$ of rational points over $\mathbb{F}_q$ of the hypersurface of the weighted projective space $\mathbb{P}(a_0, a_1, \ldots, a_n)$ defined by $F$ satisfies*

$$N(F) \leq \frac{1}{r_i} \sum_{j=0}^{r_i-1} N(\pi_i^*(F \circ \sigma_i^j)).$$

*Proof.* If $r_i = 1$, then the set $I_i$ is empty and by $(i)$ and $(ii)$ of Proposition 3.3, we have $N(F) = R_i(F) + T_i(F) = N(\pi_i^* F)$ which gives the result.

Suppose now that $r_i \neq 1$. By $(i)$ of Proposition 3.3, we have:

$$r_i N(F) = (r_i T_i(F) + R_i(F)) + (r_i I_i(F) + (r_i - 1) R_i(F)).$$

On one hand, we have by $(ii)$ of Proposition 3.3 that $r_i T_i(F) + R_i(F) = N(\pi_i^* F)$ and on the other hand, by Lemma 3.1, we can write $I_i(F) \leq \sum_{j=1}^{r_i-1} Z_i(j)(F)$. Thus, we have:

$$r_i I_i(F) + (r_i - 1) R_i(F) \leq r_i \left( \sum_{j=1}^{r_i-1} Z_i(j)(F) \right) + (r_i - 1) R_i(F) = \sum_{j=1}^{r_i-1} \left( r_i Z_i(j)(F) + R_i(F) \right).$$

Moreover, by Proposition 3.3 $(iii)$, we have:

$$r_i Z_i(j)(F) + R_i(F) = r_i T_i(F \circ \sigma_i^j) + R_i(F \circ \sigma_i^j)$$

and we obtain with Proposition 3.3 $(ii)$:

$$r_i Z_i(j)(F) + R_i(F) = N(\pi_i^*(F \circ \sigma_i^j)).$$

Thus we deduce that:

$$r_i I_i(F) + (r_i - 1) R_i(F) = \sum_{j=1}^{r_i-1} N(\pi_i^*(F \circ \sigma_i^j))$$

and we obtain the desired formula.

□

**Remark 3.5.** Note that under the additionnal assumption that $(a_i, a_j) = 1$ for any $1 \leq i \neq j \leq n$, we have equality in the above Proposition 3.4. This comes from the fact that, under this assumption, the sets $\mathcal{Z}_i(j)$ for $1 \leq j \leq r_i - 1$ form a partition of $\mathcal{I}_i$, hence both inequalities in the above proof are equalities. It remains to show that the sets $\mathcal{Z}_i(j)$ for $1 \leq j \leq r_i - 1$ are pairwise disjoint. Indeed, suppose that there is some common point with $\mathbb{F}_q$-coordinates

$$[y_0 : \cdots : y_{i-1} : \delta^{j_1} : y_{i+1} : \cdots : y_n] = [y'_0 : \cdots : y'_{i-1} : \delta^{j_2} : y'_{i+1} : \cdots : y'_n] \in \mathcal{Z}_i(j_1) \cap \mathcal{Z}_i(j_2),$$

with say $1 \leq j_1 \leq j_2 \leq r_i - 1$. Since this point does not lie in $\mathcal{R}_i$, there is at least one position $k \neq i$, such that $y_k \neq 0 \neq y'_k$. Since they are equal, there is some $\lambda \in \overline{\mathbb{F}}_q^*$ such that

$$(y'_0, \cdots, y'_{i-1}, \delta^{j_2}, y'_{i+1}, \cdots, y'_n) = (\lambda^{a_0} y_0, \cdots, \lambda^{a_{i-1}} y_{i-1}, \lambda^{a_i} \delta^{j_1}, \lambda^{a_{i+1}} y_{i+1}, \cdots, \lambda^{a_n} y_n).$$

From the $k$-th and the $i$-th position, we get $y'_k = \lambda^{a_k} y_k$ and $\delta^{j_2} = \lambda^{a_i}\delta^{j_1}$. It follows first that $\lambda^{a_k} = \frac{y'_k}{y_k} \in \mathbb{F}_q^*$, second that $\lambda^{a_i} = \delta^{j_2-j_1}$. But from a Bézout relation $ua_k + va_i = 1$, we deduce that

$$\lambda = (\lambda^{a_k})^u \times (\lambda^{a_i})^v = (\frac{y'_k}{y_k})^u \times (\delta^{j_2-j_1})^v \in \mathbb{F}_q^*.$$

Hence, we have $\lambda = \delta^m$ for some $m \in \mathbb{N}$, so that $\delta^{j_2-j_1} = \lambda^{a_i} = \delta^{ma_i}$. It follows that $j_2 - j_1 \equiv ma_i \pmod{q-1}$. Since $r_i = (a_i, q-1)$ divides both $a_i$ and $q-1$, it divides $j_2 - j_1 \in \{0, \cdots, r_i - 1\}$, hence $j_1 = j_2$ and we are done.

## 4. An upper bound for the number of rational points

We prove in this section that an hypersurface in a weighted projective space cannot have more rational point than in a standard projective space. The proof is based on an unscrewing and uses Proposition 3.4.

$$\begin{array}{c}
\mathbb{P}(1,1,1,\dots,1) = \mathbb{P}^n \\
\Big\downarrow \pi_n \\
\vdots \\
\Big\downarrow \pi_1 \\
\mathbb{P}(1, a_1, a_2 \dots, a_n) \\
\Big\downarrow \pi_0 \\
\mathbb{P}(a_0, a_1, a_2, \dots, a_n)
\end{array}$$

Figure 1. Screwing of weighted projective spaces

**Theorem 4.1.** *Let $F$ be a homogeneous polynomial in $\mathbb{F}_q[X_0, \dots, X_n]$ of $(a_0, a_1, \dots, a_n)$-weighted degree $d \leq q+1$. Then the number $N(F)$ of rational points over $\mathbb{F}_q$ of the hypersurface of the weighted projective space $\mathbb{P}(a_0, a_1, \dots, a_n)$ given by the set of zeros of $F$ satisfies:*

$$N(F) \leq dq^{n-1} + p_{n-2}.$$

*Proof.* Let $F$ be a homogeneous polynomial in $\mathbb{F}_q[X_0, \dots, X_n]$ of $(a_0, a_1, \dots, a_n)$-weighted degree $d$. We consider the successive pullbacks $\pi_0^*(F \circ \sigma_0^{j_0})$ with $j_0 \in \{0, \dots, r_0 - 1\}$, and $\pi_1^*(\pi_0^*(F \circ \sigma_0^{j_0}) \circ \sigma_1^{j_1})$ with $j_1 \in \{0, \dots, r_1 - 1\}$, and so on, of $F$.

By Proposition 3.4, considering the morphism

$$\begin{array}{cccc}
\pi_0 & : & \mathbb{P}(1, a_1, \dots, a_n) & \longrightarrow & \mathbb{P}(a_0, a_1, \dots, a_n) \\
 & & [x_0 : x_1 : \dots : x_n] & \longmapsto & [x_0^{a_0} : x_1 : \dots : x_n]
\end{array}$$

we have:

$$N(F) \leq \frac{1}{r_0} \sum_{j_0=0}^{r_0-1} N(F_0(j_0))$$

where $F_0(j_0) = \pi_0^*(F \circ \sigma_0^{j_0})$. Then, considering the morphism

$$\begin{array}{rccc} \pi_1 & : & \mathbb{P}(1,1,a_2\ldots,a_n) & \longrightarrow & \mathbb{P}(1,a_1,\ldots,a_n) \\ & & [x_0 : x_1 : x_2 : \cdots : x_n] & \longmapsto & [x_0 : x_1^{a_1} : x_2 : \cdots : x_n] \end{array}$$

we have for $0 \leq j_0 \leq r_0 - 1$:

$$N(F_0(j_0)) \leq \frac{1}{r_1} \sum_{j_1=0}^{r_1-1} N(F_1(j_1))$$

where $F_1(j_1) = \pi_1^*(F_0(j_0) \circ \sigma_1^{j_1})$.

Thus:

$$N(F) \leq \frac{1}{r_0 r_1} \sum_{j_0=0}^{r_0-1} \sum_{j_1=0}^{r_1-1} N(F_1(j_1)).$$

Continuing this process, we obtain

$$N(F) \leq \frac{1}{r_0 \ldots r_n} \sum_{j_0=0}^{r_0-1} \ldots \sum_{j_n=0}^{r_n-1} N(F_n(j_n)).$$

The last polynomials are homogeneous polynomials of degree $d$ in the standard $n$-dimensional projective space $\mathbb{P}^n = \mathbb{P}(1,\ldots,1)$. Then we apply the Serre bound

$$N(F) \leq \frac{1}{r_0 \ldots r_n} r_0 \ldots r_n (dq^{n-1} + p_{n-2}) = dq^{n-1} + p_{n-2}$$

and we get the result. □

## 5. The main result

We are now enable to state and prove Conjecture 1.1 provided $a_1 = 1$ (it was already assumed in the conjecture that $a_0 = 1$).

**Theorem 5.1.** *For any degree $d$ and for any nonnegative integers $a_2, \ldots, a_n$, we have:*

$$e_q(d; 1, 1, a_2, \ldots, a_n) = \min\{p_n, dq^{n-1} + p_{n-2}\}.$$

*In other words, Conjecture 1.1 is true for any $(a_1, a_2, \ldots, a_n)$ with $a_1 = 1$ and without any assumption on the degree $d$.*

*Proof.* By Theorem 4.1, we have $e_q(d; 1, 1, a_2, \ldots, a_n) \leq \min\{p_n, dq^{n-1} + p_{n-2}\}$. Now if $d \leq q + 1$, then the bound of Theorem 4.1 is met by the following polynomial of degree $d$ (the degree is $d$ since we have supposed that the weights of $X_0$ and $X_1$ are equal to 1 in the graded ring $\mathbb{F}_q[X_0, \ldots, X_n]$):

$$F = \prod_{i=1}^{d} (\alpha_i X_0 - \beta_i X_1)$$

where $(\alpha_1 : \beta_1), \ldots, (\alpha_d : \beta_d)$ are distinct elements of $\mathbb{P}^1(\mathbb{F}_q)$. □

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