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Abstract. The statement of item (ii) of Proposition 3.2 of the article referenced in the title is not correct. We provide a corrected version and show that, under the assumption that $\gcd(a_i, a_j, q-1) = 1$ for any pair $i \neq j$ in $\{0, \cdots, n\}$ (with the notations of the paper), our initial statement becomes valid, as does the remainder of the paper.

As pointed to us by Jade Nardi and Rodrigo San-José, Proposition 3.2 (ii) of our paper cited in the title is not correct. More precisely, let P be a point lying in the set

$$\mathcal{T}_i = \{P = [y_0 : \cdots : y_n] \in \mathbb{P}(a_0, \cdots, a_n)(\mathbb{F}_q) | y_i = 1; P \neq [0 : \cdots 0 : 1 : 0 \cdots : 0]\} .$$

Then, the number $\sharp\pi_i^{-1}(P)(\mathbb{F}_q)$ of $\mathbb{F}_q$-rational preimages of P by

$$\begin{array}{rcccc} \pi_i & : & \mathbb{P}(a_0, \cdots, a_{i-1}, 1, a_{i+1}, \cdots, a_n) & \to & \mathbb{P}(a_0, \cdots, a_n) \\ & & [x_0 : \cdots : x_n] & \mapsto & [x_0 : \cdots : x_{i-1} : x_i^{a_i} : x_{i+1} : \cdots : x_n] \end{array}$$

is not equal to

$$\gcd(a_i, q-1)$$

as claimed in our Proposition 3.2. Indeed, let us for instance consider, for fixed $a_0, a_1 \geq 1$, the case

$$\begin{array}{rccc} \pi_2 : & \mathbb{P}(a_0, a_1, 1, 4) & \to & \mathbb{P}(a_0, a_1, 2, 4) \\ & [x_0 : x_1 : x_2 : x_3] & \mapsto & [x_0 : x_1 : x_2^2 : x_3] \end{array}$$

for $q = 5$, and the point $P = [0, 0, 1, 2] \in \mathcal{T}_2 \subset \mathbb{P}(a_0, a_1, 2, 4)(\mathbb{F}_5)$. We thus have $a_2 = 2$ and $q-1 = 4$, so that our Proposition 3.2 predicts $\gcd(2, 4) = 2$ rational inverse preimages Q of P in $\mathbb{P}(a_0, a_1, 1, 4)(\mathbb{F}_5)$, while a close study shows that it has only one, namely $Q = [0 : 0 : 1 : 2] = [0 : 0 : -1 : 2]$.

Instead, the following Lemma is true.

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Lemma (Corrected form of Item (ii) of Proposition 3.2)**.** Let $i \in \{0, \ldots, n\}$ and $P = [y_0 \cdots : y_n] \in \mathcal{T}_i \subset \mathbb{P}(a_0, \cdots, a_n)(\mathbb{F}_q)$ with $y_i = 1$ and $y_j \in \mathbb{F}_q$ for $0 \leq j \leq n$.

Let $\delta_P = \gcd(a_j | j \in \mathrm{Supp}(P))$ and $\delta_{i,P} = \gcd(a_j | j \in \mathrm{Supp}(P) \text{ and } j \neq i)$ where $\mathrm{Supp}(P)$ denotes the support of P.

(1) Then, we have

$$\sharp \pi_i^{-1}(P)(\mathbb{F}_q) = \frac{\gcd(a_i, (q-1) \times \delta_{i,P})}{\delta_P}.$$

(2) Assuming that $\gcd(a_i, a_j, q-1) = 1$ for any $j \in \{1, \ldots, n\} \setminus \{i\}$, this reduces to

$$\sharp \pi_i^{-1}(P)(\mathbb{F}_q) = \gcd(a_i, q-1).$$

Note that the assumption in Item (2) is trivially satisfied, for instance

- either if the weights a_i, a_j are coprime for any $j \neq i$,
- or if a_i is coprime to $q-1$.

Under one of the above extra conditions, the result stated in Proposition 3.2 (ii) is thus correct, as well as the whole paper.

Proof of the Lemma. Let us assume for convenience that $i = 0$, and that

$$P = [1 : y_1 : \cdots : y_m : 0 : \cdots : 0] \in \mathcal{T}_0 \subset \mathbb{P}(a_0, a_1, \cdots, a_n)(\mathbb{F}_q),$$

with $y_j \in \mathbb{F}_q$ and $y_j \neq 0$ for $1 \leq j \leq m$. Note that we have necessarily $m \geq 1$ since $[1 : 0 : \cdots 0] \notin \mathcal{T}_0$.

We begin by describing the whole set $\pi_i^{-1}(P)(\overline{\mathbb{F}}_q)$. Let $Q = [x_0 : \cdots : x_n] \in \mathbb{P}(1, a_1, \cdots, a_n)(\overline{\mathbb{F}}_q)$ and let us denote by $\mu_r(\overline{\mathbb{F}}_q)$ the set of r-th roots of unity in $\overline{\mathbb{F}}_q$.

We have $Q \in \pi_i^{-1}(P)$ if and only if $[x_0^{a_0} : x_1 : \cdots : x_n] = [1 : y_1 : \cdots : y_m : 0 : \cdots : 0]$ in $\mathbb{P}(a_0, a_1, \cdots, a_n)$ which means that there exists $\lambda \in \overline{\mathbb{F}}_q^*$ such that

$$\begin{cases} x_0^{a_0} &= \lambda^{a_0} \times 1 \\ x_j &= \lambda^{a_j} \times y_j \qquad (1 \leq j \leq m) \\ x_j &= \lambda^{a_j} \times 0 \qquad (m < j). \end{cases}$$

This is equivalent to saying that there exists $\lambda \in \overline{\mathbb{F}}_q^*$ and $\zeta \in \mu_{a_0}(\overline{\mathbb{F}}_q)$ such that

$$\begin{cases} x_0 &= \lambda \times \zeta \\ x_j &= \lambda^{a_j} \times y_j \qquad (1 \leq j \leq m) \\ x_j &= 0 \qquad (m < j) \end{cases}$$

i.e. such that $Q = [x_0 : \cdots : x_n] = [\lambda \times \zeta : \lambda^{a_1} \times y_1 : \cdots : \lambda^{a_m} \times y_m : 0 \cdots 0]$.

Thus we have proved that

$$\pi_i^{-1}(P)(\overline{\mathbb{F}}_q) = \left\{ Q_\zeta = [\zeta : y_1 : \cdots : y_m : 0 \cdots 0]; \quad \zeta \in \mu_{a_0}(\overline{\mathbb{F}}_q) \right\}.$$

Next, we determine the $\mathbb{F}_q$-rational points inside the above set $\pi_i^{-1}(P)(\overline{\mathbb{F}}_q)$. Let $\zeta \in \mu_{a_0}(\overline{\mathbb{F}}_q)$. From $\zeta \neq 0$ and $y_i^q = y_i \neq 0$ for $1 \leq i \leq m$ (since $y_i \in \mathbb{F}_q^*$), we have that $Q_\zeta = [\zeta : y_1 : \cdots : y_m : 0 \cdots 0] \in \pi_i^{-1}(P)(\mathbb{F}_q)$ if and only if $[\zeta^q : y_1^q : \cdots : y_m^q : 0^q \cdots 0^q] = [\zeta : y_1 : \cdots : y_m : 0 \cdots 0]$ which is equivalent to saying that there exists $\lambda \in \overline{\mathbb{F}}_q^*$ such that

$$\begin{cases} \zeta^q &= \quad \lambda \times \zeta \\ y_j^q &= \lambda^{a_j} \times y_j \qquad (1 \leq j \leq m) \end{cases}$$

i.e. such that

$$\begin{cases} \lambda &= \zeta^{q-1} \\ \lambda^{a_j} &= 1 \qquad (1 \leq j \leq m). \end{cases}$$

This means that $\zeta^{(q-1)a_j} = 1$ for all $1 \leq j \leq m$, in other words that

$$\zeta \in \cap_{1 \leq j \leq m} \mu_{(q-1)a_j}(\overline{\mathbb{F}}_q) = \mu_{\gcd((q-1)a_1, \cdots, (q-1)a_m)}(\overline{\mathbb{F}}_q).$$

It follows that $\pi_i^{-1}(P)(\mathbb{F}_q)$ is the set of points $Q_\zeta = [\zeta : y_1 : \cdots : y_m : 0 \cdots 0]$ such that $\zeta \in \mu_{a_0}(\overline{\mathbb{F}}_q) \cap \mu_{(q-1)\times\delta_{0,P}}(\overline{\mathbb{F}}_q) = \mu_{\gcd(a_0,(q-1)\times\delta_{0,P})}(\overline{\mathbb{F}}_q)$ where $\delta_{0,P} = \gcd(a_1, \cdots, a_m)$.

We now need to determine the number of distinct elements in this set. Let $\zeta_1, \zeta_2 \in \mu_{\gcd(a_0,(q-1)\times\delta_{0,P})}(\overline{\mathbb{F}}_q)$. We have $Q_{\zeta_1} = Q_{\zeta_2}$ if and only if $[\zeta_1 : y_1 : \cdots : y_m : 0 \cdots 0] = [\zeta_2 : y_1 : \cdots : y_m : 0 \cdots 0]$ in $\mathbb{P}(1, a_1, \cdots, a_n)$. This is equivalent to the existence of $\lambda \in \overline{\mathbb{F}}_q^*$ such that

$$\begin{cases} \zeta_1 &= \quad \lambda \times \zeta_2 \\ y_j &= \lambda^{a_j} \times y_j \qquad (1 \leq j \leq m) \end{cases}$$

i.e. such that

$$\begin{cases} \lambda &= \zeta_1/\zeta_2 \\ \lambda^{a_j} &= \quad 1 \qquad (1 \leq j \leq m). \end{cases}$$

It writes $(\zeta_1/\zeta_2)^{a_j} = 1$ for all $1 \leq j \leq m$, hence we have proved that

$$Q_{\zeta_1} = Q_{\zeta_2} \iff \zeta_1/\zeta_2 \in \mu_{\gcd(a_0,(q-1)\times\delta_{0,P})}(\overline{\mathbb{F}}_q) \cap \mu_{\delta_{0,P}}(\overline{\mathbb{F}}_q) = \mu_{gcd(a_0,\delta_{0,P})}(\overline{\mathbb{F}}_q).$$

We deduce that

$$\sharp\pi_i^{-1}(P)(\mathbb{F}_q) = \frac{\gcd(a_0, (q-1) \times \delta_{0,P})}{\gcd(a_0, \delta_{0,P})},$$

which proves Item (1).

In order to prove Item (2), let ℓ be any prime number and let us set

$$\alpha := v_\ell(a_0), \quad \kappa := v_\ell(q-1) \text{ and } \delta := v_\ell(\delta_{0,P})$$

where v_ℓ stands for the ℓ-adic valuation. We have

$$(1) \qquad v_\ell\left(\frac{\gcd(a_0, (q-1) \times \delta_{0,P})}{\gcd(a_0, \delta_{0,P})}\right) = \min(\alpha, \kappa + \delta) - \min(\alpha, \delta)$$

while

$$v_\ell(\gcd(a_0, q-1)) = \min(\alpha, \kappa). \tag{2}$$

Under the extra assumption that $\gcd(a_i, a_j, q-1) = 1$, at least one of the three valuations α, κ or δ do vanish. In each case, it is a trivial matter to observe that the right hand side in Equations (1) and (2) are equal, so as their left hand side which proves Item (2).

We end this note by drawing attention to the preprint "Maximum number of zeroes of polynomials on weighted projective spaces over a finite field", arXiv:2507.22597v1 [math.AG] 30 Jul 2025, in which the authors Jade Nardi and Rodrigo San-José present a proof of the conjecture in the general case.

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