

# **Equality and dependent type theory**

CIRM, May 31

## The Axiom of Univalence, a type-theoretic view point

In type theory, we reduce proof-checking to type-checking

Hence we want type-checking to be decidable

This holds as soon as we have the normalization property

To add an axiom does not destroy the normalization property

## A type-theoretic view point

Normalization property is obtained by giving a computational justification of each new construct

Another property that follows from this method is that we know that any closed term of type  $N$  reduces to a numeral, and any term of type  $\sum_{x:A} B$  reduces to a pair

If we add an axiom, we destroy these properties

## This talk

We present various axiomatizations of the equality type (this is complete, and has been formally checked by Nils Anders Danielsson)

## Equality in type theory

First edition of *Principia Mathematica* (1910): no axiom of extensionality, but axiom of reducibility (propositions form a type, and we can quantify over any type, also known as *impredicativity*)

Second edition (1925): under the influence of Wittgenstein, Russell introduces the principle of extensionality

*a function of propositions is always a truth function, and a function occurs only in a proposition through its values*

and sees this as a (partial) replacement of the axiom of reducibility

## Equality in type theory

*A function can only appear in a matrix though its values*

“This assumption is fundamental in the following theory. It has its difficulties, but for the moment, we ignore them. It takes the place (not quite adequately) of the axiom of reducibility”

## Church's formulation of type theory

Simplification of Russell's theory of types

A type of proposition  $o$ , a type of individuals and function type  $A \rightarrow B$

For instance  $o \rightarrow o$  is the type of the operation of negation

We have the usual connectives on propositions

$p \rightarrow q : o$  for the implication if  $p, q : o$

quantifiers at *any* type  $\forall x : A. \varphi : o$  if  $\varphi : o [x : A]$

## Church's formulation

Uses  $\lambda$ -calculus to represent terms (implicit in *Principia Mathematica*)

If  $f : A \rightarrow B$  and  $a : A$  then  $f\ a : B$  the application of the function  $f$  to the argument  $a$

If  $t : B\ [x : A]$  then  $\lambda x. t : A \rightarrow B$

The terms of type  $o$  are the propositions

Usual connectives and (classical) logical rules



## Equality in Church's formulation

We can *define* an equality (Leibnitz equality)  $\text{Id}_A\ a_0\ a_1$  as

$$\forall P : A \rightarrow o. P(a_0) \rightarrow P(a_1)$$

This definition is *impredicative*

One can show that this is a reflexive, symmetric and transitive relation

The *axiom of extensionality* has then two forms

on propositions:  $(p \leftrightarrow q) \rightarrow \text{Id}_o\ p\ q$

on functions:  $(\forall x : A. \text{Id}_B\ (f\ x)\ (g\ x)) \rightarrow \text{Id}_{A \rightarrow B}\ f\ g$

## Equality in Church's formulation

Axiomatic presentation

$$\text{ax}_1 : \forall x : A. \text{Id}_A \ x \ x$$

$$\text{ax}_2 : \text{Id}_A \ a_0 \ a_1 \rightarrow P(a_0) \rightarrow P(a_1)$$

$$\text{ax}_3 : (p \leftrightarrow q) \rightarrow \text{Id}_o \ p \ q$$

$$\text{ax}_4 : (\forall x : A. \text{Id}_B \ (f \ x) \ (g \ x)) \rightarrow \text{Id}_{A \rightarrow B} \ f \ g$$

## Dependent Type Theory

Uniform foundation for logic and type theory: True = Provable = Inhabited

(In Church's type theory, one needs to add logical rules to the type structure)

For instance

$$\prod_{A\ B:U} (A \rightarrow B \rightarrow A)$$

is true because it is inhabited by  $\lambda A\ B\ x\ y. x$

$$A : U, B : U \vdash \lambda x\ y. x : A \rightarrow B \rightarrow A$$

$$A : U, B : U, x : A, y : B \vdash x : A$$

## Equality in Dependent Type Theory

Martin-Löf 1971 no equality (can be defined as Leibnitz equality)

1972 predicative system no equality (can be defined by recursion on natural numbers)

1973 equality type, least reflexive relation

1979-1986 “extensional” version

1986 equality type like in 1973

This equality captures some important properties, but does not seem a correct representation of equality for function types and for the universe

## Equality in Dependent Type Theory

We follow an *axiomatic* approach: what should be the property of equality?

We should have a *type* of equality proofs  $\text{Id}_A\ a_0\ a_1$  if  $A$  type and  $a_0\ a_1 : A$

How to think equality in the framework of propositions-as-types?

We write  $\alpha, \beta, \dots$  equality proofs

Some axioms

$1_a : \text{Id}_A\ a\ a$  if  $a : A$

$(\cdot) : B(a_0) \rightarrow \text{Id}_A\ a_0\ a_1 \rightarrow B(a_1)$  given  $B(x)$  type over  $x : A$

We have  $b \cdot \alpha : B(a_1)$  if  $b : B(a_0)$  and  $\alpha : \text{Id}_A\ a_0\ a_1$

## Equality as Path

We think of a type  $A$  as a *space*

A proof  $\alpha : \text{Id}_A a_0 a_1$  is thought of as a *path* between  $a_0$  and  $a_1$

The operation  $b \cdot \alpha : B(a_1)$  for  $b : B(a_0)$  corresponds then to the *path lifting property*

(For a covering space, this lifting property provides a bijection between two fibers of two connected points)

We expect to have  $\text{Id}_{B(a_0)} (b \cdot 1_{a_0}) b$

## Equality as Path

3 axioms so far

$$1_a : \text{Id}_A \ a \ a \text{ if } a : A$$

$$(\cdot) : B(a_0) \rightarrow \text{Id}_A \ a_0 \ a_1 \rightarrow B(a_1)$$

$$\text{ax}_3 : \text{Id}_{B(a_0)} \ (b \cdot 1_{a_0}) \ b$$

## Contractible Spaces

If  $A$  is a type we define a new type  $\text{iscontr } A$  to be  $\sum_{a:A} \prod_{x:A} \text{Id}_A a x$

This means that  $A$  has *exactly one element*

In term of space,  $A$  is *contractible*

The justification of this last point is subtle:  $\text{iscontr } A$  seems at first to only say that  $A$  is inhabited and (path) connected



## A further axiom

(J.P.Serre) *when I was working on homotopy groups (around 1950), I convinced myself that, for a space  $X$ , there should exist a fibre space  $E$ , with base  $X$ , which is *contractible*; such a space would allow me (using Leray's methods) to do lots of computations on homotopy groups... But how to find it? It took me several weeks (a very long time, at the age I was then) to realize that the *space of "paths"* on  $X$  had all the necessary properties-if only I dared call it a "fiber space". This was the starting point of the loop space method in algebraic topology.*

(Interview in the Mathematical Intelligencer, 1986)

## A further axiom

Given a point  $a$  in  $X$ , J.P. Serre was considering the space  $E$  of paths  $\alpha$  from  $a$  to another point  $x$  of  $A$ , with the map  $E \rightarrow A$ ,  $\alpha \mapsto x$

$E$  is contractible, and we have a contractible fibre space  $E$  with base  $X$

In type theory, this translates to

For  $a : X$ , the type  $E = \sum_{x:X} \text{Id}_A a x$  should be contractible

Any element  $(x, \alpha) : E$  is equal to  $(a, 1_a)$

## Equality as Path

4 axioms

$$1_a : \text{Id}_A \ a \ a \text{ if } a : A$$

$$(\cdot) : B(a_0) \rightarrow \text{Id}_A \ a_0 \ a_1 \rightarrow B(a_1)$$

$$\text{ax}_3 : \text{Id}_{B(a_0)} \ (b \cdot 1_{a_0}) \ b$$

$$\text{ax}_4 : \text{iscontr} \left( \sum_{x:A} \text{Id}_A \ a \ x \right)$$

## Equivalent formulation

introduction rule  $1_a : \text{Id}_A a a$

elimination rule: given  $C(x, \alpha)$  for  $x : A$  and  $\alpha : \text{Id}_A a x$  then we have

$$\text{elim} : C(a, 1_a) \rightarrow \prod_{x:A} \prod_{\alpha:\text{Id}_A a x} C(x, \alpha)$$

(C. Paulin's formulation of equality in type theory)

“computation” rule:  $\text{Id}_{C(a, 1_a)} (\text{elim } c a 1_a) c$  for any  $c : C(a, 1_a)$

Dependent type version of  $\text{Id}_A a x \rightarrow P(a) \rightarrow P(x)$

## Equivalent formulation

introduction rule  $1_a : \text{Id}_A a a$

elimination rule: given  $C(x_0, x_1, \alpha)$  for  $x_0 x_1 : A$  and  $\alpha : \text{Id}_A x_0 x_1$  we have

$$J : \left( \prod_{x:A} C(x, x, 1_x) \right) \rightarrow \prod_{x_0 x_1:A} \prod_{\alpha:\text{Id}_A x_0 x_1} C(x_0, x_1, \alpha)$$

“computation” rule:  $\text{Id}_{C(x,x,1_x)} (J d x x 1_x) (d x)$  for any  $d : \prod_{x:A} C(x, x, 1_x)$

This is P. Martin-Löf’s formulation of equality in type theory

It expresses in type theory that  $\text{Id}_A$  is the least reflexive relation on  $A$

## Consequences of these axioms

All these different formulations are equivalent axiom systems (proved formally in type theory)

Given these axioms any type has automatically a *groupoid structure*

Proofs-as-programs version of the fact that equality is symmetric and transitive

Any function  $f : A \rightarrow B$  defines a functor

Hofmann-Streicher 1992

## Equality as Path

Most topological intuitions have a direct formal expression in type theory, e.g.

for any type  $X$  and  $a : X$  the loop space  $\Omega_1(X, a) = \text{Id}_X \ a \ a$  has a group structure

$$\Omega_2(X, a) = \Omega_1(\text{Id}_X \ a \ a, 1_a), \dots$$

## Equality as Path

We have (proved formally)

**Proposition:** (Čech, 1932)  $\Omega_n(X, a)$  is commutative for  $n \geq 2$

This is a corollary of the following fact.

**Proposition:** If  $X$  with a binary operation and an element  $e : X$  which is both a left and right unit for this operation then the group  $\Omega_1(X, e) = \text{Id}_X \, e \, e$  is commutative



## Equality as Path

*Warning!* Our statement is actually different from the usual statement on higher homotopy group

$\Omega_1(X, a)$  is defined as a space, which may have a complex equality

To get the usual statement, we would have to consider the set (as defined later)  $\pi_1(X, x)$  associated to it

This is needed if we want to state that  $f : X \rightarrow Y$  is a weak equivalence iff all  $\pi_n(f)$  are isomorphisms

## Equality as Path

Voevodsky has represented the construction of a long exact sequence

$$\cdots \rightarrow \Omega_n(B(a), b) \rightarrow \Omega_n\left(\sum_{x:A} B, (a, b)\right) \rightarrow \Omega_n(A, a) \rightarrow \cdots$$

from the sequence  $B(a) \rightarrow \sum_{x:A} B \rightarrow A$

## Axiom of extensionality

The usual formulation of this axiom is, with  $F = \prod_{x:A} B(x)$

$$\left( \prod_{x:A} \text{Id}_{B(x)} (f\ x) (g\ x) \right) \rightarrow \text{Id}_F f\ g$$

(V. Voevodsky) This is equivalent to

*A product of contractible types is contractible*

$$\left( \prod_{x:A} \text{iscontr } (B(x)) \right) \rightarrow \text{iscontr } \left( \prod_{x:A} B(x) \right)$$

## Equality as Path

5 axioms

$$1_a : \text{Id}_A \ a \ a \text{ if } a : A$$

$$(\cdot) : B(a_0) \rightarrow \text{Id}_A \ a_0 \ a_1 \rightarrow B(a_1)$$

$$\text{ax}_3 : \text{Id}_{B(a_0)} \ (b \cdot 1_{a_0}) \ b$$

$$\text{ax}_4 : \text{iscontr} \left( \sum_{x:A} \text{Id}_A \ a \ x \right)$$

$$\text{ax}_5 : \left( \prod_{x:A} \text{iscontr} \ (B(x)) \right) \rightarrow \text{iscontr} \left( \prod_{x:A} B(x) \right)$$

## Stratification of types

$A$  is of h-level 0 iff  $A$  is contractible

$A$  is of h-level 1 iff  $\text{Id}_A\ a_0\ a_1$  is contractible for any  $a_0\ a_1 : A$

$A$  is a *proposition* iff  $A$  is of h-level 1

$A$  is of h-level 2 iff  $\text{Id}_A\ a_0\ a_1$  is a proposition for any  $a_0\ a_1 : A$

$A$  is a *set* iff  $A$  is of h-level 2

...

## Stratification of types

These definitions can be internalised in type theory

$$\text{isprop } A = \prod_{x_0 \ x_1 : A} \text{iscontr } (\text{Id}_A \ x_0 \ x_1)$$

$$\text{isset } A = \prod_{x_0 \ x_1 : A} \text{isprop } (\text{Id}_A \ x_0 \ x_1)$$

There is no “global” type of all propositions like in an impredicative framework or a type of all sets

## Extensionality and impredicativity

The extensionality axiom implies

-a product of propositions is always a proposition

$$\prod_{x:A} \text{isprop } (B(x)) \rightarrow \text{isprop } \left( \prod_{x:A} B(x) \right)$$

-a product of sets is always a set

$$\prod_{x:A} \text{isset } (B(x)) \rightarrow \text{isset } \left( \prod_{x:A} B(x) \right)$$

The first implication confirms Russell's remark that the principle of extensionality can replace in some cases the axiom of reducibility

## Propositions

If we have  $\text{isprop } (B(x))$  for all  $x : A$  then the canonical projection

$$\left( \sum_{x:A} B(x) \right) \rightarrow A$$

is a mono, and we can think of  $\sum_{x:A} B(x)$  as the *subset* of elements in  $A$  satisfying the property  $B(x)$



## Unique Existence

$\text{iscontr}(\sum_{x:A} B(x))$  a generalisation of unique existence  $\exists!x : A.B(x)$

If  $B(x)$  is a proposition,  $\text{iscontr}(\sum_{x:A} B(x))$  reduces to unique existence on  $x$

More refined in general than to state that only one element in  $A$  satisfies  $B(x)$

We always have  $\text{iscontr}(\sum_{x:A} \text{Id}_A a x)$  but  $\text{Id}_A a x$  may not be a proposition

## Hedberg's Theorem

Define  $\text{isdec } A$  to be  $\prod_{x_0 \ x_1 : A} \text{Id}_A \ x_0 \ x_1 + \neg (\text{Id}_A \ x_0 \ x_1)$

$\neg C$  denotes  $C \rightarrow N_0$ , where  $N_0$  is the empty type

M. Hedberg noticed (1995) that we have

$$\text{isdec } A \rightarrow \text{isset } A$$

In particular  $N$  the type of natural numbers is decidable

So  $N$  is a *set* but it is not a *proposition* (since  $\neg (\text{Id}_N \ 0 \ 1)$  is inhabited)

## Other properties

$\text{isprop } N_0, \text{ iscontr } N_1, \text{ isset } N_2$

$\neg A \rightarrow \text{isprop } A$

$\text{isprop } (\text{iscontr } A)$  for all type  $A$

$\text{isprop } (\text{isprop } A)$  for all type  $A$

$\text{isprop } (\text{isset } A)$  for all type  $A$

$\text{isprop } A \text{ iff } \prod_{x_0 \ x_1:A} \text{iscontr}(\text{Id}_A \ x_0 \ x_1) \text{ iff } \prod_{x_0 \ x_1:A} \text{Id}_A \ x_0 \ x_1$

## Axiom of extensionality

In Church's type theory  $(p \leftrightarrow q) \rightarrow \text{Id}_o p q$

What about adding as an axiom  $(X \leftrightarrow Y) \rightarrow \text{Id}_U X Y$ ?

S. Berardi noticed that this is contradictory (with dependent type theory):

If  $X$  inhabited  $X$  is logically equivalent to  $X \rightarrow X$

We would have  $\text{Id}_U X (X \rightarrow X)$  and then  $X$  and  $X \rightarrow X$  are isomorphic

$X$  model of  $\lambda$ -calculus, hence any map on  $X$  has a fixed-point

and we get a contradiction if  $X = N$  or  $X = N_2$

## Axiom of extensionality

In ordinary type theory, one can notice directly that if  $X$  is inhabited then  $X$  is logically equivalent to  $N_1$  and hence  $X$  is a singleton

## Axiom of extensionality

So we need a more subtle formulation

Define  $\text{Isom } X \ Y$  to be

$$\sum_{f:X \rightarrow Y} \sum_{g:Y \rightarrow X} \left( \prod_{x:X} \text{Id}_X (g (f x)) x \right) \times \left( \prod_{y:Y} \text{Id}_Y (f (g y)) y \right)$$

Extensionality axiom for small types (Hofmann-Streicher 1996)

$$\text{Isom } X \ Y \rightarrow \text{Id}_U X \ Y$$

## Other properties

A consequence of this axiom is

$$\neg(\text{isset } U)$$

Indeed,  $\text{id}_U : N_2 \rightarrow N_2$  has two distinct elements

We have

If  $\text{isset } A$  and  $\prod_{x:A} \text{isset } (B(x))$  then  $\text{isset } (\sum_{x:A} B(x))$

$\text{isset } A$  is not connected to the size of  $A$  but with the complexity of the equality on  $A$

## Equality as Path

6 axioms

$$1_a : \text{Id}_A \ a \ a \text{ if } a : A$$

$$(\cdot) : B(a_0) \rightarrow \text{Id}_A \ a_0 \ a_1 \rightarrow B(a_1)$$

$$\text{ax}_3 : \text{Id}_{B(a_0)} \ (b \cdot 1_{a_0}) \ b$$

$$\text{ax}_4 : \text{iscontr} \left( \sum_{x:A} \text{Id}_A \ a \ x \right)$$

$$\text{ax}_5 : \left( \prod_{x:A} \text{iscontr} \ (B(x)) \right) \rightarrow \text{iscontr} \left( \prod_{x:A} B(x) \right)$$

$$\text{ax}_6 : \text{Isom} \ X \ Y \rightarrow \text{Id}_U \ X \ Y$$



## Univalence Axiom

For  $f : Y \rightarrow X$  and  $x_0 : X$ , the *fiber* of  $f$  above  $x_0$  is

$$f^{-1}(x_0) =_{def} \sum_{y:Y} \text{Id}_X x_0 (f y)$$

$$\sum_{x:X} f^{-1}(x) = \sum_{x:X} \sum_{y:Y} \text{Id}_X x (f y) \text{ is the graph of } f$$

Any map  $f : Y \rightarrow X$  is isomorphic to a fibration  $(\sum_{x:X} f^{-1}(x)) \rightarrow X$

## Univalence Axiom

We define what should be a “path” between two types  $X$  and  $Y$

If  $f : X \rightarrow Y$  we define when  $f$  is a *weak equivalence*

$$\text{isweq } f \stackrel{\text{def}}{=} \prod_{y:Y} \text{iscontr } (f^{-1}(y))$$

**Theorem:** *To be a weak equivalence is always a proposition, i.e.*  
 $\text{isprop } (\text{isweq } f)$

We define  $\text{Weq } X \ Y$  to be  $\sum_{f:X \rightarrow Y} \text{isweq } f$

## Univalence Axiom

Let  $\text{isiso } f$  be

$$\sum_{g:Y \rightarrow X} \left( \prod_{x:X} \text{Id}_X (g (f x)) x \right) \times \left( \prod_{y:Y} \text{Id}_Y (f (g y)) y \right)$$

$$\text{isiso } f \leftrightarrow \text{isweq } f$$

However  $\text{isweq } f$  is always a proposition while

$\text{isiso } f$  may not be a proposition in general

## Univalence Axiom

*Warning!* Weak equivalence is *stronger* than logical equivalence, e.g.

$$\prod_{x:A} \sum_{y:B} R(x, y) \text{ and } \sum_{f:A \rightarrow B} \prod_{x:A} R(x, f\ x)$$

are weakly equivalent, since they are isomorphic

This is more precise than only to state logical equivalence

## Univalence Axiom

Clearly we have  $\text{Weq } X \ X$ , because the identity map is a weak equivalence

Hence we have a map

$$\text{Id}_U \ X \ Y \rightarrow \text{Weq } X \ Y$$

The *Univalence Axiom* states that this map is a weak equivalence

V. Voevodsky has shown that this implies functional extensionality

This axiom does not hold for the set-theoretic interpretation of type theory

## Equality as Path

6 axioms

$$1_a : \text{Id}_A \ a \ a \text{ if } a : A$$

$$(\cdot) : B(a_0) \rightarrow \text{Id}_A \ a_0 \ a_1 \rightarrow B(a_1)$$

$$\text{ax}_3 : \text{Id}_{B(a_0)} \ (b \cdot 1_{a_0}) \ b$$

$$\text{ax}_4 : \text{iscontr} \left( \sum_{x:A} \text{Id}_A \ a \ x \right)$$

$$\text{ax}_5 : \left( \prod_{x:A} \text{iscontr} \ (B(x)) \right) \rightarrow \text{iscontr} \left( \prod_{x:A} B(x) \right)$$

$$\text{ax}_6 : \textit{The canonical map } \text{Id}_U \ X \ Y \rightarrow \text{Weq } X \ Y \text{ is a weak equivalence}$$

## Invariance under isomorphisms

We get a formalism where two *isomorphic* mathematical structures are *equal*

For instance on the type  $S = \sum_{X:U} X \times (X \rightarrow X)$  we have (proved formally)

$\text{Id}_S (X, a, f) (Y, b, g)$  iff the structures  $(X, a, f)$  and  $(Y, b, g)$  are isomorphic

*This invariance property does not hold for set theory*

Is this theory *consistent*?

## Model

Since the paper

D. Kan *A combinatorial definition of homotopy groups*, Annals of Mathematics, 1958, 67, 282-312

a way to represent spaces is to use (Kan) simplicial sets

This model satisfies (and suggested?) the univalence axiom



## Implementation

Nils Anders Danielsson has proved formally that most properties proved by V. Voevodsky can be proved from a purely axiomatic presentation (no new computational rules)

This fact has been used crucially in this presentation

See [www.cse.chalmers.se/~nad/listings/equality/README.html](http://www.cse.chalmers.se/~nad/listings/equality/README.html)