THE UNDECIDABILITY OF SECOND ORDER LINEAR LOGIC WITHOUT EXPONENTIALS

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ABSTRACT. Recently, Lincoln, Scedrov and Shankar showed that the multiplicative fragment of second order intuitionistic linear logic is undecidable, using an encoding of second order intuitionistic logic. Their argument applies to the multiplicative-additive fragment, but it does not work in the classical case, because second order classical logic is decidable. Here we show that the multiplicative-additive fragment of second order classical linear logic is also undecidable, using an encoding of two-counter machines originally due to Kanovich. The faithfulness of this encoding is proved by means of the phase semantics.

In this paper, we write **LL** for the full propositional fragment of linear logic, **MLL** for the multiplicative fragment, **MALL** for the multiplicative-additive fragment, and **MELL** for the multiplicative-exponential fragment. Similarly, we write **ILL**, **IMLL**, etc. for the fragments of intuitionistic linear logic, **LL2**, **MLL2**, etc. for the second order fragments of linear logic, and **ILL2**, **IMLL2**, etc. for the second order fragments of intuitionistic linear logic. If $i \leq j$ are integers, we write $[i, j]$ for the set $\{i, i+1, \ldots, j\}$. If A is a formula and $n \in \mathbb{N}$, we write A^n for the multiset consisting of n occurrences of A. If Γ is a multiset of formulas $A_1, \ldots, A_n$, we write $!\Gamma$ for the multiset $!A_1, \ldots, !A_n$ and $\Gamma \mathbin{\&} \mathbf{1}$ for $A_1 \mathbin{\&} \mathbf{1}, \ldots, A_n \mathbin{\&} \mathbf{1}$.

In the first section, we describe Kanovich's encoding of two-counter machines in propositional linear logic [K]. It allows to show that both **ILL** and **LL** are undecidable, a result which was first established by Lincoln, Mitchell, Scedrov, and Shankar [LMSS]. In the second section, we present a similar encoding for **MALL2** and in the third section, we introduce the phase semantics to show that our encoding works properly. For self-containedness, the syntax of linear logic is given in the appendices. For more information on linear logic, we refer to [G1, G2, Sv].

1. ENCODING TWO-COUNTER MACHINES IN LL

A *deterministic two-counter machine* is given by an integer m and a map

$$\tau : [1, m] \to \{+\} \times \{1, 2\} \times [0, m] \ \cup\ \{-\} \times \{1, 2\} \times [0, m] \times [0, m].$$

If, at a given time, the machine is in the *configuration* $(i, p, q) \in [0, m] \times \mathbb{N} \times \mathbb{N}$, its next configuration is:

- $(j, p+1, q)$ if $\tau(i) = (+, 1, j)$ (increment first counter),
- $(j, p-1, q)$ if $\tau(i) = (-, 1, j, k)$ and $p > 0$ (decrement first counter),
- (k, p, q) if $\tau(i) = (-, 1, j, k)$ and $p = 0$ (test that first counter is zero),

- $(j, p, q+1)$ if $\tau(i) = (+, 2, j)$ (increment second counter),
- $(j, p, q-1)$ if $\tau(i) = (-, 2, j, k)$ and $q > 0$ (decrement second counter),
- (k, p, q) if $\tau(i) = (-, 2, j, k)$ and $q = 0$ (test that second counter is zero).

Since τ is only defined for $i \in [1, m]$, the machine stops when $i = 0$. We say that a configuration (i, p, q) is *accepted* by the machine if, starting from (i, p, q), it eventually stops on $(0, 0, 0)$.

Theorem 1. (Lambek [L], Minsky [M]) There is a deterministic two-counter machine for which the set of accepted configurations is not recursive.

Using such a machine, we can show that both **ILL** and **LL** are undecidable. For this, we introduce atoms $a, b, a', b', c_0, c_1, \ldots, c_m$ and we write Θ for the following multiset of formulae:

- $c_i \multimap c_j \otimes a$ for $i \in [1, m]$ and $\tau(i) = (+, 1, j)$,
- $c_i \otimes a \multimap c_j$ and $c_i \multimap c_k \oplus a'$ for $i \in [1, m]$ and $\tau(i) = (-, 1, j, k)$,
- $c_i \multimap c_j \otimes b$ for $i \in [1, m]$ and $\tau(i) = (+, 2, j)$,
- $c_i \otimes b \multimap c_j$ and $c_i \multimap c_k \oplus b'$ for $i \in [1, m]$ and $\tau(i) = (-, 2, j, k)$,
- $a' \otimes b \multimap a'$, $a' \multimap c_0$, $b' \otimes a \multimap b'$, $b' \multimap c_0$.

Theorem 2. (Kanovich [K]) For any $(i, p, q) \in [0, m] \times \mathbb{N} \times \mathbb{N}$, the following assertions are equivalent:

(1) The configuration (i, p, q) is accepted by the machine;
(2) The sequent $!\Theta, c_i, a^p, b^q \vdash c_0$ is provable in **ILL**;
(3) The latter sequent is provable in **LL**.

Proof. First we show that (1) implies (2) by induction on the length of the computation. If this length is 0, then $(i, p, q) = (0, 0, 0)$ and the sequent $!\Theta, c_0 \vdash c_0$ is provable. Otherwise $i > 0$, and there are six possible cases:

- If $\tau(i) = (+, 1, j)$, then $!\Theta, c_j, a^{p+1}, b^q \vdash c_0$ is provable by induction hypothesis and we build the following proof (for conciseness, contraction and dereliction are combined in a single rule):

$$\dfrac{\dfrac{\dfrac{}{c_i \vdash c_i} \qquad \dfrac{\begin{array}{c}\vdots\\ !\Theta,\ c_j,\ a^{p+1},\ b^q \vdash c_0\end{array}}{!\Theta,\ c_j \otimes a,\ a^p,\ b^q \vdash c_0}}{!\Theta,\ c_i \multimap c_j \otimes a,\ c_i,\ a^p,\ b^q \vdash c_0}}{!\Theta,\ c_i,\ a^p,\ b^q \vdash c_0}$$

- If $\tau(i) = (-, 1, j, k)$ and $p > 0$, then $!\Theta, c_j, a^{p-1}, b^q \vdash c_0$ is provable by induction hypothesis, and we build the following proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{c_i \vdash c_i} \qquad \dfrac{}{a \vdash a}}{c_i,\ a \vdash c_i \otimes a} \qquad \begin{array}{c}\vdots\\ !\Theta,\ c_j,\ a^{p-1},\ b^q \vdash c_0\end{array}}{!\Theta,\ c_i \otimes a \multimap c_j,\ c_i,\ a^p,\ b^q \vdash c_0}}{!\Theta,\ c_i,\ a^p,\ b^q \vdash c_0}$$

- If $\tau(i) = (-, 1, j, k)$ and $p = 0$, then $!\Theta, c_k, b^q \vdash c_0$ is provable by induction hypothesis, and we build the following proof:

$$
\dfrac{\dfrac{}{c_i \vdash c_i} \qquad \dfrac{\dfrac{\vdots}{!\Theta,\, c_k,\, b^q \vdash c_0} \qquad \dfrac{\vdots}{!\Theta,\, a',\, b^q \vdash c_0}}{!\Theta,\, c_k \oplus a',\, b^q \vdash c_0}}{\dfrac{!\Theta,\, c_i \multimap c_k \oplus a',\, c_i,\, b^q \vdash c_0}{!\Theta,\, c_i,\, b^q \vdash c_0}}
$$

  The derivability of $!\Theta, a', b^q \vdash c_0$ is easily proved by induction on q.

The three other cases are obtained by exchanging the roles of the counters. The fact that (2) implies (1) can be proved by a careful analysis of cut-free proofs in **ILL**. Finally, the equivalence between (2) and (3) comes from the fact that, in the absence of additive units and of second order quantifiers, classical linear logic is a conservative extension of intuitionistic linear logic [Sx]. □

This encoding uses exponentials and additives in an essential way. Exponentials are really necessary because **MALL** is decidable [LMSS], but the decidability of **MELL** is still an open problem.

2. Encoding two-counter machines in **MALL2**

Now we want to adapt the above encoding to **MALL2**. This means that we must recover the expressive power of exponentials by means of second order quantification. We may first replace $!\Theta$ by $\Theta \,\&\, \mathbf{1}$, so that we automatically get dereliction and weakening. We must add an explicit formula for contraction. Unfortunately, the formula

$$\varphi = \forall\alpha.\, (\alpha \multimap \alpha \otimes \alpha)$$

used in [LSS] is too strong: For instance, it implies $a \multimap a \otimes a$ which does not correspond to any transition of the machine. Instead, we shall use

$$\psi = \forall\alpha.\, (\alpha \,\&\, \mathbf{1} \multimap \alpha \otimes \alpha).$$

Now we can state the central result of this paper:

Theorem 3. For any $(i, p, q) \in [0, m] \times \mathbb{N} \times \mathbb{N}$, the following assertions are equivalent:

(1) The configuration (i, p, q) is accepted by the machine;
(2) The sequent $\psi^2, \Theta \,\&\, \mathbf{1}, c_i, a^p, b^q \vdash c_0$ is provable in **IMALL2**;
(3) The latter sequent is provable in **MALL2**.

To prove that (1) implies (2), we proceed exactly as in Theorem 2, using the fact that the following rules are derivable in **IMALL2**:

$$
\dfrac{\Gamma \vdash C}{\psi, \Gamma \vdash C} \qquad \dfrac{\psi^2, A \,\&\, \mathbf{1}, A, \Gamma \vdash C}{\psi^2, A \,\&\, \mathbf{1}, \Gamma \vdash C}
$$

We derive the first rule as follows:

$$\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1}} \qquad \dfrac{}{\vdash \mathbf{1}}}{\vdash \mathbf{1} \,\&\, \mathbf{1}} \qquad \dfrac{\dfrac{\dfrac{\Gamma \vdash C}{\mathbf{1},\, \Gamma \vdash C}}{\mathbf{1},\, \mathbf{1},\, \Gamma \vdash C}}{\mathbf{1} \otimes \mathbf{1},\, \Gamma \vdash C}}{\dfrac{\mathbf{1} \,\&\, \mathbf{1} \multimap \mathbf{1} \otimes \mathbf{1},\, \Gamma \vdash C}{\psi,\, \Gamma \vdash C}}$$

Now, we write B for $\psi \otimes (A \,\&\, \mathbf{1})$ and, using the first rule, we derive the second one as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\psi \vdash \psi} \qquad \dfrac{}{A \,\&\, \mathbf{1} \vdash A \,\&\, \mathbf{1}}}{\psi,\, A \,\&\, \mathbf{1} \vdash B} \qquad \dfrac{\dfrac{\dfrac{}{\mathbf{1} \vdash \mathbf{1}}}{A \,\&\, \mathbf{1} \vdash \mathbf{1}}}{\psi,\, A \,\&\, \mathbf{1} \vdash \mathbf{1}}}{\psi,\, A \,\&\, \mathbf{1} \vdash B \,\&\, \mathbf{1}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\psi^2,\, A \,\&\, \mathbf{1},\, A,\, \Gamma \vdash C}{\psi^2,\, (A \,\&\, \mathbf{1})^2,\, \Gamma \vdash C}}{\psi,\, A \,\&\, \mathbf{1},\, B,\, \Gamma \vdash C}}{B,\, B,\, \Gamma \vdash C}}{B \otimes B,\, \Gamma \vdash C}}{\dfrac{B \,\&\, \mathbf{1} \multimap B \otimes B,\, \psi,\, A \,\&\, \mathbf{1},\, \Gamma \vdash C}{\psi^2,\, A \,\&\, \mathbf{1},\, \Gamma \vdash C}}$$

MALL2 is not a conservative extension of **IMALL2**, but it is clear that (2) implies (3). Therefore, it only remains to show that (3) implies (1), and for this, we shall use the phase semantics.

3. Phase semantics

A *phase space* [G1, G2] is a commutative monoid M endowed with a subset $\perp$ of M. If X and Y are any subsets of M, we define:

- $XY = \{xy;\ x \in X \text{ and } y \in Y\}$,
- $X \multimap Y = \{z \in M;\ xz \in Y \text{ for all } x \in X\}$,
- $X^\perp = X \multimap \perp$.

A *fact* is a subset X of M such that $X^{\perp\perp} = X$. It is easy to see that $X \subset X^{\perp\perp}$ and $X^\perp = X^{\perp\perp\perp}$ for any $X \subset M$, so that $X^\perp$ is always a fact. In particular $\perp = \{1\}^\perp$ is a fact. Also, $X \multimap Y$ is a fact whenever Y is a fact, and any intersection of facts is a fact. In particular, $X^{\perp\perp}$ is the intersection of all facts containing X, that is the smallest fact containing X. Now, define $\mathbf{1} = \perp^\perp = \{1\}^{\perp\perp}$, and for any facts X, Y:

- $X \parr Y = (X^\perp Y^\perp)^\perp$ and $X \otimes Y = (X^\perp \parr Y^\perp)^\perp = (XY)^{\perp\perp}$,
- $X \,\&\, Y = X \cap Y$ and $X \oplus Y = (X^\perp \,\&\, Y^\perp)^\perp = (X \cup Y)^{\perp\perp}$.

From those definitions, we deduce easily, for any $X, Y \subset M$:

- $X^{\perp\perp} \otimes Y^{\perp\perp} = (XY)^{\perp\perp}$,
- $X^{\perp\perp} \oplus Y^{\perp\perp} = (X \cup Y)^{\perp\perp}$,
- $X^\perp \parr Y^{\perp\perp} = X \multimap Y^{\perp\perp}$.

In particular, $X^{\perp} \parr Y = X \multimap Y$ whenever Y is a fact, which justifies the notation a posteriori. If a fact $\alpha^{\bullet}$ is given for each atom α occurring in a formula A of **MALL**, we define a fact $A^{\bullet}$ by applying the above definitions inductively, and we extend this interpretation to **MALL2** as follows:

$$(\forall\alpha.\, A)^{\bullet} = \bigcap_{X \in \mathcal{F}(M)} A^{\bullet}_X$$

where $\mathcal{F}(M)$ is the set of all facts of M and $A^{\bullet}_X$ is the interpretation defined by $\alpha^{\bullet}_X = X$. Now, we say that M *satisfies* a formula A when $1 \in A^{\bullet}$.

Theorem 4. (Girard [G1]) If A is provable in classical linear logic, then M satisfies A.

This is a *soundness* result, which is easily proved by induction on proofs. In fact, there is also a *completeness* result: If a formula is satisfied by all phase spaces for all interpretations of the atoms, then it is provable in classical linear logic, at least in the propositional case. For second order linear logic, a more general notion of model is needed, but it does not matter, since we shall only need the soundness result. An easy consequence of Theorem 4 is that $A_1^{\bullet} \cdots A_n^{\bullet} \subset B^{\bullet}$ whenever $A_1, \ldots, A_n \vdash B$ is provable in classical linear logic. This holds more generally whenever $\Gamma, A_1, \ldots, A_n \vdash B$ is provable and M satisfies all the formulas of Γ.

Lemma. If all elements of M (respectively of $\mathbf{1}$) are idempotent, then M satisfies φ (respectively ψ).

Proof. If all elements of M are idempotent, then $X \subset XX \subset (XX)^{\perp\perp} = X \otimes X$ for any fact X, which means that M satisfies φ. Similarly, if all elements of $\mathbf{1}$ are idempotent, then $X \,\&\, \mathbf{1} = X \cap \mathbf{1} \subset XX \subset X \otimes X$ for any fact X, which means that M satisfies ψ. $\square$

To complete the proof of Theorem 3, we use a particular phase space: M is the free commutative monoid generated by the atoms $a, b, a', b', c_0, c_1, \ldots, c_m$ and $\perp$ is

$$\{c_i a^p b^q;\ (i,p,q) \text{ is accepted by the machine}\} \cup \{a' b^q;\ q \in \mathbb{N}\} \cup \{b' a^p;\ p \in \mathbb{N}\}.$$

We define an interpretation by $s^{\bullet} = \{s\}^{\perp\perp}$ for each atom s. We have $c_0 x \in \perp$ if and only if $x = a^0 b^0 = 1$. Therefore $\{c_0\}^{\perp} = \{1\}$, $c_0^{\bullet} = \{1\}^{\perp} = \perp$, and $\mathbf{1} = \perp^{\perp} = \{1\}$. Since 1 is the only element of $\mathbf{1}$, we know that M satisfies ψ by the lemma. Also, M satisfies all formulas of Θ:

- If A is of the form $c_i \multimap c_j \otimes a$ then $A^{\bullet} = \{c_i\} \multimap \{c_j a\}^{\perp\perp}$ and we must show that $c_i \in \{c_j a\}^{\perp\perp}$. But if $x \in \{c_j a\}^{\perp}$, then $c_j a x \in \perp$, which means that x is of the form $a^p b^q$ where $(j, p+1, q)$ is accepted by the machine. Therefore, (i,p,q) is also accepted by the machine and $c_i x = c_i a^p b^q \in \perp$.
- If A is of the form $c_i \multimap c_k \oplus a'$ then $A^{\bullet} = \{c_i\} \multimap \{c_k, a'\}^{\perp\perp}$ and we must show that $c_i \in \{c_k, a'\}^{\perp\perp}$. But if $x \in \{c_k, a'\}^{\perp}$, then $a'x \in \perp$, which means that x is of the form b^q, and $c_k x = c_k b^q \in \perp$, which means that $(k, 0, q)$ is accepted by the machine. Therefore $(i, 0, q)$ is also accepted by the machine and $c_i x = c_i b^q \in \perp$.

The other cases are similar. Now, if the sequent $\psi^2, \Theta \,\&\, \mathbf{1}, c_i, a^p, b^q \vdash c_0$ is provable in classical linear logic, then Theorem 4 gives $c_i a^p b^q \in c_i^\bullet (a^\bullet)^p (b^\bullet)^q \subset c_0^\bullet = \bot$, which means that the configuration (i, p, q) is accepted by the machine, and Theorem 3 is proved.

Remarks

We have used the phase semantics because it seemed difficult to analyze proofs in second order linear logic. However, our method works perfectly well in the propositional case and leads to an alternative proof of Theorem 2, using the above model M. The exponential connective is interpreted by

$$!X = (X \cap I)^{\bot\bot}$$

where I is the set of idempotents of $\mathbf{1}$ [G2]. This I is not a fact in general, but in our model $I = \{1\} = \mathbf{1}$, so that $!X = X \,\&\, \mathbf{1}$ for any fact X. This remark explains the choice of ψ for encoding contraction.

After this paper has been submitted for publication, it has been proved that **MLL2** is also undecidable [LS]. The encoding is more complicated, but the ideas are similar.

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Appendix A. Intuitionistic Linear Logic

Sequents are of the form $\Gamma \vdash C$ where Γ is a multiset of formulas and C is a formula. A formula is an atom α or is of one of the following forms (assuming that A and B are formulas):

$$A \otimes B, \quad \mathbf{1}, \quad A \multimap B, \quad A \oplus B, \quad \mathbf{0}, \quad A \,\&\, B, \quad \top, \quad !A, \quad \exists\alpha.\, A, \quad \forall\alpha.\, A.$$

$\otimes$, $\mathbf{1}$ and $\multimap$ are the *multiplicatives*. $\oplus$, $\mathbf{0}$, $\&$ and $\top$ are the *additives*. ! is the *exponential*. $\exists$ and $\forall$ are the *second order quantifiers*. Proofs are built with the following rules:

- Identity and cut:

$$\frac{}{A \vdash A} \qquad \frac{\Gamma \vdash A \qquad \Delta, A \vdash C}{\Gamma, \Delta \vdash C}$$

- Multiplicatives:

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C} \qquad \frac{}{\vdash \mathbf{1}} \qquad \frac{\Gamma \vdash C}{\Gamma, \mathbf{1} \vdash C}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \qquad \frac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}$$

- Additives:

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B} \qquad \frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C} \qquad \frac{}{\Gamma, \mathbf{0} \vdash C}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B} \qquad \frac{\Gamma, A \vdash C}{\Gamma, A \,\&\, B \vdash C} \qquad \frac{\Gamma, B \vdash C}{\Gamma, A \,\&\, B \vdash C} \qquad \frac{}{\Gamma \vdash \top}$$

- Exponentials:

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A} \qquad \frac{\Gamma, A \vdash C}{\Gamma, !A \vdash C} \qquad \frac{\Gamma, !A, !A \vdash C}{\Gamma, !A \vdash C} \qquad \frac{\Gamma \vdash C}{\Gamma, !A \vdash C}$$

  Those rules are respectively called *promotion*, *dereliction*, *contraction* and *weakening*.
- Second order quantifiers:

$$\frac{\Gamma \vdash A[B/\alpha]}{\Gamma \vdash \exists\alpha.\, A} \qquad \frac{\Gamma, A[\beta/\alpha] \vdash C}{\Gamma, \exists\alpha.\, A \vdash C} \qquad \frac{\Gamma \vdash A[\beta/\alpha]}{\Gamma \vdash \forall\alpha.\, A} \qquad \frac{\Gamma, A[B/\alpha] \vdash C}{\Gamma, \forall\alpha.\, A \vdash C}$$

  In the second and in the third rule, β must be a fresh variable.

Note that all connectives are definable in terms of $\forall$, $\multimap$ and ! :

- $\exists\alpha.\, A \sim \forall\beta.\, ((\forall\alpha.\, (A \multimap \beta)) \multimap \beta)$,
- $A \otimes B \sim \forall\alpha.\, ((A \multimap B \multimap \alpha) \multimap \alpha)$ and $\mathbf{1} \sim \forall\alpha.\, (\alpha \multimap \alpha)$,
- $A \,\&\, B \sim \exists\alpha.\, (!(\alpha \multimap A) \otimes !(\alpha \multimap B) \otimes \alpha)$ and $\top \sim \exists\alpha.\, \alpha$,
- $A \oplus B \sim \forall\alpha.\, ((A \multimap \alpha) \,\&\, (B \multimap \alpha) \multimap \alpha)$ and $\mathbf{0} \sim \forall\alpha.\, \alpha$.

In the last case, β must be a fresh variable. Note also that the additive units $\top$ and $\mathbf{0}$ are definable even in the absence of exponentials.

Appendix B. Classical Linear Logic

Sequents are of the form $\vdash \Gamma$ where Γ is a multiset of formulas. A formula is a positive literal α, a negative literal $\alpha^\perp$, or is of one of the following forms (assuming that A and B are formulas):

$$A \otimes B, \quad \mathbf{1}, \quad A \parr B, \quad \perp, \quad A \oplus B, \quad \mathbf{0}, \quad A \mathbin{\&} B, \quad \top, \quad !A, \quad ?A, \quad \exists \alpha.\, A, \quad \forall \alpha.\, A.$$

Negation is extended to all formulas by $\alpha^{\perp\perp} = \alpha$ and:

- $(A \otimes B)^\perp = A^\perp \parr B^\perp$ and $(A \parr B)^\perp = A^\perp \otimes B^\perp$,
- $\mathbf{1}^\perp = \perp$ and $\perp^\perp = \mathbf{1}$,
- $(A \oplus B)^\perp = A^\perp \mathbin{\&} B^\perp$ and $(A \mathbin{\&} B)^\perp = A^\perp \oplus B^\perp$,
- $\mathbf{0}^\perp = \top$ and $\top^\perp = \mathbf{0}$,
- $(!A)^\perp = ?A^\perp$ and $(?A)^\perp = !A^\perp$,
- $(\exists \alpha.\, A)^\perp = \forall \alpha.\, A^\perp$ and $(\forall \alpha.\, A)^\perp = \exists \alpha.\, A^\perp$.

Proofs are built with the following rules:

- Identity and cut:

$$\frac{}{\vdash A, A^\perp} \qquad \frac{\vdash A, \Gamma \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}$$

- Multiplicatives:

$$\frac{\vdash A, \Gamma \qquad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta} \qquad \frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma} \qquad \frac{}{\vdash \mathbf{1}} \qquad \frac{\vdash \Gamma}{\vdash \perp, \Gamma}$$

- Additives:

$$\frac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma} \qquad \frac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma} \qquad \frac{\vdash A, \Gamma \qquad \vdash B, \Gamma}{\vdash A \mathbin{\&} B, \Gamma} \qquad \frac{}{\vdash \top, \Gamma}$$

- Exponentials:

$$\frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma} \qquad \frac{\vdash A, \Gamma}{\vdash ?A, \Gamma} \qquad \frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma} \qquad \frac{\vdash \Gamma}{\vdash ?A, \Gamma}$$

- Second order quantifiers:

$$\frac{\vdash A[\beta/\alpha], \Gamma}{\vdash \forall \alpha.\, A, \Gamma} \qquad \frac{\vdash A[B/\alpha], \Gamma}{\vdash \exists \alpha.\, A, \Gamma}$$

  In the first rule, β must be a fresh variable.

In this calculus, a two-sided sequent $\Gamma \vdash \Delta$ is interpreted as $\vdash \Gamma^\perp, \Delta$ and the linear implication is defined by:

$$A \multimap B = A^\perp \parr B.$$

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