# THE FINITE MODEL PROPERTY FOR VARIOUS FRAGMENTS OF LINEAR LOGIC

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To show that a formula $A$ is not provable in propositional classical logic, it suffices to exhibit a finite boolean model which does not satisfy $A$. A similar property holds in the intuitionistic case, with Kripke models instead of boolean models (see for instance [TvD88]). One says that the propositional classical logic and the propositional intuitionistic logic satisfy a *finite model property*. In particular, they are decidable: there is a semi-algorithm for provability (proof search) and a semi-algorithm for non provability (model search). For that reason, a logic which is undecidable, such as first order logic, cannot satisfy a finite model property.

The case of *linear logic* is more complicated. The full propositional fragment **LL** has a complete semantics in terms of *phase spaces* [Gir87, Gir95], but it is undecidable [LMSS92]. The multiplicative additive fragment **MALL** is decidable, in fact PSPACE-complete [LMSS92], but the decidability of the multiplicative exponential fragment **MELL** is still an open problem. For *affine logic*, that is, linear logic with weakening, the situation is somewhat better: the full propositional fragment **LLW** is decidable [Kop95a].

Here, we show that the finite phase semantics is complete for **MALL** and for **LLW**, but not for **MELL**. In particular, this gives a new proof of the decidability of **LLW**. The noncommutative case is mentioned, but not handled in detail.

## 1. Syntax of linear logic

Roman capitals $A$, $B$ stand for formulas. The connectives of propositional linear logic are:

- the *multiplicatives* $A \parr B$, $A \otimes B$, $\bot$, $\mathbf{1}$;
- the *additives* $A \,\&\, B$, $A \oplus B$, $\top$, $\mathbf{0}$;
- the *exponentials* $?A$, $!A$.

Linear negation $A^\bot$ is only given for positive atoms. It is extended to all formulas by $A^{\bot\bot} = A$ and by

$$(A \parr B)^\bot = A^\bot \otimes B^\bot, \qquad \bot^\bot = \mathbf{1}, \qquad (A \,\&\, B)^\bot = A^\bot \oplus B^\bot, \qquad \top^\bot = \mathbf{0}, \qquad (?A)^\bot = !A^\bot.$$

One writes $A \multimap B$ for $A^\bot \parr B$. Greek capitals $\Gamma$, $\Delta$ stand for sequents, which are multisets of formulas, so that exchange is implicit. *Identity* and *cut* are written as follows:

$$\frac{}{\vdash A, A^\bot} \qquad \frac{\vdash A, \Gamma \qquad \vdash A^\bot, \Delta}{\vdash \Gamma, \Delta}$$

The *logical rules* are:

$$\frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma} \qquad \frac{\vdash A, \Gamma \qquad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta} \qquad \frac{\vdash \Gamma}{\vdash \bot, \Gamma} \qquad \frac{}{\vdash \mathbf{1}}$$

$$\frac{\vdash A, \Gamma \qquad \vdash B, \Gamma}{\vdash A \,\&\, B, \Gamma} \qquad \frac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma} \qquad \frac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma} \qquad \frac{}{\vdash \top, \Gamma}$$

$$\frac{\vdash A, \Gamma}{\vdash ?A, \Gamma} \qquad \frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma} \qquad \frac{\vdash \Gamma}{\vdash ?A, \Gamma} \qquad \frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}$$

In the last rule, $?\Gamma$ stands for a sequent of the form $?A_1, \ldots, ?A_n$. The last four rules are respectively called *dereliction*, *(logical) contraction*, *(logical) weakening*, and *promotion*.

## 2. Semantics of multiplicatives and additives

If $M$ is a commutative monoid, and $X, Y \subset M$, define

$$XY = \{xy;\ x \in X \text{ and } y \in Y\}, \qquad X \multimap Y = \{z \in M;\ xz \in Y \text{ for all } x \in X\}.$$

A *phase space* is a commutative monoid $M$ endowed with a subset $\bot^\bullet$ of $M$. In that case, one writes $X^\perp$ for $X \multimap \bot^\bullet$. Clearly, $X \subset X^{\perp\perp}$ for any $X$, and $Y^\perp \subset X^\perp$ whenever $X \subset Y$. A *fact* is an $X \subset M$ such that $X^{\perp\perp} = X$. This amounts to say that $X$ is of the form $Y^\perp$ for some $Y \subset M$. In particular $\bot^\bullet = \{1\}^\perp$ is a fact. Also, $X \multimap Y$ is a fact whenever $Y$ is a fact, and any intersection of facts is a fact. In particular, $X^{\perp\perp}$ is the intersection of all facts containing $X$. Define $\mathbf{1}^\bullet = \bot^{\bullet\perp} = \{1\}^{\perp\perp}$, $\top^\bullet = M$, $\mathbf{0}^\bullet = \top^{\bullet\perp} = \emptyset^{\perp\perp}$, and for any facts $X, Y$,

$$X \parr Y = (X^\perp Y^\perp)^\perp, \qquad X \otimes Y = (X^\perp \parr Y^\perp)^\perp = (XY)^{\perp\perp},$$
$$X \mathbin{\&} Y = X \cap Y, \qquad X \oplus Y = (X^\perp \mathbin{\&} Y^\perp)^\perp = (X \cup Y)^{\perp\perp}.$$

One sees easily that $X \multimap Y = X^\perp \parr Y$ whenever $Y$ is a fact, which justifies the notation a posteriori.

A *phase model* is given by a phase space $M$ and an *interpretation* which maps each positive atom $\alpha$ to a fact $\alpha^\bullet$ of $M$. In that case, one defines a fact $A^\bullet$ for any formula $A$ of **MALL** by applying the above definitions inductively, and one says that $M$ *satisfies* $A$ if $1 \in A^\bullet$. The *syntactical model* $\mathbf{MALL}^\bullet$ is defined as follows:

- $\mathbf{MALL}^\bullet$ is the free commutative monoid generated by all formulas in **MALL**. In other words, $\mathbf{MALL}^\bullet$ consists of all sequents in **MALL**, the product of $\Gamma$ by $\Delta$ is $\Gamma, \Delta$, and the unit is the empty sequent;
- $\bot^\bullet$ is the set of all provable sequents in *cut-free* **MALL**;
- $\alpha^\bullet = \{\alpha\}^\perp$ for any positive atom $\alpha$.

Note that $\{A\}^\perp$ is the set of all sequents $\Gamma$ such that $A, \Gamma$ is provable in cut-free **MALL**.

**Lemma 1.** (Okada) For any formula $A$ of **MALL**, one has $A^\bullet \subset \{A\}^\perp$ in $\mathbf{MALL}^\bullet$. In particular, any formula which is satisfied by $\mathbf{MALL}^\bullet$ is provable in cut-free **MALL**.

*Proof.* By induction on formulas. If $\alpha$ is a positive atom, then:

- $\alpha^\bullet = \{\alpha\}^\perp$ by definition of the model;
- $\alpha^{\perp\bullet} = \{\alpha\}^{\perp\perp} \subset \{\alpha^\perp\}^\perp$ since $\alpha \in \{\alpha^\perp\}^\perp$ by the identity axiom.

If $A^\bullet \subset \{A\}^\perp$ and $B^\bullet \subset \{B\}^\perp$, then:

- $(A \parr B)^\bullet = (A^{\bullet\perp} B^{\bullet\perp})^\perp \subset (\{A\}^{\perp\perp}\{B\}^{\perp\perp})^\perp = (\{A\}\{B\})^\perp \subset \{A \parr B\}^\perp$ by the $\parr$-rule;
- $(A \otimes B)^\bullet = (A^\bullet B^\bullet)^{\perp\perp} \subset (\{A\}^\perp\{B\}^\perp)^{\perp\perp} \subset \{A \otimes B\}^\perp$ since $\{A\}^\perp\{B\}^\perp \subset \{A \otimes B\}^\perp$ by the $\otimes$-rule;
- $(A \mathbin{\&} B)^\bullet = A^\bullet \cap B^\bullet \subset \{A\}^\perp \cap \{B\}^\perp \subset \{A \mathbin{\&} B\}^\perp$ by the $\&$-rule;
- $(A \oplus B)^\bullet = (A^\bullet \cup B^\bullet)^{\perp\perp} \subset (\{A\}^\perp \cup \{B\}^\perp)^{\perp\perp} \subset \{A \oplus B\}^\perp$ since $\{A\}^\perp \cup \{B\}^\perp \subset \{A \oplus B\}^\perp$ by the $\oplus$-rules.

Similarly, $\bot^\bullet \subset \{\bot\}^\perp$ by the $\bot$-rule, $\mathbf{1}^\bullet = \bot^{\bullet\perp} \subset \{\mathbf{1}\}^\perp$ since $\mathbf{1} \in \bot^\bullet$ by the $\mathbf{1}$-rule, $\top^\bullet \subset \{\top\}^\perp$ by the $\top$-rule, and $\mathbf{0}^\bullet = M^\perp \subset \{\mathbf{0}\}^\perp$. $\square$

It is possible to show that the inclusion of this lemma is in fact an equality (see [Gir87]), but the nice point about this proof is that it does not use the cut rule (see also [Oka94]). The following theorem summarizes three properties: *soundness* of the phase semantics, *completeness* of the phase semantics, and *cut elimination*.

**Theorem 1.** For any formula $A$ in **MALL**, the following statements are equivalent:

(1) $A$ is provable in **MALL**;
(2) $A$ is satisfied by all phase models;
(3) $A$ is satisfied by $\mathbf{MALL}^\bullet$;
(4) $A$ is provable in cut-free **MALL**.

*Proof.* (2) implies (3) and (4) implies (1) trivially. (1) implies (2) by induction on proofs, as in [Gir87], and (3) implies (4) by lemma 1. $\square$

The soundness of the phase semantics has been used in [Laf95], to show that second order multiplicative additive logic is undecidable, and subsequently in [LS95, Kan95, Kop95b], to show that various fragments of second order linear logic are undecidable.

The proof of theorem 1 works with smaller models than $\mathbf{MALL}^\bullet$. For instance, let $\mathbf{MALL}[A]$ be $\mathbf{MALL}$ restricted to the subformulas of $A$ and let $\mathbf{MALL}^\bullet[A]$ be the following phase model:

- $\mathbf{MALL}^\bullet[A]$ is the free commutative monoid generated by all subformulas of $A$;
- $\perp^\bullet$ is the set of all provable sequents in cut-free $\mathbf{MALL}[A]$;
- $\alpha^\bullet = \{\alpha\}^\perp$ if $\alpha$ occurs in $A$, and $\alpha^\bullet = \{\alpha^\perp\}^{\perp\perp}$ if $\alpha^\perp$ occurs in $A$, but not $\alpha$.

Then $A$ is provable in $\mathbf{MALL}$ if and only if $A$ is satisfied by this $\mathbf{MALL}^\bullet[A]$. As immediate consequences, one gets the subformula property, and the completeness of the finitely generated phase semantics.

A *logical congruence* on a phase model $M$ is a congruence $\sim$ on $M$ such that $\perp^\bullet$ is closed for $\sim$. This means that, if $x \in \perp^\bullet$ and $x \sim y$, then $y \in \perp^\bullet$. For instance:

- the finest logical congruence is equality;
- the coarsest logical congruence is $\equiv$, defined by $x \equiv y$ whenever $\{x\}^\perp = \{y\}^\perp$.

If $\sim$ is a logical congruence, then all facts are closed for $\sim$, so that $\sim$ induces a structure of phase model on the quotient monoid $M/\!\sim$. Let us write $\pi : M \to M/\!\sim$ for the canonical map, and $A^\bullet_\sim$ for the interpretation of a formula $A$ in the quotient model $M/\!\sim$.

**Lemma 2.** For any formula $A$ in $\mathbf{MALL}$, one has $\pi^{-1}(A^\bullet_\sim) = A^\bullet$. In particular, $M/\!\sim$ satisfies the same formulas as $M$.

*Proof.* By construction, $\pi^{-1}(\perp^\bullet_\sim) = \perp^\bullet$ and $\pi^{-1}(\alpha^\bullet_\sim) = \alpha^\bullet$ for each positive atom $\alpha$. Since $\pi$ is a surjective morphism, the following equality holds for any $X, Y \subset M/\!\sim$:

$$\pi^{-1}(X \multimap Y) = \pi^{-1}(X) \multimap \pi^{-1}(Y).$$

The lemma follows by easy induction on $A$. □

Let $\sqsubseteq$ be the smallest preordering on sequents which satisfies the following properties:

$$\Gamma \sqsubseteq \Gamma, \Delta \qquad A, B, \Gamma \sqsubseteq A \parr B, \Gamma \qquad A, \Gamma \sqsubseteq A * B, \Gamma \qquad B, \Gamma \sqsubseteq A * B, \Gamma$$

where $*$ stands for $\otimes$, $\&$ or $\oplus$. It is easy to see that $\sqsubseteq$ is in fact a partial ordering, and for any formula $A$, there are finitely many sequents $\Gamma$ such that $\Gamma \sqsubseteq A$. Let $\mathbf{MALL}\langle A\rangle$ be the fragment $\mathbf{MALL}[A]$ extended with an axiom $\vdash \Gamma$ for each sequent $\Gamma$ such that $\Gamma \not\sqsubseteq A$, and let $\mathbf{MALL}^\bullet\langle A\rangle$ be the corresponding syntactical model. The equivalence $\sim_A$ which collapses all sequents $\Gamma$ such that $\Gamma \not\sqsubseteq A$ is a logical congruence on $\mathbf{MALL}^\bullet\langle A\rangle$, and the quotient $\mathbf{MALL}^\bullet\langle A\rangle/\!\sim_A$ is finite. The following theorem gives the completeness of the finite phase semantics for $\mathbf{MALL}$.

**Theorem 2.** For any formula $A$ in $\mathbf{MALL}$, the following statements are equivalent:

1. $A$ is provable in $\mathbf{MALL}$;
2. $A$ is satisfied by all finite phase models;
3. $A$ is satisfied by $\mathbf{MALL}^\bullet\langle A\rangle/\!\sim_A$;
4. $A$ is satisfied by $\mathbf{MALL}^\bullet\langle A\rangle$;
5. $A$ is provable in cut-free $\mathbf{MALL}\langle A\rangle$.

*Proof.* (1) implies (2) by theorem 1, and (2) implies (3) trivially. (3) implies (4) by lemma 2, and (4) implies (5) by the analogue of lemma 1 for $\mathbf{MALL}\langle A\rangle$. Finally, (5) implies (1) because $\sqsubseteq$ has been defined in such a way that the premises of a logical rule are always smaller than its conclusion. □

This proof can be adapted to the noncommutative versions of $\mathbf{MALL}$. For the syntax and the semantics of noncommutative linear logic, we refer to [Abr91].

## 3. Semantics of exponentials

If $M$ is a phase space, it is easy to see that $J(M) = \{x \in \mathbf{1}^\bullet;\ x \in \{x^2\}^{\perp\perp}\}$ is a submonoid of $M$. An *enriched phase space* is a phase space $M$ endowed with a submonoid $K$ of $J(M)$. This $K$ is not required to be a fact. For instance, $K$ may be $\{1\}$, or $J(M)$ itself, or $I(M) = \{x \in \mathbf{1}^\bullet;\ x = x^2\}$. In the latter case, one recovers Girard's phase semantics [Gir95]. For any fact $X$ in an enriched phase space, define

$$?X = (X^\perp \cap K)^\perp, \qquad !X = (?X^\perp)^\perp = (X \cap K)^{\perp\perp}.$$

From these definitions, one deduces easily, for any facts $X$ and $X_1, \dots, X_n$:

$$!X \subset X, \qquad !X \subset !X \otimes !X, \qquad !X \subset \mathbf{1}^\bullet, \qquad !X_1 \cdots !X_n \subset !X \text{ whenever } !X_1 \cdots !X_n \subset X.$$

From now on, all phase models are supposed to be enriched. The syntactical model $\mathbf{LL}^\bullet$ is defined as follows:

- $\mathbf{LL}^\bullet$ is the free commutative monoid generated by all formulas in $\mathbf{LL}$;
- $\perp^\bullet$ is the set of all provable sequents in cut-free $\mathbf{LL}$;
- $\alpha^\bullet = \{\alpha\}^\perp$ for any positive atom $\alpha$;
- $K$ is the submonoid of $\mathbf{LL}^\bullet$ generated by all formulas of the form $?A$.

By contraction and weakening, $K$ is indeed a submonoid of $J(M)$. Moreover, the analogue of lemma 1 holds:

**Lemma 3.** For any formula $A$ of $\mathbf{LL}$, one has $A^\bullet \subset \{A\}^\perp$ in $\mathbf{LL}^\bullet$. In particular, any formula which is satisfied by $\mathbf{LL}^\bullet$ is provable in cut-free $\mathbf{LL}$.

*Proof.* By induction on formulas. If $A^\bullet \subset \{A\}^\perp$, then:

- $(?A)^\bullet = (A^{\bullet\perp} \cap K)^\perp \subset (\{A\}^{\perp\perp} \cap K)^\perp \subset \{?A\}^\perp$ since $?A \in K$ and $\{A\}^\perp \subset \{?A\}^\perp$ by dereliction;
- $(!A)^\bullet = (A^\bullet \cap K)^{\perp\perp} \subset (\{A\}^\perp \cap K)^{\perp\perp} \subset \{!A\}^\perp$ since $\{A\}^\perp \cap K \subset \{!A\}^\perp$ by promotion.

The other cases are handled as in lemma 1. $\square$

From this lemma, one deduces the analogue of theorem 1, and in particular, the completeness of the phase semantics for $\mathbf{LL}$. Similarly, one shows the completeness of the finitely generated phase semantics, but not the completeness of the finite phase semantics: if one extends $\sqsubseteq$ to $\mathbf{LL}$ in such a way that the premises of a logical rule are always smaller than its conclusion, then it is no more the case that there are finitely many sequents $\Gamma$ such that $\Gamma \sqsubseteq A$. To show that the finite phase semantics is not complete, even for the multiplicative exponential fragment **MELL**, it suffices to consider the following formula:

$$\varphi = !a \otimes !(a \otimes b) \otimes !(a \otimes b \multimap \mathbf{1}) \multimap b = ?a^\perp \parr ?(a^\perp \parr b^\perp) \parr ?(a \otimes b \otimes \perp) \parr b,$$

where $a$ and $b$ are positive atoms.

**Theorem 3.** $\varphi$ is satisfied by all finite phase models, but it is not provable.

*Proof.* One writes $b^n$ for $b \otimes \cdots \otimes b$ ($n$ times). If $M$ is a finite phase model, there are finitely many facts in $M$, so that $(b^p)^\bullet = (b^q)^\bullet$ for some $p < q$. In particular, the formula $\psi = b^p \multimap b^q$ is satisfied by $M$, and since it is easy to see that the formula $\psi \multimap \varphi$ is provable, $\psi \multimap \varphi$ is also satisfied by $M$, and so is $\varphi$. To see that $\varphi$ is not provable, consider the following infinite phase model:

- $M$ is the free group generated by the symbol $a$. This means that any element of $M$ can be uniquely written as $a^n$ with $n \in \mathbb{Z}$;
- $\perp^\bullet = \{a^n;\ n \geq 0\}$, $a^\bullet = \{a\}^\perp = \{a^n;\ n \geq -1\}$, $b^\bullet = \{a^{-1}\}^\perp = \{a^n;\ n \geq 1\}$, and $K = \{1\}$.

It is easy to see that $1 \notin \varphi^\bullet = b^\bullet$, which means that $\varphi$ is not satisfied. $\square$

In fact, $\varphi$ belongs to the so-called *multiplicative exponential Horn fragment* of linear logic, which encodes the reachability problem for Petri nets. This means that this fragment does not satisfy the finite model property, whereas it is decidable.

If $\sim$ is a logical congruence on a phase model $M$, then $K$ is not necessarily closed for $\sim$, but one can always replace $K$ by its *closure* $\overline{K} = \{x \in M;\ x \sim y \text{ for some } y \in K\}$. It is indeed easy to see that, for any fact $X$, one has $(X \cap K)^\perp = (X \cap \overline{K})^\perp$. Moreover, $\pi(K) = \pi(\overline{K}) \subset \pi(J(M)) = J(M/{\sim})$, so that $\sim$ induces a structure of enriched phase space on $M/{\sim}$, and the analogue of lemma 2 holds for $\mathbf{LL}$. For instance, the smallest congruence $\sim$ such that $?A \sim ?A, ?A$ for any $A$ is a logical congruence on $\mathbf{LL}^\bullet$, and $K$ is already closed for $\sim$. In that case, $\pi(K) = I(\mathbf{LL}^\bullet/\sim)$, and since $\mathbf{LL}^\bullet/\sim$ satisfies the same formulas as $\mathbf{LL}^\bullet$, one gets the completeness of Girard's phase semantics for **LL** [Gir95]. Note that Girard's phase semantics is not closed under quotient by a logical congruence, because it is not true in general that $\pi(I(M)) = I(M/\sim)$.

## 4. Semantics of affine logic

If $M$ is a commutative monoid, an *ideal* of $M$ is an $X \subset M$ such that $XM \subset X$, or equivalently, $XM = X$. Clearly, $X \multimap Y$ is an ideal whenever $Y$ is an ideal, and any union of ideals is an ideal. Say that an ideal is *principal* if it is of the form $xM$ for some $x \in M$, and say that it is *of finite type* if it is a finite union of principal ideals. Finally, say that $M$ is *noetherian* if all its ideals are of finite type. We shall need the following classical result, which is also crucial in [Kop95a]:

**Lemma 4.** Any finitely generated free commutative monoid is noetherian.

*Proof.* Such a monoid is isomorphic to $\mathbb{N}^k$ with addition, and an ideal of $\mathbb{N}^k$ is a subset of $\mathbb{N}^k$ which is upwards closed for the usual ordering. $\mathbb{N}$ is clearly noetherian: its ideals are empty or principal. To see that $\mathbb{N}^2$ is noetherian, consider an ideal $X$ of $\mathbb{N}^2$. If $X$ is nonempty, one can choose $(p, q) \in X$, and one has

$$X = X_0 \ \cup \ \bigcup_{i<p} \{i\} \times Y_i \ \cup \ \bigcup_{j<q} Z_j \times \{j\},$$

where $X_0$ is the principal ideal generated by $(p, q)$, $Y_i = \{n \in \mathbb{N};\ (i, n) \in X\}$, and $Z_j = \{n \in \mathbb{N};\ (n, j) \in X\}$. Since the $Y_i$ and the $Z_j$ are ideals of the noetherian monoid $\mathbb{N}$, one sees easily that $X$ is of finite type. More generally, one proves that $\mathbb{N}^k$ is noetherian by induction on $k$. $\square$

An *affine phase space* is a phase space $M$ such that $\bot^\bullet$ is an ideal of $M$, or equivalently, $\mathbf{1}^\bullet = M$. In that case, all facts of $M$ are ideals, and $M$ satisfies all formulas which are provable in **LLW**, that is, **LL** extended with the *(structural) weakening rule*:

$$\frac{\vdash \Gamma}{\vdash A, \Gamma}$$

The syntactical model $\mathbf{LLW}^\bullet$ is an affine phase model. Again, one shows the analogue of theorem 1, and the completeness of the finitely generated affine phase semantics. More precisely, a formula $A$ is provable in **LLW** if and only if $A$ is satisfied by the phase model $\mathbf{LLW}^\bullet[A]$ whose underlying monoid is a finitely generated free commutative monoid.

**Lemma 5.** If $M$ is an affine phase model whose underlying monoid is a finitely generated free commutative monoid, then $M/\!\equiv$ is finite.

*Proof.* We can assume that the underlying monoid is $\mathbb{N}^k$ with addition. By lemma 4,

$$\bot^\bullet = \bigcup_{i=1,\dots,n} \{x \in \mathbb{N}^k;\ x \geq u_i\}$$

for some $u_1, \dots, u_n \in \mathbb{N}^k$. Therefore, $\{x\}^\bot$ is completely determined by $x' = \inf(x, \sup(u_1, \dots, u_n))$. Since there are finitely many such $x'$, this means that $M/\!\equiv$ is finite. $\square$

This lemma allows to prove the following theorem, which gives the completeness of the finite affine phase semantics for **LLW**. In particular, **LLW** is decidable.

**Theorem 4.** For any formula $A$ in **LL**, the following statements are equivalent:

(1) $A$ is provable in **LLW**;
(2) $A$ is satisfied by all finite affine phase models;
(3) $A$ is satisfied by $\mathbf{LLW}^\bullet[A]/\!\equiv$;
(4) $A$ is satisfied by $\mathbf{LLW}^\bullet[A]$;
(5) $A$ is provable in cut-free $\mathbf{LLW}[A]$.

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