# Diagrammatic syntax for algebra

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# Presentations (of groups) by generators and relations

- $\mathbf{S}_3 = \langle a,b \mid a^2, b^2, (ab)^3 \rangle = \langle a,b \mid a^2, b^2, abab^{-1}a^{-1}b^{-1} \rangle$
- $\mathbf{S}_2 = \langle a \mid a^2 \rangle \hookrightarrow \mathbf{S}_3$
- $\mathbf{B}_3 = \langle a,b \mid abab^{-1}a^{-1}b^{-1} \rangle \twoheadrightarrow \mathbf{S}_3$
- $G = \langle a_1,\dots,a_p \mid r_1,\dots,r_q \rangle = F/N$ where $F = \langle a_1,\dots,a_p \rangle$ and N is the normal subgroup of F generated by $r_1,\dots,r_q$

# Presentations of monoids

- $\mathbf{B}_3 = \langle a,b \mid bab = aba \rangle$
- $\mathbf{B}_3 = \langle a,a',b,b' \mid aa' = a'a = bb' = b'b = 1,\ bab = aba \rangle^+$
- $\mathbf{B}_3^+ = \langle a,b \mid bab = aba \rangle^+$
- $M = \langle a_1,\dots,a_p \mid u_1 = v_1,\dots,\ u_q = v_q \rangle^+ = W/{\leftrightarrow_R}^*$ where $W = \{a_1\dots,a_p\}^*$ and ${\leftrightarrow_R}^*$ is the congruence generated by the pairs $(u_1,v_1)$, …, $(u_q,v_q)$
- This leads to the notion of reduction: $u \to_R^* v$.
- In fact, reductions define a monoidal category.

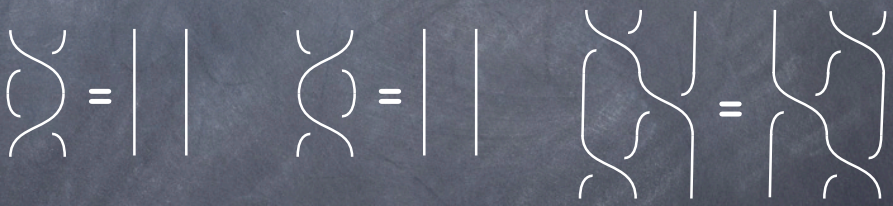


# Presentations of PROs

A PRO is a strict monoidal category whose objects are natural numbers (with sum).

- generators:

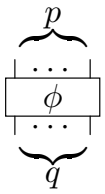
- relations:



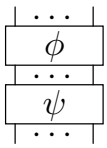
- The PRO is the free PRO D of diagrams built with the generators, quotiented by the congruence ${\leftrightarrow_R}^*$ generated by the relations.

# Diagrams

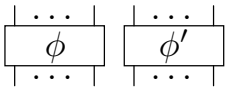
inputs/outputs :



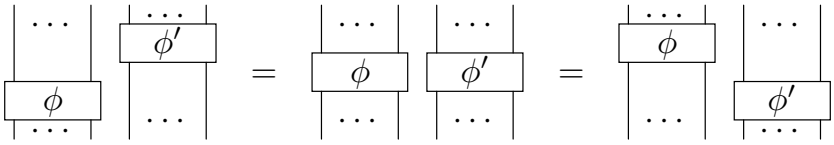

sequential composition:




parallel composition:



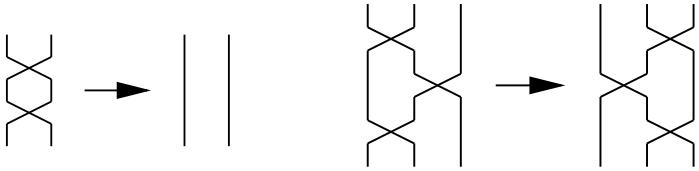

interchange:






# Diagram rewriting

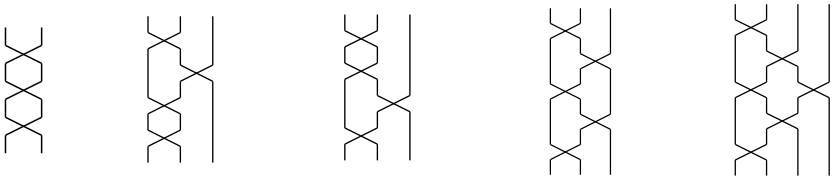
Rules:



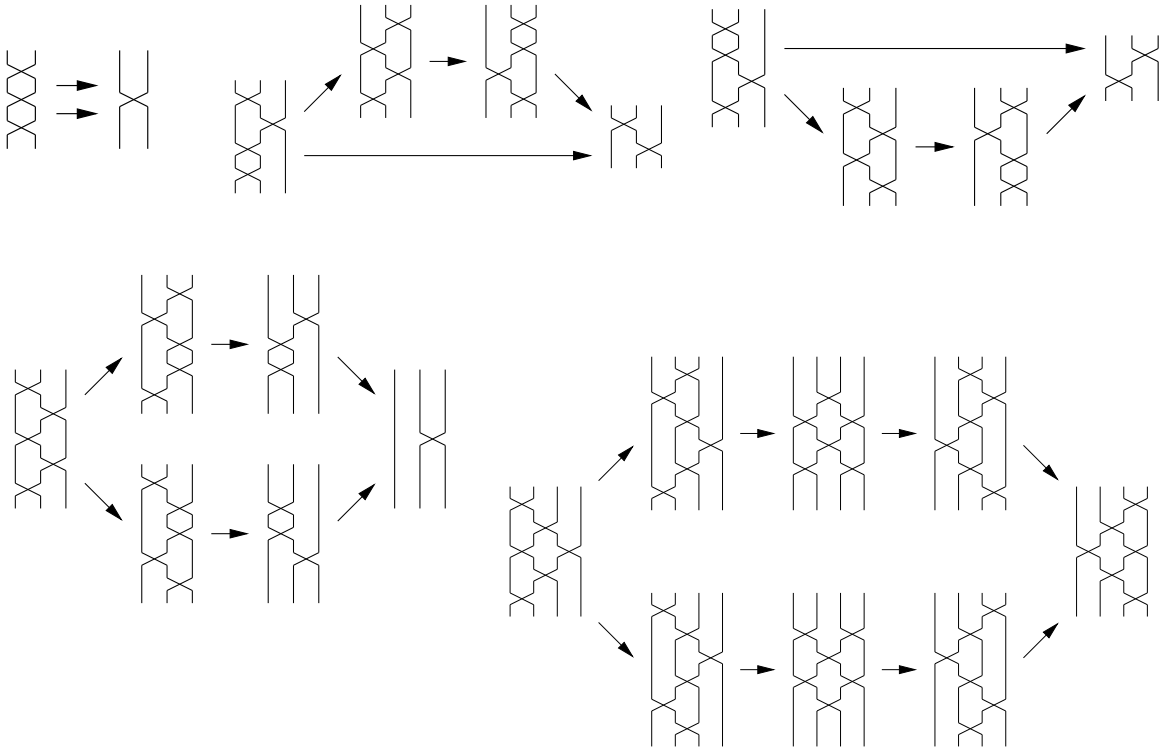
This system is *convergent*:

- *Termination* (existence of normal form)
- *Confluence* (uniqueness of normal form)

Conflicts (*critical peaks*) :

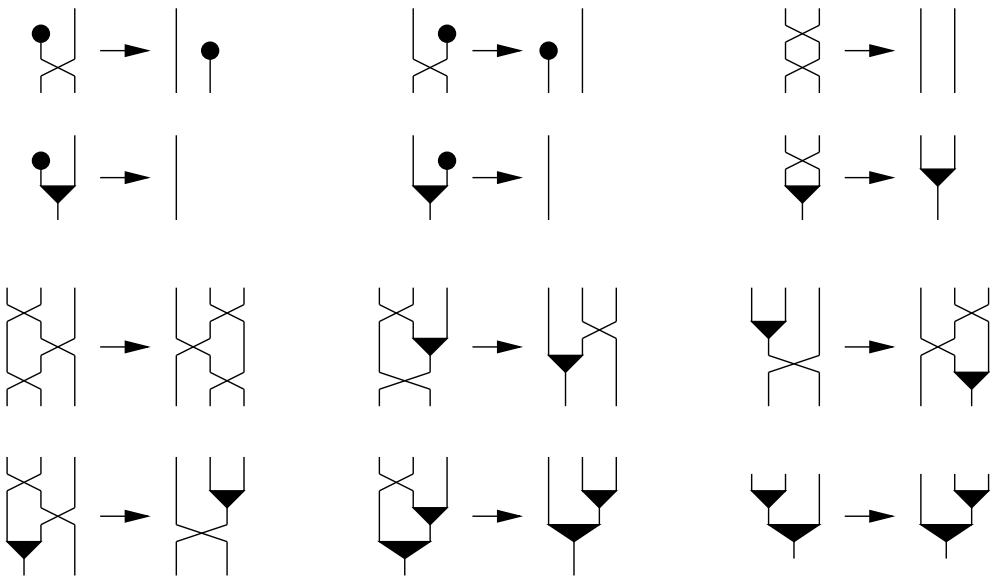


# Confluence of critical peaks



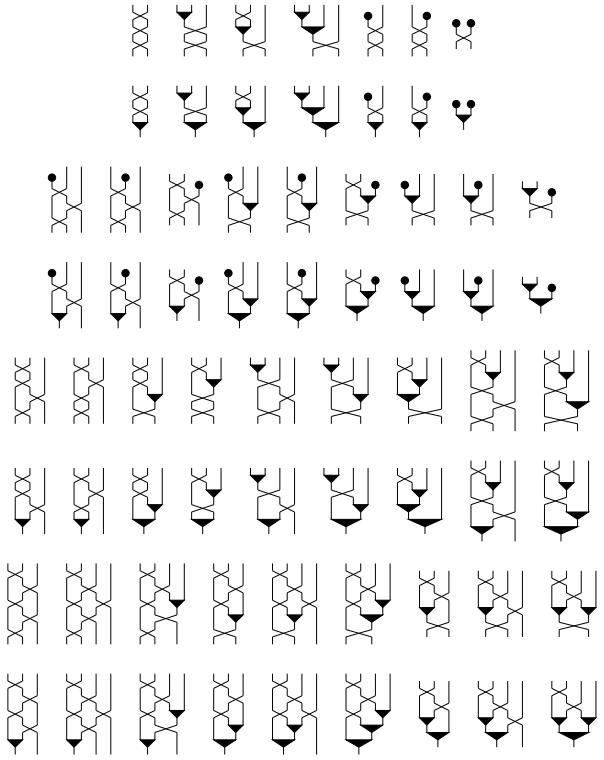


# Rewrite system for the PRO of finite maps



This rewrite system is convergent.

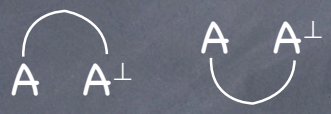
# The 68 critical peaks



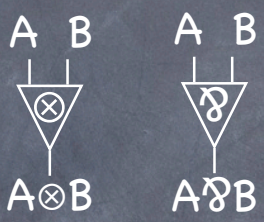


# Proof nets (by Girard)

- axiom and cut:




- connectives:



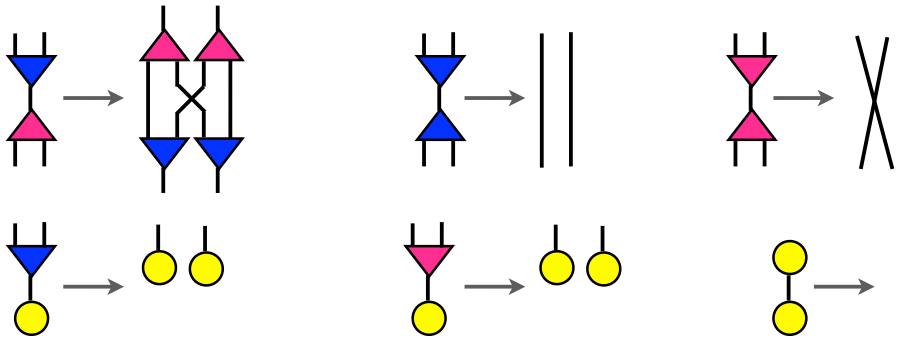

- cut elimination:

- + correctness criterion
- + boxes
- but no associativity (and no commutativity)

# Interaction combinators

*constructor*: *duplicator*: *eraser*:

*interaction rules*:



Theorem: this interaction system is *universal*.



# Remarks

- This has nothing to do with algebra.
- But those rules are very similar to the definition of a Hopf algebra: it is a generalized bialgebra.
- There is no conflict (except if we insist on explicite encoding of axiom, cut, and exchange).
- Interaction nets are a paradigm of distributed (strongly deterministic) computation.

# Computation with Σ-diagrams

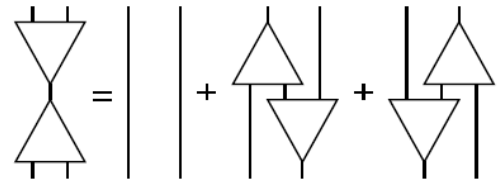
- Many computations happen in algebras rather than in groups or in monoids.
- The syntax must be linearized: use linear combinations of diagrams (called Σ-diagrams) instead of diagrams.
- In that case, a congruence corresponds to a two-sided ideal.
- Sometimes, relations need to be consider as rules (explicitely oriented). Sometimes, they do not.



# Examples of Σ-diagrams (Rannou 2010)

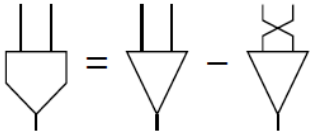
nonunitary infinitesimal

$$\delta(XY) = X \otimes Y + X_1 \otimes X_2 Y + XY_1 \otimes Y_2$$



Lie bracket

$$[X, Y] = XY - YX$$



Lie cobracket

$$\delta_{[,]}(X) = \delta(X) - \tau\delta(X)$$