# Signal Processing Tutorial

## Research School: Mathematics, Signal Processing and Learning

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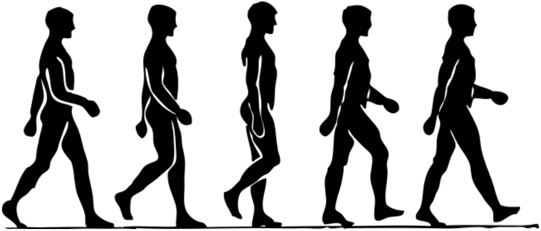
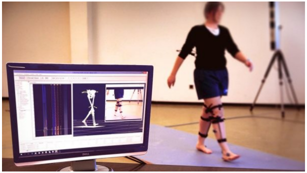
# Gait analysis



Why is is important to study locomotion?

- Most common dynamic human activity
- Can reveal a large number of neurological, orthopedic, rheumatological disorders...
- Strong influence on daily life : risk of falling, frailty, autonomy, dependency...

# Gait analysis



How can we study locomotion?

- Early tests: clinical examination by the physician, functional tests, clinical questionnaires

| | |
|---|---|
| + | Easy to perform, use of clinical expertise |
| - | Lack of precision, difficult to objectively compare two sessions |

- Dedicated platforms for the study of locomotion: instrumented mats, video/optical systems

| | |
|---|---|
| + | Great precision, extraction of a large number of useful features, objective quantification |
| - | Expensive, difficult to put in practice |

# Main principles

★ Objective quantification of human gait

→ Use of sensors and physiological measurements

★ Longitudinal follow-up and inter-individual comparison

→ Need for a fixed protocol

★ Experimentation outside the laboratory and on the field

→ User-mounted sensors and fully automatic device for consultation and routine use

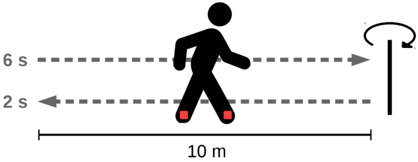
★ Clean data

→ Control of the entire measurement chain, robust and reproducible algorithms

★ Willingness to capture the expertise of the clinician

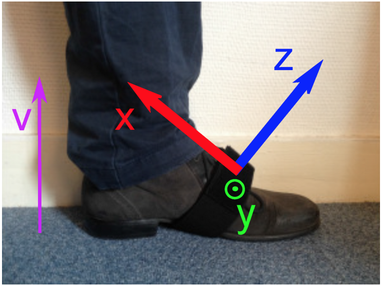
→ Clinical annotations and metadata

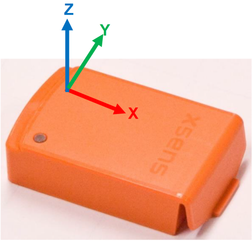
# Protocol




- Sequence of activities:
  - stand for 6 s,
  - walk 10 m at preferred walking speed on a level surface to a previously shown turn point,
  - turn around (without previous specification of a turning side),
  - walk back to the starting point,
  - stand for 2 s.
- Subjects walked at their comfortable speed with their shoes and without walking aid.

# Sensors



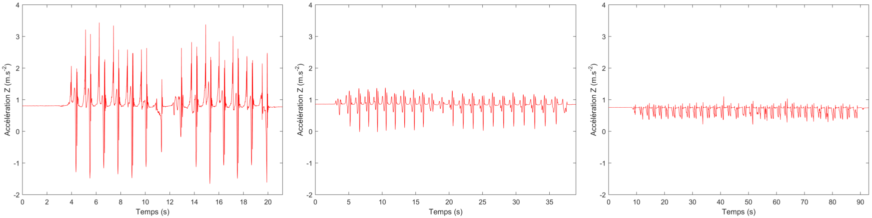

- IMU (Inertial Measurement Unit) record linear accelerations (3D), angular velocities (3D) and magnetic fields (3D) on each foot
- Sensor frame consists of 3-axis $(X, Y, Z)$
- **For this tutorial: angular velocity aroung axis $Y$**

# Database

221 recordings:

- Healthy subjects had no known medical impairment (labelled as ”T” for Témoin).
- The orthopedic group is composed of 3 cohorts of distinct pathologies: lower limb osteoarthrosis (ArtH, ArtG), cruciate ligament injury (LCA), knee injury (Genou)
- The neurological group is composed of 2 cohorts: cerebellar disorder (CER) and radiation induced leukoencephalopathy (LER)

# Main scientific questions




Sujet sain — Pathologie neurologique peu sévère — Pathologie neurologique sévère

## Non-stationary signals

→ How can be detect the different regimes (stop, walking, U-turn...)?

## Presence of repetitive patterns: the steps

→ What are they? How could we automatically extract them?

## Robust feature extraction

→ How could be extract relevant features for longitudinal follow-up and inter-individual comparison?

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# What is signal?

- A time series (or signal) is a series of data points indexed in time order
- In practice, array of real numbers of size $D \times N$ where $D$ is the number of dimensions and $N$ the number of samples
  - Sample number $n$

| $n$ | | | | | | | |
|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 |

  - Time series values $x[n]$

| $x[n]$ | | | | | | | |
|---|---|---|---|---|---|---|---|
| | 0.7 | 0.2 | 0.8 | 0.9 | 0.3 | 0.2 | 0.7 |
| | 0.4 | 0.1 | 0.6 | 0.2 | 0.5 | 0.6 | 0.3 |

  - Time stamps $t[n]$

| $t[n]$ | | | | | | | |
|---|---|---|---|---|---|---|---|
| | 16:30:01 | 16:30:23 | 16:31:43 | 16:32:38 | 16:33:06 | 16:33:16 | 16:33:56 |

# Two visions: physics vs. statistics

- The notion of time have been used and modeled in physics since 18th century and before (eg. Fourier transform).
  **First vision :** a time series $x[1:N]$ is the result of the digitization of a physical phenomenon $x(t)$. Physical properties of this phenomenon can be retrieved and analyzed through the study of $x[1:N]$ (and vice/versa).
- Randomness can also play a part to model a wider class of signals.
  **Second vision :** a time series $x[1:N]$ is a realization of a stochastic process $X[1:N]$. Statistical properties of this phenomenon can be retrieved and analyzed through the study of $x[1:N]$ (and vice/versa).

In most cases, both approaches can be combined.

# Some useful signal processing tools

In the following, we will introduce basic signal processing tools and apply them to our signals:

- Discrete Fourier Transform (DFT)
- Notion of stationarity, ergodicity and autocorrelation function
- Spectrogram

# Sampling and Fourier analysis

- Most tools for signal processing are derived from Fourier analysis
- In this context, we assume that $\mathbf{x}$ corresponds to the discrete measurement of a continuous signal $x(t)$
- **Sampling theory:** uniform sampling period $T_s$ and sampling frequency $F_s = \frac{1}{T_s}$

$$x[n] = x(nT_s)$$

# Discrete Fourier Transform (DFT)

$$\boxed{X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi \frac{kn}{N}} \text{ pour } 0 \leq k \leq N-1}$$

where $N$ is the number of samples

- The space between two observable frequencies is called **frequency resolution**

$$\boxed{\Delta f = \frac{F_s}{N}}$$

- $X[k]$ corresponds to the DFT for the physical frequency

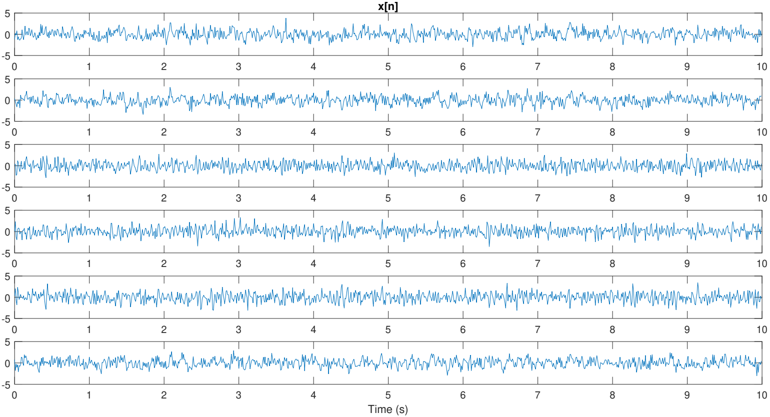
$$\boxed{f[k] = k\frac{F_s}{N} \text{ for } 0 \leq k \leq N-1}$$

- No physical frequency greater than $\frac{F_s}{2}$ can be observed (Nyquist theorem).

# Spectral analysis

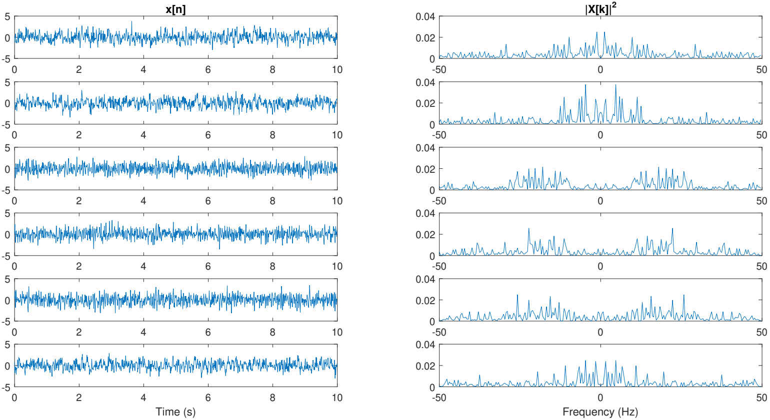
- $X[k]$ is a complex quantity: most of the time, we use the squared absolute values $|X[k]|^2$ instead
- The analysis of the quantity $|X[k]|^2$ can allow to discover interesting properties of the time series
- $|X[k]|^2$ with low frequencies $f_k$ correspond to phenomena with smooth variations
- $|X[k]|^2$ with large frequencies $f_k$ correspond to phenomena with fast variations
- One very useful plot consists in plotting $|X[k]|^2$ as a function of $f[k]$: such plot is often referred to as **spectrum** (hence spectral analysis)

# Example




Two classes of signals?

# Example




Can be distinguished based on their DFT coefficients

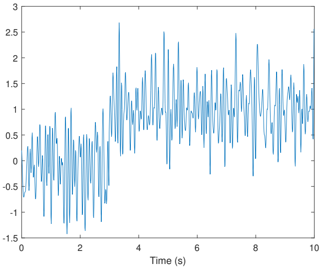
# Statistical vision

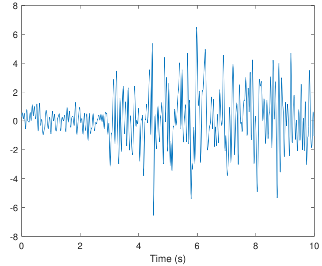
- In order to better understand the properties of a signal, deterministic analysis such as Fourier has been extended to probabilistic and statistical analysis
- In this context, we assume that $x[1:N]$ corresponds to a realization of a stochastic process $X[1:N]$
- Each $X[n]$ can be seen as a random variable
- Statistical properties of $X[1:n]$ can be retrieved from estimates based on $x[1:n]$

# Two fundamental properties

- **Stationarity :** The statistical properties of the time series do not change over time
  - Order 1
  $$\forall n, \quad \mathbb{E}\left[X[n]\right] = \mu$$
  - Order 2
  $$\forall n_1, n_2, \quad \mathbb{E}\left[X[n_1]X[n_2]\right] = \gamma_X[|n_2 - n_1|]$$
  - Order 1 + Order 2 $\rightarrow$ wide-sense stationarity (most common assumption)
- **Ergodicity :** Ensemble mean and the mean over time are equal (implies stationarity of order 1)

# Stationarity vs. non-stationarity




- None of these signals are stationary
- In order to prevent this to happen, two solutions exist
  - Divide the signals into small frames where the signal is assumed to be stationary and ergodic (see later: spectrogram)
  - Use a change-point detection algorithm to detect these changes and work separately on each segment (see later: change-point detection)

# Autocorrelation

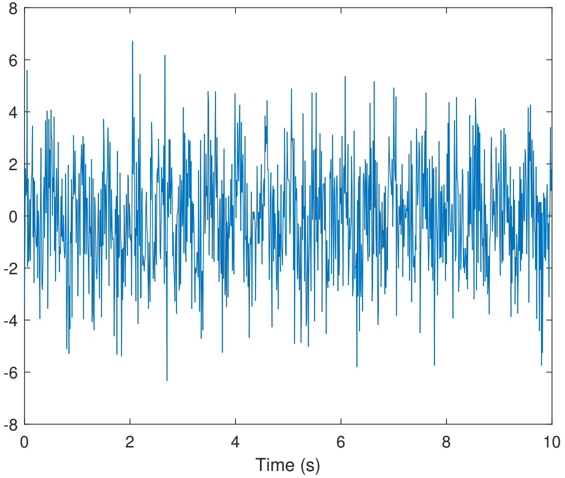
Assuming that $X[1:n]$ is ergodic and wide-sense stationary, we can estimate from $x[1:n]$ the autocorrelation function

- Autocorrelation function

$$\boxed{\hat{\gamma}_x^{biased}[m] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]x[n+m]} \quad \text{where } x[n] = 0 \text{ for } n \neq 0 \ldots N-1$$

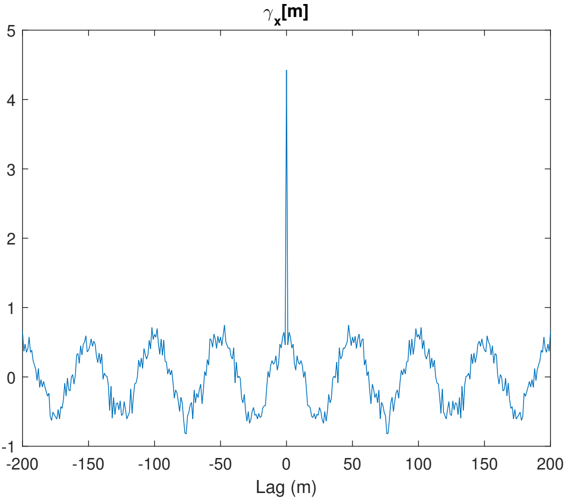
- This function helps (among other things) to discover the presence of periodic components within a signal

# How to use the autocorrelation function



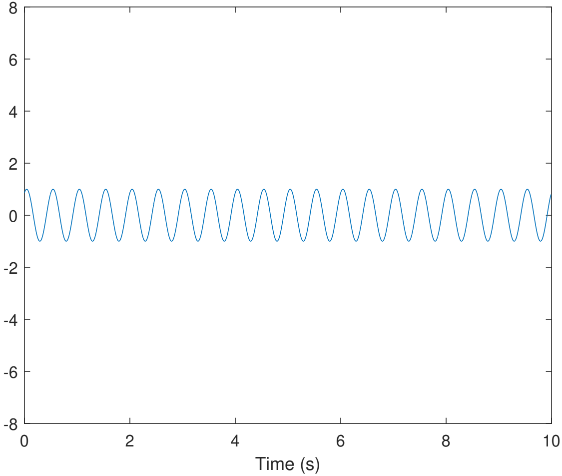

Original signal, sampling frequency 100 Hz

# How to use the autocorrelation function




Autocorrelation function, peaks are visible for lags multiple of 50

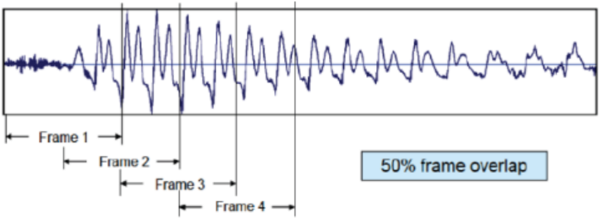
# How to use the autocorrelation function




A periodic signal with period $50 \times \frac{1}{100} = 0.5$ sec was hiding!

# Spectrogram

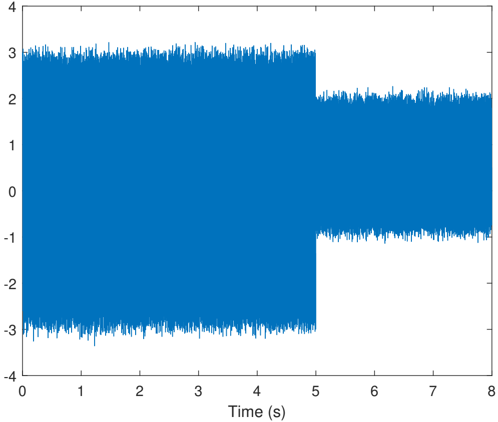
- When the properties of the time series tend to change with time (non-stationary signals), it is more careful to compute the DFT on sliding windows
- By sliding the window along the signal, we recover a time-frequency representation called **spectrogram**




- Matrix representation: each column corresponds to the DFT on the window of interest.

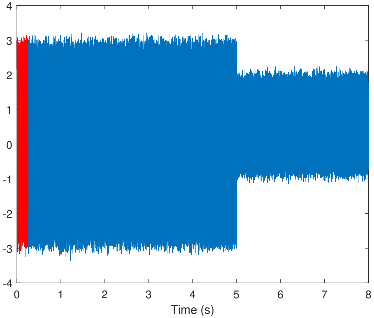
X-axis: frame number, Y-axis: frequency bin

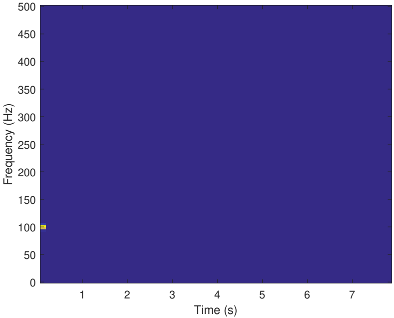
# Example




Original signal, $F_s = 1000$ Hz

# Example

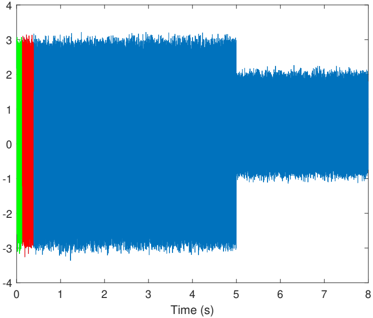


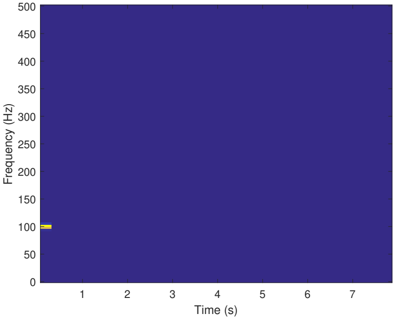




Window length $N_w = 256$

Computation of the DFT on the first frame and storage in the spectrogram matrix...

# Example

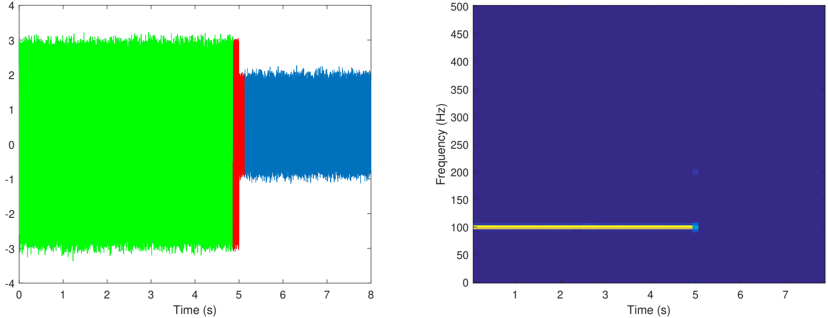



Window length $N_w = 256$

Computation of the DFT on the second frame and storage in the spectrogram matrix...

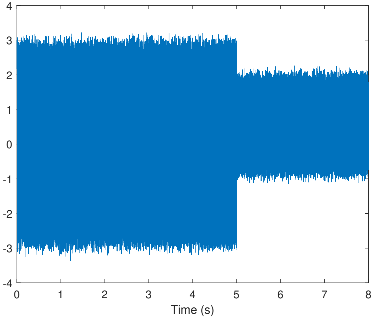
# Example

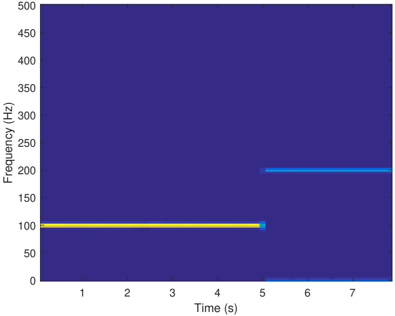



Window length $N_w = 256$

Same process...

# Example



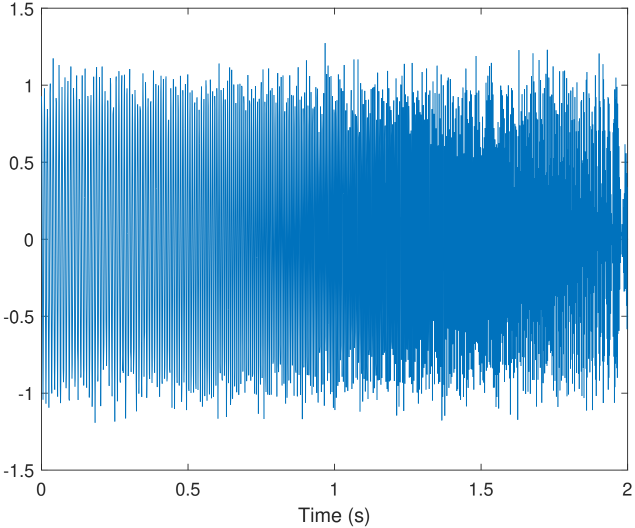




Window length $N_w = 256$
Final result

# DFT vs. Spectrogram

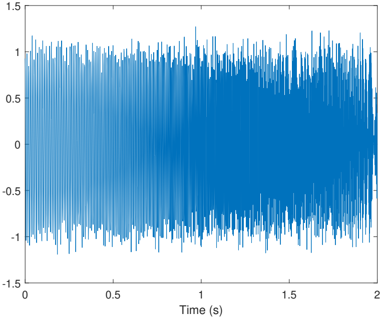
- Only use DFT when you are sure that there is no abrupt changes in the time series
- Note that using DFT will tend to average the frequency content on the whole time series, which can be tricky in some application contexts
- For safety, always first visualize the spectrogram to make sure that no significant changes occur

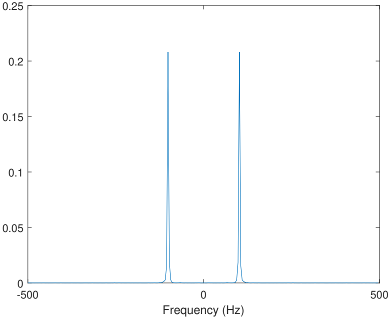
# DFT vs. Spectrogram




Periodic phenomenon ? (sinusoid ?)

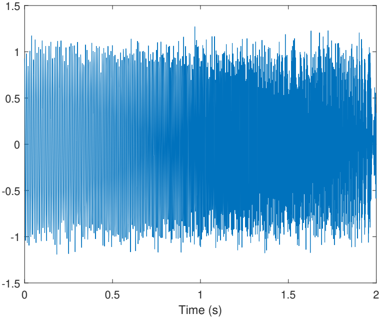
# DFT vs. Spectrogram

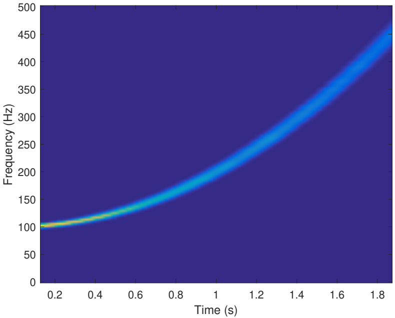



DFT suggests a sinusoidal phenomenon around frequency 100 Hz

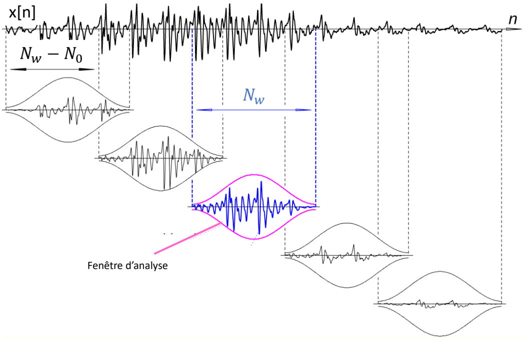
# DFT vs. Spectrogram




In fact, chirp signal between 100 and 500 Hz ‼

# Hyperparameters for spectrogram




- $N_w$ : window length (in samples)
  Often taken as a power of 2 (for FFT) and linked to the desired frequency resolution.
- $N_o$ : overlap between two successive frames (in samples)
  Often taken as 50% or 75% of the window length and characterizes the time resolution (optimal when $N_o = N_w - 1$)
- $w$ : analysis window (Hann, Hamming, Blackman...)
  Traditionally, in order to limit side effects, the signal frame is multiplied by an analysis window

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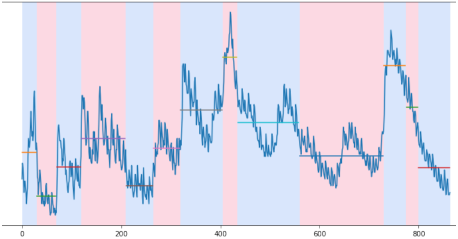
# Problem 1: Change-Point Detection

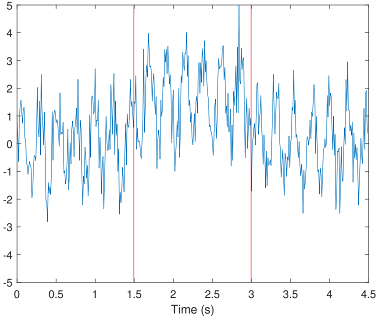



## Change-Point Detection

**Given a time series x, retrieve the times $(t_1, \ldots, t_K)$ where a significant change occurs**

- Necessitates to estimate both the change-points but also the number of changes $K$
- Highly depends on the meaning given to change

# Problem statement



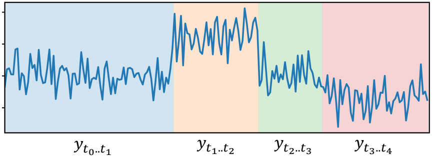

- When the changes are abrupt or when the estimation of the change-points is relevant in the context, we can use **change-point detection** methods
- Let assume that signal $x[n]$ undergoes abrupt changes at times

$$\mathcal{T}^* = (t_1^*, \dots, t_{K^*}^*)$$

- Goal: retrieve the number of change-points and $K^*$ and their times $\mathcal{T}^*$
- One assumption: offline segmentation (but can easily be adapted to online setting) [Truong et al., 2020]

# Problem statement

$$(\hat{t}_1, \ldots, \hat{t}_K) = \underset{(t_1, \ldots, t_K)}{\text{argmin}} \sum_{k=0}^{K} c(x[t_k : t_{k+1}])$$

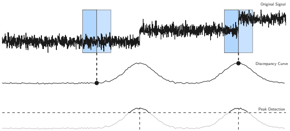



## Cost function $c(.)$

- Measures the homogeneity of the segments
- Choosing $c(.)$ conditions the type of change-points that we want to detect
- Often based on a probabilistic model for the data

## Problem solving

- Optimal resolution with dynamic programming
- Approximate resolution (sliding windows...)



# Cost function

$$(\hat{t}_1, \ldots, \hat{t}_K) = \underset{(t_1, \ldots, t_K)}{\text{argmin}} \sum_{k=0}^{K} c(x[t_k : t_{k+1}])$$

Convention : $t_0 = 0$, $t_{K+1} = N$

- Function $c(.)$ is characteristic of the notion of *homogeneity*
- The most common cost functions are linked to parametric probabilistic models: in this case change-points are defined as changes in the parameters of a probability density function [Basseville et al., 1993]
- Non parametric cost functions can also be introduced when no model is available

# Maximum likelihood estimation

Given a parametric family of distribution densities $f(\cdot|\theta)$ parametrized with $\theta \in \Theta$, a cost function can be derived:

$$c_{ML}(x[a:b]) = -\sup_{\theta} \sum_{n=a+1}^{b} \log f\left(x[n]|\theta\right)$$

- Corresponds to the assumption that on a regime, samples are i.i.d. according to a parametric distribution density
- On each regime, the parameters are estimated through maximum likelihood estimation
- This model can be adapted to several situations: change in mean, change in variance, change in both mean and variance...

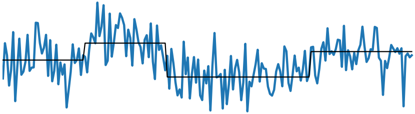
# Change in mean

The most popular is indubitably the L2 norm [Page, 1955]

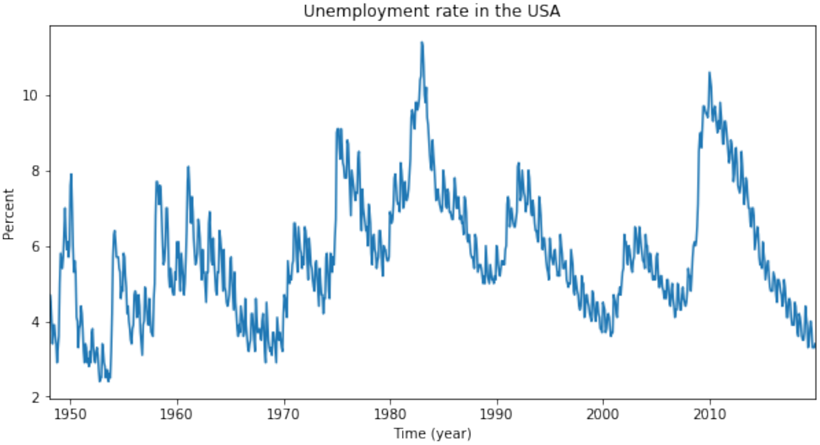
$$c_{L_2}(x[a:b]) = \sum_{n=a+1}^{b} \|x[n] - \mu_{a:b}\|_2^2$$

where $\mu_{a:b}$ is the empirical mean of the segment $x[a:b]$.

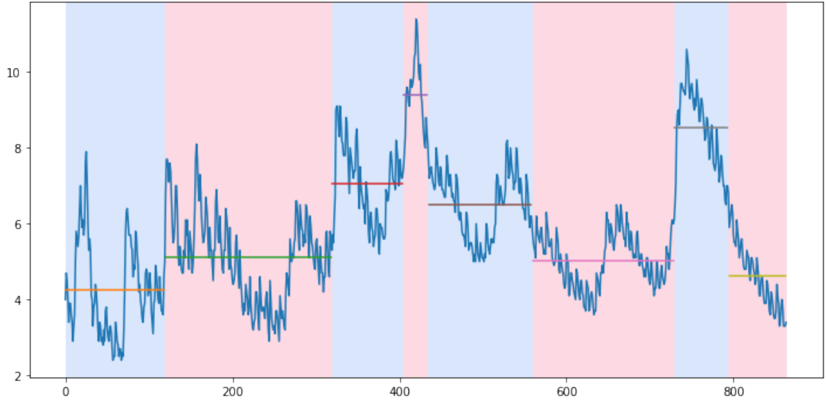
- Particular case of $c_{ML}$ with Gaussian model with fixed variance
- Allows to detect changes in mean



# Example

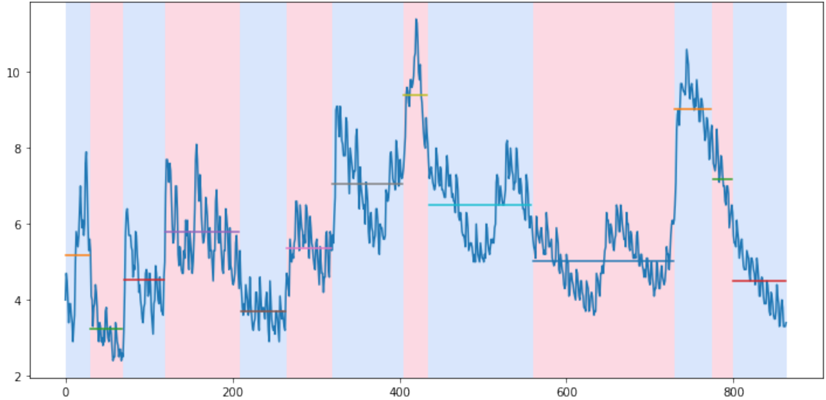

Unemployment rate in the USA
Percent
Time (year)
2
4
6
8
10
1950
1960
1970
1980
1990
2000
2010

# Example: Change-Point Detection with $c_{L_2}$




$K = 7$

# Example: Change-Point Detection with $c_{L_2}$




$K = 12$

# Search method

$$\boxed{(\hat{t}_1, \ldots, \hat{t}_K) = \underset{(t_1,\ldots,t_K)}{\text{argmin}} \sum_{k=0}^{K} c(x[t_k : t_{k+1}])}$$

Convention : $t_0 = 0$, $t_{K+1} = N$

- Several methods can be used to solve this problem with a fixed $K$
- Optimal resolution with dynamic programming: find the true solution of the problem (but costly)
- Approximated resolution with windows: test for one unique change-point on a window

# Optimal resolution

- By denoting

$$\mathcal{V}(\mathcal{T}, \mathbf{x}) = \sum_{k=0}^{K} c(x[t_k : t_{k+1}])$$

we can see that

$$\begin{aligned}
\min_{|\mathcal{T}|=K} V(\mathcal{T}, \mathbf{x}) &= \min_{0=t_0<t_1<\cdots<t_K<t_{K+1}=N} \sum_{k=0}^{K} c(x[t_k : t_{k+1}]) \\
&= \min_{t \leq T-K} \left[ c(x[0:t]) + \min_{t_0=t<t_1<\cdots<t_{K-1}<t_K=T} \sum_{k=0}^{K-1} c(x[t_k : t_{k+1}]) \right] \\
&= \min_{t \leq T-K} \left[ c(x[0:t]) + \min_{|\mathcal{T}|=K-1} V(\mathcal{T}, x[t:N]) \right]
\end{aligned}$$

- Recursive problem: resolution with dynamic programming [Bai et al., 2003]
- Two steps: computation of the cumulative costs + determination of the change-points

# Optimal resolution

**Algorithm 1** Algorithm `Opt`

**Input:** signal $\{y_t\}_{t=1}^{T}$, cost function $c(\cdot)$, number of regimes $K \geq 2$.
**for all** $(u, v)$, $1 \leq u < v \leq T$ **do**
  Initialize $C_1(u, v) \leftarrow c(\{y_t\}_{t=u}^{v})$.
**end for**
**for** $k = 2, \ldots, K-1$ **do**
  **for all** $u, v \in \{1, \ldots, T\}, v - u \geq k$ **do**
    $C_k(u, v) \leftarrow \min\limits_{u+k-1 \leq t < v} C_{k-1}(u, t) + C_1(t+1, v)$
  **end for**
**end for**
Initialize $L$, a list with $K$ elements.
Initialize the last element: $L[K] \leftarrow T$.
Initialize $k \leftarrow K$.
**while** $k > 1$ **do**
  $s \leftarrow L(k)$
  $t^* \leftarrow \operatorname{argmin}_{k-1 \leq t < s} C_{k-1}(1, t) + C_1(t+1, s)$
  $L(k-1) \leftarrow t^*$
  $k \leftarrow k - 1$
**end while**
Remove $T$ from $L$
**Output:** set $L$ of estimated breakpoint indexes.

Complexity of $\mathcal{O}(KN^2)$

# Contents

1. Introduction and data

2. Basic tools and notions for signal processing

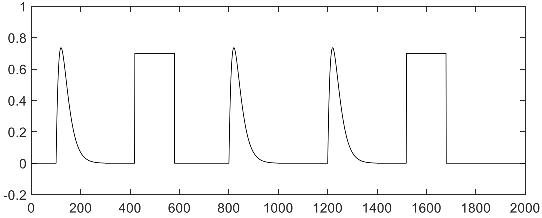
3. Change-point detection

4. Convolutional dictionary learning

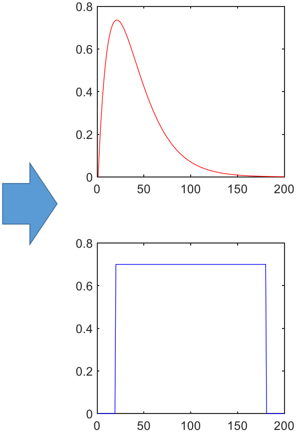
4.1 Problem

4.2 Alternated resolution

# Problem 2: Pattern Extraction




Input time series




Extracted patterns

# Problem 2: Pattern Extraction

**Pattern Extraction**

**Given an input time series x (or a set of time series), learn a dictionary of patterns $\mathcal{P}$**

- A template is a *shape* that appear repetitively in the time series (but kinda blurry notion)
- All templates are supposed to have the same length (for sake of simplicity)
- The extracted patterns can be used to characterize the time series, or studied individually

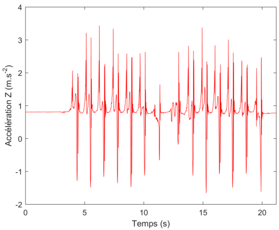
# Dictionary-based pattern extraction

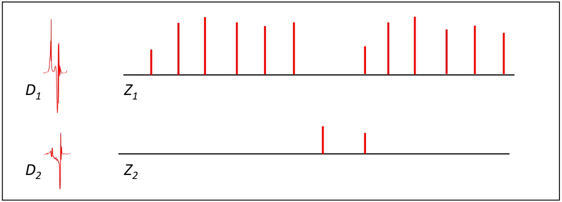
- Finding patterns in a time series can be seen as a dictionary learning optimization problem
- Given an input time series $\mathbf{x}$, learn a dictionary of $K$ patterns $\mathbf{d}_k$ of length $L$
- These patterns can be activated : activations $\mathbf{z}_k$ of length $N - L + 1$

$$z_k[n] \neq 0 \text{ if pattern } \mathbf{d}_k \text{ is activated at time } n$$

[Grosse et al., 2007 ; Wohlberg, 2014]

# Convolutional dictionary learning




## Convolutional dictionary learning

Given a time series $\mathbf{x}$, number of pattern $K$ and pattern length $L$, learn

- Patterns $\mathbf{d}_k$ of length $L$
- Activation signals $\mathbf{z}_k$ of length $N - L + 1$

$$x[n] = \sum_{k=1}^{K} (\mathbf{z}_k * \mathbf{d}_k)[n] + e[n]$$

# Optimization problem

$$\min_{\substack{(\mathbf{d}_k),(\mathbf{z}_k)\\ \forall k, \|\mathbf{d}_k\|_2^2 \leq 1}} \left\| \mathbf{x} - \sum_{k=1}^{K} \mathbf{z}_k * \mathbf{d}_k \right\|_2^2 + \lambda \sum_{k=1}^{K} \|\mathbf{z}_k\|_1$$

- **Normalization constraint** for the dictionary atoms $\mathbf{d}_k$, that prevents numerical instabilities (otherwise setting $\alpha\mathbf{d}_k$ and $\alpha^{-1}\mathbf{z}_k$ gives the same result)
- **Sparsity constraint** for the activations $\mathbf{z}_k$, that improves the interpretability of the learned patterns (see Lecture 3)

Not convex with respect to the couple $(\mathbf{d}_k), (\mathbf{z}_k)$ but convex when the subproblems are taken individually

# Alternated resolution

Alternated resolution of two subproblems

## Dictionary learning

$$\mathbf{D}^* = \underset{\substack{\mathbf{D}=(\mathbf{d}_1,\dots,\mathbf{d}_K)\\ \forall k, \|\mathbf{d}_k\|_2^2 \leq 1}}{\operatorname{argmin}} \left\| \mathbf{x} - \sum_{k=1}^{K} \mathbf{z}_k * \mathbf{d}_k \right\|_2^2$$

## Convolutional sparse coding

$$\mathbf{Z}^* = \underset{\mathbf{Z}=(\mathbf{z}_1,\dots,\mathbf{z}_K)}{\operatorname{argmin}} \left\| \mathbf{x} - \sum_{k=1}^{K} \mathbf{z}_k * \mathbf{d}_k \right\|_2^2 + \lambda \sum_{k=1}^{K} \|\mathbf{z}_k\|_1$$

# Alternated resolution

$$\min_{\substack{(\mathbf{d}_k),(\mathbf{z}_k)\\ \forall k, \|\mathbf{d}_k\|_2^2 \leq 1}} \underbrace{\left\| \mathbf{x} - \sum_{k=1}^{K} \mathbf{z}_k * \mathbf{d}_k \right\|_2^2}_{f(\mathbf{Z},\mathbf{D})} + \lambda \sum_{k=1}^{K} \|\mathbf{z}_k\|_1$$

- Both these problems can be solved with Proximal Gradient Descent algorithms
- Two main steps :
  1. Gradient descent step w.r.t. $\nabla_{\mathbf{D}} f(\mathbf{Z}, \mathbf{D})$ or $\nabla_{\mathbf{Z}} f(\mathbf{Z}, \mathbf{D})$
  2. Proximal step to *project* the update on the constraint set
- The main question is therefore: how can we easily compute the gradients?

# Formulation of the gradients

$$f(\mathbf{Z},\mathbf{D}) = \left\| \mathbf{x} - \sum_{k=1}^{K} \mathbf{z}_k * \mathbf{d}_k \right\|_2^2$$

- By using the convolution theorem and the Parseval theorem, we have that

$$f(\mathbf{Z},\mathbf{D}) = \left\| \hat{\mathbf{x}} - \sum_{k=1}^{K} \widehat{\mathbf{z}_k} \odot \widehat{\mathbf{d}_k} \right\|_2^2$$

  where $\hat{\cdot}$ denotes the Discrete Fourier Transform (DFT) and $\odot$ the component-wise product
- As $\mathbf{u} \odot \mathbf{d} = \mathrm{diag}(\mathbf{u})\mathbf{v}$, this expression can be rewritten in the matrix form as

$$f(\mathbf{Z},\mathbf{D}) = \left\| \hat{\mathbf{x}} - \widehat{\mathbf{D}}\hat{\mathbf{z}} \right\|_2^2$$

- The gradient of this quantity is easy to compute w.r.t. $\widehat{\mathbf{D}}$ and $\hat{\mathbf{z}}$
- Updates for $\mathbf{D}$ and $\mathbf{Z}$ can then be estimated by performing inverse DFT

# Proximal operators

- For the unit ball constraint (dictionary), the proximal operator writes as

$$\mathrm{proj}_{\|.\|_2^2 \leq 1}(\mathbf{y}) = \frac{\mathbf{y}}{\max(\|\mathbf{y}\|_2^2, 1)}$$

Projection on the unit ball

- For the L1-sparsity constraint (atoms), the proximal operator writes as

$$\mathcal{S}\gamma(\mathbf{y})[n] = \mathrm{sign}(y[n]) \times \max(|y[n]| - \gamma, 0)$$

**Soft thresholding operator**

# Dictionary learning

$$\mathbf{D}^* = \underset{\substack{\mathbf{D}=(\mathbf{d}_1,\ldots,\mathbf{d}_K) \\ \forall k, \|\mathbf{d}_k\|_2^2 \leq 1}}{\operatorname{argmin}} \underbrace{\left\| \mathbf{x} - \sum_{k=1}^{K} \mathbf{z}_k * \mathbf{d}_k \right\|_2^2}_{f(\mathbf{Z},\mathbf{D})}$$

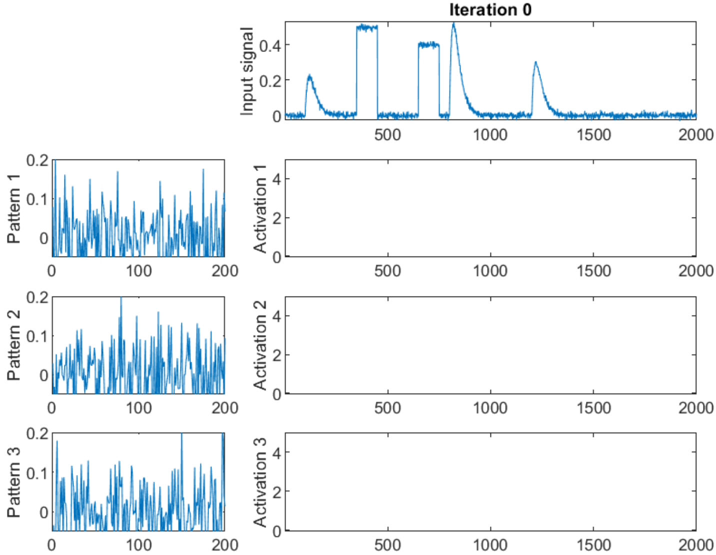
- Basic approach : **Proximal Gradient Descent** with fixed $\mathbf{Z}$
  1. Gradient step :
  $$\mathbf{D} \leftarrow \mathbf{D} - \alpha \nabla_{\mathbf{D}} f(\mathbf{Z}, \mathbf{D})$$
  2. Proximal projection step :
  $$\mathbf{d}_k \leftarrow \operatorname{proj}_{\|.\|_2^2 \leq 1}(\mathbf{d}_k) = \frac{\mathbf{d}_k}{\max(\|\mathbf{d}_k\|_2^2, 1)}$$
- Other approaches : Alternate Direction Method of Multiplier (ADMM), K-SVD... (see [Mairal et al., 2010] and Lecture 3 for more details)

# Convolutional sparse coding

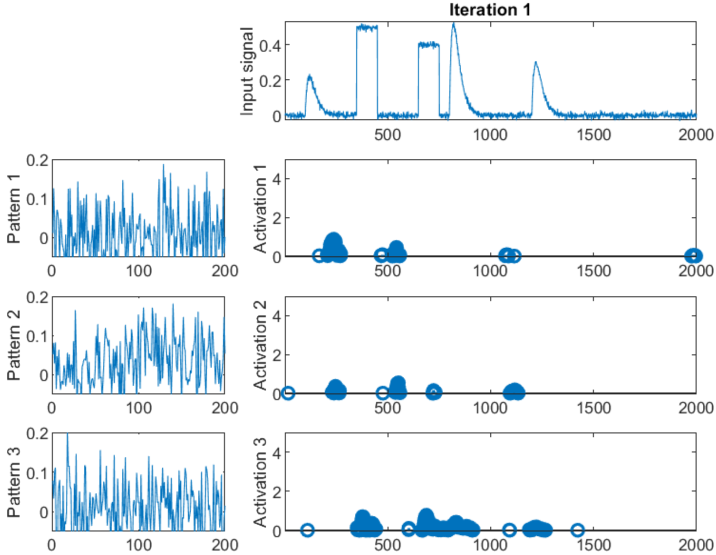
$$\mathbf{Z}^* = \underset{\mathbf{Z}=(\mathbf{z}_1,\ldots,\mathbf{z}_K)}{\operatorname{argmin}} \underbrace{\left\| \mathbf{x} - \sum_{k=1}^{K} \mathbf{z}_k * \mathbf{d}_k \right\|_2^2}_{f(\mathbf{Z},\mathbf{D})} + \underbrace{\lambda \sum_{k=1}^{K} \|\mathbf{z}_k\|_1}_{\psi(\mathbf{Z})}$$

- Basic approach : **Iterative Soft Thresholding Algorithm (ISTA)** with fixed **D** [Daubechies et al., 2004]
    1. Gradient step
    $$\mathbf{Z} \leftarrow \mathbf{Z} - \alpha \nabla_{\mathbf{Z}} f(\mathbf{Z}, \mathbf{D})$$
    2. Proximal step
    $$\mathbf{Z} = \mathcal{S}_{\lambda\alpha}(\mathbf{Z})$$
- Other approaches: Alternate Direction Method of Multiplier (ADMM), Fast Iterative Soft Thresholding Algorithm (FISTA), Coordinate Descent (CD) (see [Mairal et al., 2010] and Lecture 3 for more details)

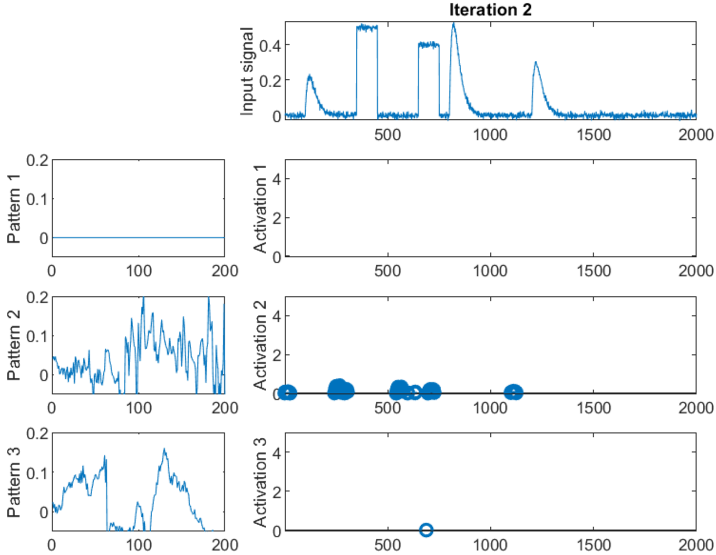
# Results


Iteration 0
Input signal
Pattern 1
Pattern 2
Pattern 3
Activation 1
Activation 2
Activation 3
0.4
0.2
0
500
1000
1500
2000
0.1
100
200
4
2

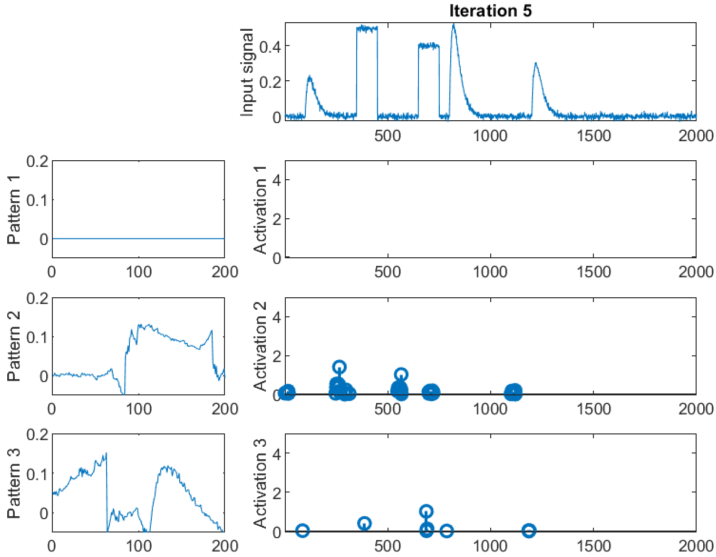
# Results


Iteration 1
Input signal
0.4
0.2
0
500
1000
1500
2000
Pattern 1
Activation 1
Pattern 2
Activation 2
Pattern 3
Activation 3
0.2
0.1
0
100
200
4
2

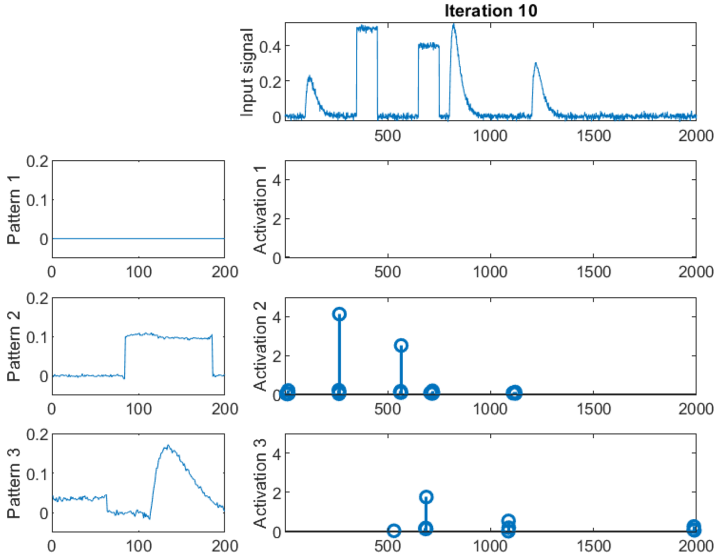
# Results


Iteration 2
Input signal
Pattern 1
Pattern 2
Pattern 3
Activation 1
Activation 2
Activation 3
0.4
0.2
0
500
1000
1500
2000
0.1
100
200
4
2

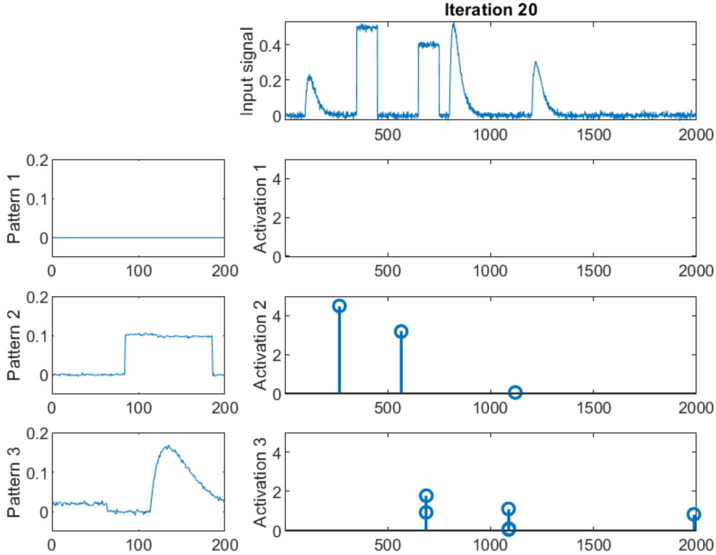
# Results


Iteration 5
Input signal
0.4
0.2
0
500
1000
1500
2000
Pattern 1
0.2
0.1
0
0
100
200
Activation 1
4
2
0
Pattern 2
Activation 2
Pattern 3
Activation 3

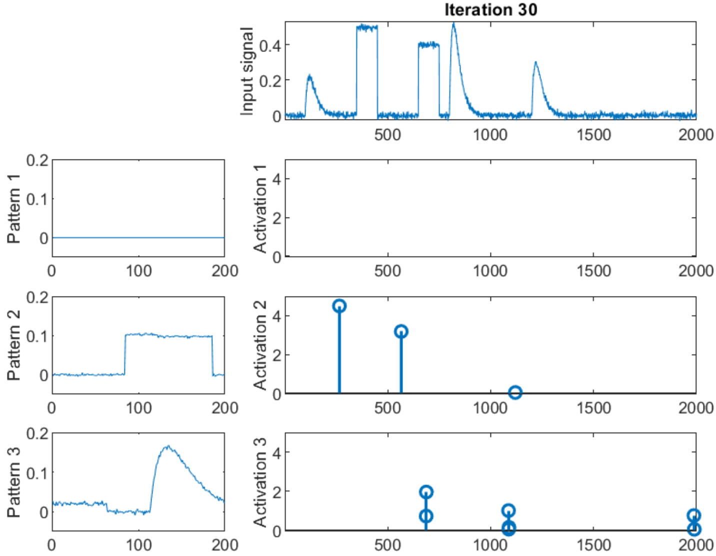
# Results


Iteration 10
Input signal
0.4
0.2
0
500
1000
1500
2000
Pattern 1
0.2
0.1
0
0
100
200
Activation 1
4
2
0
Pattern 2
Activation 2
Pattern 3
Activation 3

# Results

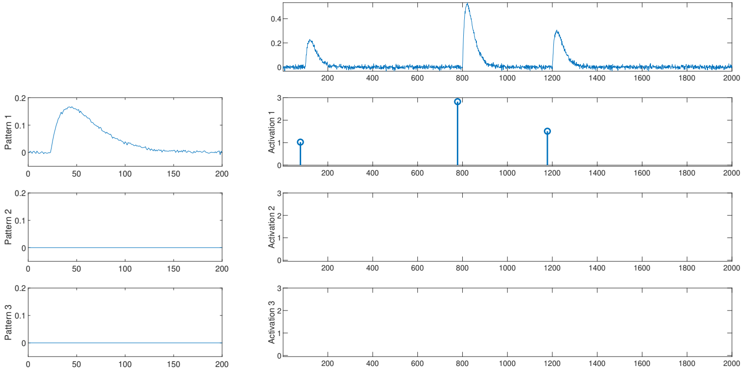

Iteration 20
Input signal
0.4
0.2
0
500
1000
1500
2000
Pattern 1
Activation 1
Pattern 2
Activation 2
Pattern 3
Activation 3
0.2
0.1
0
100
200
4
2

# Results



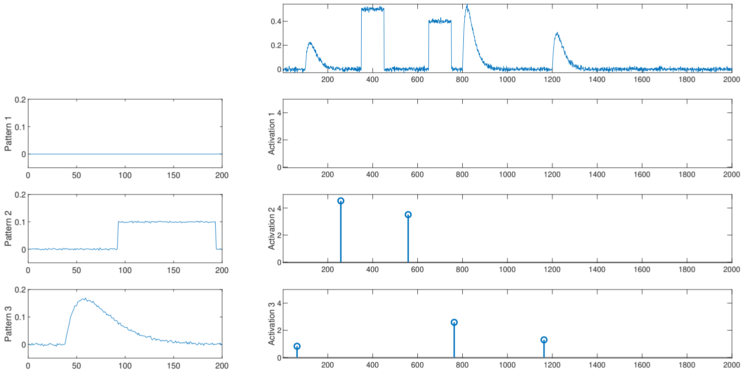
Iteration 30
Input signal
0.4
0.2
0
500
1000
1500
2000
Pattern 1
Activation 1
Pattern 2
Activation 2
Pattern 3
Activation 3
0.2
0.1
0
100
200
4
2

# Results



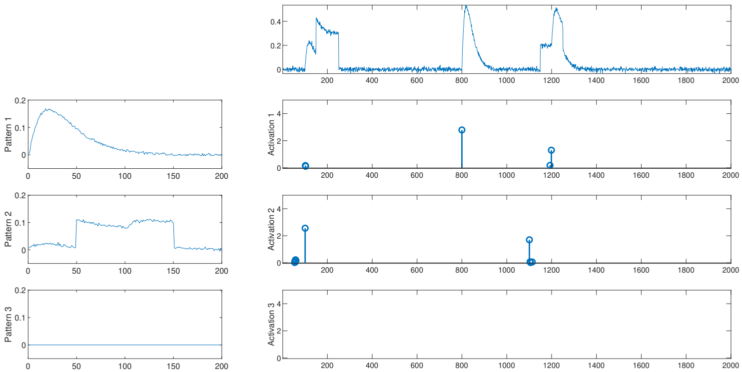

One pattern

# Results




Two patterns

# Results




Mixture of patterns

# Summary

- CDL allows to detect mixture of patterns thanks to the additive model
- CDL is traditionally quite sensitive to initialization and/or hyperparameters : random initializations (normalized gaussian noise), randomly chosen data frames, Lipschitz constant for gradient descent...
- CDL can also be used to search patterns from a fixed dictionary of templates : in this case, only perform the convolutional sparse coding step