

Optimization

Part VI: Conjugate

Nelly Pustelnik

CNRS, Laboratoire de Physique de l'ENS de Lyon, France



(several slides in this part traced back Tutorial ICASSP 2014 written in collaboration with **Jean-Christophe Pesquet** from Centre de Vision Numérique, CentraleSupélec, University Paris-Saclay, Inria, France.)

Conjugate



Adrien-Marie Legendre
(1752–1833)



Werner Fenchel
(1905–1988)

Conjugate



Adrien-Marie Legendre
(1752–1833)



Werner Fenchel
(1905–1988)

Conjugate: definition

Let \mathcal{H} be a Hilbert space and $f: \mathcal{H} \rightarrow]-\infty, +\infty]$.

The **conjugate** of f is the function $f^*: \mathcal{H} \rightarrow [-\infty, +\infty]$ such that

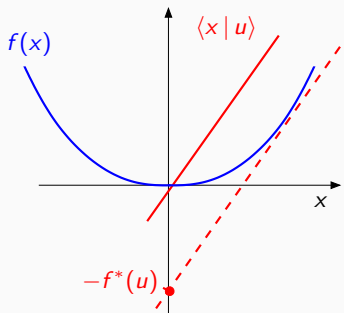
$$(\forall u \in \mathcal{H}) \quad f^*(u) = \sup_{x \in \mathcal{H}} (\langle x | u \rangle - f(x)) .$$

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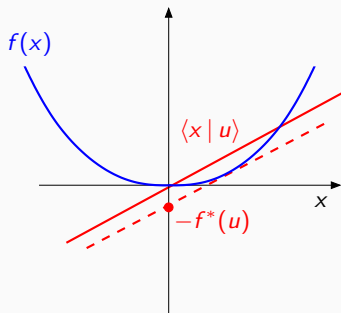


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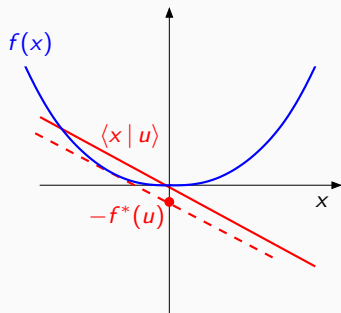


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Examples :

- $f = \frac{1}{2} \|\cdot\|^2 \Rightarrow f^* = \frac{1}{2} \|\cdot\|^2$

Proof : For every $(x, u) \in \mathcal{H}^2$, $\langle x | u \rangle - \frac{1}{2} \|x\|^2 = \frac{1}{2} \|u\|^2 - \frac{1}{2} \|u - x\|^2$
is maximum at $x = u$.

Consequently, $f^*(u) = \frac{1}{2} \|u\|^2$.

Conjugate: definition

Let \mathcal{H} be a Hilbert space and $f: \mathcal{H} \rightarrow]-\infty, +\infty]$.

The **conjugate** of f is the function $f^*: \mathcal{H} \rightarrow [-\infty, +\infty]$ such that

$$(\forall u \in \mathcal{H}) \quad f^*(u) = \sup_{x \in \text{dom } f} (\langle x | u \rangle - f(x))$$

Examples :

- $f = \frac{1}{2} \|\cdot\|^2 \Rightarrow f^* = \frac{1}{2} \|\cdot\|^2$.
- $(\forall x \in \mathbb{R}^N) f(x) = \frac{1}{q} \|x\|_q^q$ with $q \in]1, +\infty[$
 $\Rightarrow (\forall u \in \mathbb{R}^N) f^*(u) = \frac{1}{q^*} \|u\|_{q^*}^{q^*}$ with $\frac{1}{q} + \frac{1}{q^*} = 1$

Conjugate: definition

Let \mathcal{H} be a Hilbert space and $f: \mathcal{H} \rightarrow]-\infty, +\infty]$.

The **conjugate** of f is the function $f^*: \mathcal{H} \rightarrow [-\infty, +\infty]$ such that

$$(\forall u \in \mathcal{H}) \quad f^*(u) = \sup_{x \in \text{dom } f} (\langle x | u \rangle - f(x))$$

- If f is even, then f^* is even.

Conjugate: definition

Let \mathcal{H} be a Hilbert space and $f: \mathcal{H} \rightarrow]-\infty, +\infty]$.

The **conjugate** of f is the function $f^*: \mathcal{H} \rightarrow [-\infty, +\infty]$ such that

$$(\forall u \in \mathcal{H}) \quad f^*(u) = \sup_{x \in \text{dom } f} (\langle x | u \rangle - f(x))$$

- If f is even, then f^* is even.
- For every $\alpha \in]0, +\infty[$, $(\alpha f)^* = \alpha f^*(\cdot/\alpha)$.
- For every $(y, v) \in \mathcal{H}^2$ et $\alpha \in \mathbb{R}$,
 $(f(\cdot - y) + \langle \cdot | v \rangle + \alpha)^* = f^*(\cdot - v) + \langle y | \cdot - v \rangle - \alpha$.
- Let \mathcal{G} be a Hilbert space and $L \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ be an isomorphism.
 $(f \circ L)^* = f^* \circ (L^{-1})^*$.
- f^* is l.s.c. and convex.

Conjugate: definition

Let \mathcal{H} be a Hilbert space and $f: \mathcal{H} \rightarrow]-\infty, +\infty]$.

The **conjugate** of f is the function $f^*: \mathcal{H} \rightarrow [-\infty, +\infty]$ such that

$$(\forall u \in \mathcal{H}) \quad f^*(u) = \sup_{x \in \text{dom } f} (\langle x | u \rangle - f(x))$$

Moreau-Fenchel theorem

Let \mathcal{H} be a Hilbert space and $f: \mathcal{H} \rightarrow]-\infty, +\infty]$ be a proper function.

$$f \text{ is l.s.c. and convex} \Leftrightarrow f^{**} = f.$$

Conjugate: definition

Let \mathcal{H} be a Hilbert space and $f: \mathcal{H} \rightarrow]-\infty, +\infty]$.

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Moreau-Fenchel theorem

Let \mathcal{H} be a Hilbert space and $f: \mathcal{H} \rightarrow]-\infty, +\infty]$ be a proper function.

$$f \text{ is l.s.c. and convex} \Leftrightarrow f^{**} = f.$$

- Consequence: If $f \in \Gamma_0(\mathbb{R})$, then f^* is proper, hence $f^* \in \Gamma_0(\mathbb{R})$.

Conjugate: properties

Fenchel-Young inequality: If f is proper, then

$$1. (\forall (x, u) \in \mathcal{H}^2) \quad f(x) + f^*(u) \geq \langle x | u \rangle$$

$$2. (\forall (x, u) \in \mathcal{H}^2) \quad u \in \partial f(x) \Leftrightarrow f(x) + f^*(u) = \langle x | u \rangle.$$

If $f \in \Gamma_0(\mathcal{H})$, then

$$(\forall (x, u) \in \mathcal{H}^2) \quad u \in \partial f(x) \Leftrightarrow x \in \partial f^*(u).$$

Proximity operator: Moreau decomposition

Moreau decomposition formula

Let \mathcal{H} be a Hilbert space, $f \in \Gamma_0(\mathcal{H})$ and $\gamma \in]0, +\infty[$.

$$(\forall x \in \mathcal{H}) \quad \text{prox}_{\gamma f^*} x = x - \gamma \text{prox}_{\gamma^{-1} f}(\gamma^{-1} x).$$

Proof:

$$\begin{aligned} p = \text{prox}_{\gamma f^*} x &\Leftrightarrow x - p \in \gamma \partial f^*(p) \\ &\Leftrightarrow p \in \partial f\left(\frac{x - p}{\gamma}\right) \\ &\Leftrightarrow \frac{x}{\gamma} - \frac{x - p}{\gamma} \in \frac{1}{\gamma} \partial f\left(\frac{x - p}{\gamma}\right) \\ &\Leftrightarrow \frac{x - p}{\gamma} = \text{prox}_{\gamma^{-1} f}(\gamma^{-1} x) \\ &\Leftrightarrow p = x - \gamma \text{prox}_{\gamma^{-1} f}(\gamma^{-1} x). \end{aligned}$$

Proximity operator: Moreau decomposition

Moreau decomposition formula

Let \mathcal{H} be a Hilbert space, $f \in \Gamma_0(\mathcal{H})$ and $\gamma \in]0, +\infty[$.

$$(\forall x \in \mathcal{H}) \quad \text{prox}_{\gamma f^*} x = x - \gamma \text{prox}_{\gamma^{-1} f}(\gamma^{-1} x).$$

Example: If $\mathcal{H} = \mathbb{R}^N$, $f = \frac{1}{q} \|\cdot\|_q^q$ with $q \in]1, +\infty[$, then $f^* = \frac{1}{q^*} \|\cdot\|_{q^*}^{q^*}$ with $1/q + 1/q^* = 1$, and

$$(\forall x \in \mathbb{R}^N) \quad \text{prox}_{\frac{\gamma}{q^*} \|\cdot\|_{q^*}^{q^*}} x = x - \gamma \text{prox}_{\frac{1}{\gamma q} \|\cdot\|_q^q}(\gamma^{-1} x).$$

Conjugate: properties

Let $(\mathcal{H})_{i \in I}$ where $I \subset \mathbb{N}$ be Hilbert spaces and let $\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i$.
For every $i \in I$, let $f_i: \mathcal{H}_i \rightarrow]-\infty, +\infty]$. Let

$$f: \mathcal{H} \rightarrow]-\infty, +\infty] : x = (x_i)_{i \in I} \mapsto \sum_{i \in I} f_i(x_i)$$

Then,

$$(\forall u = (u_i)_{i \in I} \in \mathcal{H}) \quad f^*(u) = \sum_{i \in I} f_i^*(u_i) .$$

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Then,

$$(\forall u = (u_i)_{i \in I} \in \mathcal{H}) \quad f^*(u) = \sum_{i \in I} f_i^*(u_i).$$

Proof: Let $u = (u_i)_{i \in I} \in \mathcal{H}$. We have

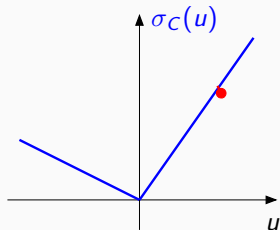
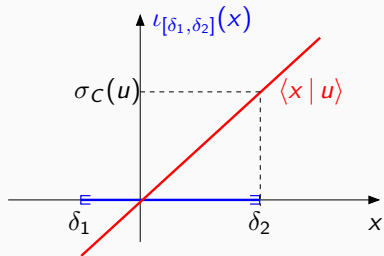
$$\begin{aligned} f^*(u) &= \sup_{x \in \mathcal{H}} \langle x \mid u \rangle - f(x) \\ &= \sup_{x=(x_i)_{i \in I} \in \mathcal{H}} \sum_{i \in I} \langle x_i \mid u_i \rangle - f_i(x_i) \\ &= \sum_{i \in I} \sup_{x_i \in \mathcal{H}_i} \langle x_i \mid u_i \rangle - f_i(x_i) \\ &= \sum_{i \in I} f_i^*(u_i). \end{aligned}$$

Conjugate: example

Let \mathcal{H} be a Hilbert space and $C \subset \mathcal{H}$.

σ_C is the **support function** of C if

$$\begin{aligned}(\forall u \in \mathcal{H}) \quad \sigma_C(u) &= \sup_{x \in C} \langle x | u \rangle \\ &= \iota_C^*(u).\end{aligned}$$

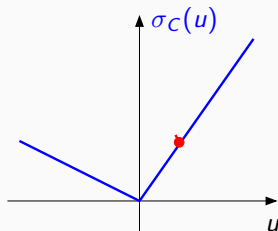
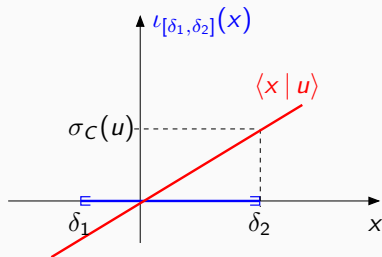


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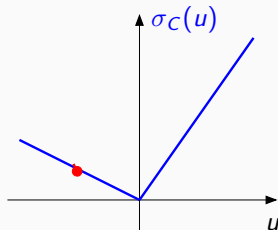
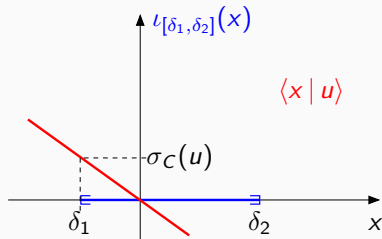


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Proximity operator: support function

Support function :

Let \mathcal{H} be a Hilbert space and $C \subset \mathcal{H}$ be nonempty closed convex.

$$(\forall x \in \mathcal{H}) \quad \text{prox}_{\sigma_C} = \text{Id} - P_C.$$

Conjugate: example

- Let $f: \mathbb{R} \rightarrow]-\infty, +\infty] : x \mapsto \begin{cases} \delta_1 x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \delta_2 x & \text{if } x > 0 \end{cases}$

with $-\infty \leq \delta_1 < \delta_2 \leq +\infty$.

Then, $f = \sigma_C$ where C is the closed real interval such that $\inf C = \delta_1$ et $\sup C = \delta_2$.

Conjugate: example

- Let $f: \mathbb{R} \rightarrow]-\infty, +\infty]: x \mapsto \begin{cases} \delta_1 x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \delta_2 x & \text{if } x > 0 \end{cases}$

with $-\infty \leq \delta_1 < \delta_2 \leq +\infty$.

Then, $f = \sigma_C$ where C is the closed real interval such that $\inf C = \delta_1$ et $\sup C = \delta_2$.

- Let f be a ℓ^q norm of \mathbb{R}^N with $q \in [1, +\infty]$.

We have $f = \sigma_C$ where

$$C = \{y \in \mathbb{R}^N \mid \|y\|_{q^*} \leq 1\} \quad \text{with } \frac{1}{q} + \frac{1}{q^*} = 1.$$

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$$C = \{y \in \mathbb{R}^N \mid \|y\|_{q^*} \leq 1\} \quad \text{with } \frac{1}{q} + \frac{1}{q^*} = 1.$$

Particular case: ℓ^1 norm of \mathbb{R}^N : $C = [-1, 1]^N$.

Proximity operator: existence and uniqueness

Let $f \in \Gamma_0(\mathcal{H})$ and $\gamma \in]0, +\infty[$.

For every $x \in \mathcal{H}$, there exists a unique vector $p \in \mathcal{H}$ such that

$$f(p) + \frac{1}{2\gamma} \|p - x\|^2 = \inf_{y \in \mathcal{H}} f(y) + \frac{1}{2\gamma} \|y - x\|^2.$$

Proof: $f \in \Gamma_0(\mathcal{H}) \Leftrightarrow f^* \in \Gamma_0(\mathcal{H})$. Thus, there exists $u \in \mathcal{H}$ such that $f^*(u) \in \mathbb{R}$. According to Fenchel-Young inequality, we have

$$(\forall y \in \mathcal{H}) \quad f(y) \geq \langle u \mid y \rangle - f^*(u).$$

Then, $f(y) + (2\gamma)^{-1} \|y - x\|^2 \rightarrow +\infty$ when $\|y\| \rightarrow +\infty$.

Furthermore $(2\gamma)^{-1} \|\cdot - x\|^2$ being strictly convex, $f + (2\gamma)^{-1} \|\cdot - x\|^2$ is a strictly convex coercive function.

α -averaged operator: example

Let \mathcal{H} be a Hilbert space, $f \in \Gamma_0(\mathcal{H})$, $\nu \in]0, 2[$.

If f is differentiable with a ν -lipschitzian gradient then, $\text{Id} - \nabla f$ is a $\nu/2$ -averaged operator.

Remark : $\text{Id} - \nabla f$ denotes the gradient descent operator.

α -averaged operator: example

Proof : 1) Descent lemma

For every $(x, y) \in \mathcal{H}^2$ and $t \in \mathbb{R}$, let $\varphi(t) = f(x + t(y - x))$.

φ is differentiable and $\varphi'(t) = \langle y - x \mid \nabla f(x + t(y - x)) \rangle$. We have then

$$\varphi(1) - \varphi(0) = \int_0^1 \varphi'(t) dt$$

$$\Leftrightarrow f(y) - f(x) - \langle y - x \mid \nabla f(x) \rangle = \int_0^1 \langle y - x \mid \nabla f(x + t(y - x)) - \nabla f(x) \rangle dt.$$

But, according to the Cauchy-Schwarz inequality,

$$\begin{aligned} & \langle y - x \mid \nabla f(x + t(y - x)) - \nabla f(x) \rangle \\ & \leq \|y - x\| \|\nabla f(x + t(y - x)) - \nabla f(x)\| \leq tv \|y - x\|^2. \end{aligned}$$

This leads to

$$\boxed{(\forall (x, y) \in \mathcal{H}^2) \quad f(y) \leq f(x) + \langle y - x \mid \nabla f(x) \rangle + \frac{\nu}{2} \|y - x\|^2.}$$

α -averaged operator: example

Proof : 2) $\text{Id} - \nabla f$ is α -averaged

From the descent lemma, for every $(x, y, z) \in \mathcal{H}^3$,

$$\begin{aligned} f^*(\nabla f(y)) &\geq \langle z \mid \nabla f(y) \rangle - f(z) \\ &\geq \langle z \mid \nabla f(y) - \nabla f(x) \rangle + \langle x \mid \nabla f(x) \rangle - f(x) - \frac{\nu}{2} \|z - x\|^2. \end{aligned}$$

Moreover, according to the Fenchel-Young inequality,

$$\langle x \mid \nabla f(x) \rangle - f(x) = f^*(\nabla f(x)).$$

Thus,

$$f^*(\nabla f(y)) \geq \langle z \mid \nabla f(y) - \nabla f(x) \rangle + f^*(\nabla f(x)) - \frac{\nu}{2} \|z - x\|^2$$

α -averaged operator: example

Proof : 2) $\text{Id} - \nabla f$ is α -averaged

Thus,

$$\begin{aligned} f^*(\nabla f(y)) &\geq \langle z \mid \nabla f(y) - \nabla f(x) \rangle + f^*(\nabla f(x)) - \frac{\nu}{2} \|z - x\|^2 \\ &= f^*(\nabla f(x)) + \langle x \mid \nabla f(y) - \nabla f(x) \rangle + \langle z - x \mid \nabla f(y) - \nabla f(x) \rangle - \frac{\nu}{2} \|z - x\|^2. \end{aligned}$$

This yields

$$\begin{aligned} f^*(\nabla f(y)) &\geq f^*(\nabla f(x)) + \langle x \mid \nabla f(y) - \nabla f(x) \rangle \\ &\quad + (\nu \| \cdot \|^2 / 2)^*(\nabla f(y) - \nabla f(x)) \\ &\geq f^*(\nabla f(x)) + \langle x \mid \nabla f(y) - \nabla f(x) \rangle + \frac{1}{2\nu} \|\nabla f(y) - \nabla f(x)\|^2. \end{aligned}$$

Consequently,

$$f^*(\nabla f(y)) \geq f^*(\nabla f(x)) + \langle x \mid \nabla f(y) - \nabla f(x) \rangle + \frac{1}{2\nu} \|\nabla f(y) - \nabla f(x)\|^2.$$

α -averaged operator: example

Proof : 2) $\text{Id} - \nabla f$ is α -averaged

For every $(x, y) \in \mathcal{H}^2$,

$$f^*(\nabla f(y)) \geq f^*(\nabla f(x)) + \langle x \mid \nabla f(y) - \nabla f(x) \rangle + \frac{1}{2\nu} \|\nabla f(y) - \nabla f(x)\|^2$$

and symmetrically

$$f^*(\nabla f(x)) \geq f^*(\nabla f(y)) + \langle y \mid \nabla f(x) - \nabla f(y) \rangle + \frac{1}{2\nu} \|\nabla f(x) - \nabla f(y)\|^2.$$

By summing

$$-\langle y - x \mid \nabla f(y) - \nabla f(x) \rangle + \frac{1}{\nu} \|\nabla f(x) - \nabla f(y)\|^2 \leq 0.$$

It results that

$$\|(\text{Id} - \nabla f)x - (\text{Id} - \nabla f)y\|^2 + \frac{1 - \nu/2}{\nu/2} \|\nabla f(x) - \nabla f(y)\|^2 \leq \|x - y\|^2.$$

Optimization

Part VII: Duality

Nelly Pustelnik

CNRS, Laboratoire de Physique de l'ENS de Lyon, France



Fenchel-Rockafellar duality

Primal problem

Let \mathcal{H} and \mathcal{G} be two real Hilbert spaces.

Let $f: \mathcal{H} \rightarrow]-\infty, +\infty]$, $g: \mathcal{G} \rightarrow]-\infty, +\infty]$. Let $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$.

We want to

$$\underset{x \in \mathcal{H}}{\text{minimize}} \quad f(x) + g(Lx).$$

Dual problem

Let \mathcal{H} and \mathcal{G} be two real Hilbert spaces.

Let $f: \mathcal{H} \rightarrow]-\infty, +\infty]$, $g: \mathcal{G} \rightarrow]-\infty, +\infty]$. Let $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$.

We want to

$$\underset{v \in \mathcal{G}}{\text{minimize}} \quad f^*(-L^*v) + g^*(v).$$

Fenchel-Rockafellar duality

Weak duality

Let \mathcal{H} and \mathcal{G} be two real Hilbert spaces.

Let f be a proper function from \mathcal{H} to $] -\infty, +\infty]$, g be a proper function from \mathcal{G} to $] -\infty, +\infty]$, and $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$. Let

$$\mu = \inf_{x \in \mathcal{H}} f(x) + g(Lx) \quad \text{and} \quad \mu^* = \inf_{v \in \mathcal{G}} f^*(-L^*v) + g^*(v).$$

We have $\mu \geq -\mu^*$. If $\mu \in \mathbb{R}$, $\mu + \mu^*$ is called the **duality gap**.

Fenchel-Rockafellar duality

Weak duality

Let \mathcal{H} and \mathcal{G} be two real Hilbert spaces.

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$$\mu = \inf_{x \in \mathcal{H}} f(x) + g(Lx) \quad \text{and} \quad \mu^* = \inf_{v \in \mathcal{G}} f^*(-L^*v) + g^*(v).$$

We have $\mu \geq -\mu^*$. If $\mu \in \mathbb{R}$, $\mu + \mu^*$ is called the **duality gap**.

Proof: According to Fenchel-Young inequality, for every $x \in \mathcal{H}$ and $v \in \mathcal{G}$,

$$f(x) + g(Lx) + f^*(-L^*v) + g^*(v) \geq \langle x \mid -L^*v \rangle + \langle Lx \mid v \rangle = 0.$$

Fenchel-Rockafellar duality

Strong duality

Let \mathcal{H} and \mathcal{G} be two real Hilbert spaces.

Let $f \in \Gamma_0(\mathcal{H})$, $g \in \Gamma_0(\mathcal{G})$, and $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$.

If $\text{int}(\text{dom } g) \cap L(\text{dom } f) \neq \emptyset$ or $\text{dom } g \cap \text{int}(L(\text{dom } f)) \neq \emptyset$, then

$$\mu = \inf_{x \in \mathcal{H}} f(x) + g(Lx) = - \min_{v \in \mathcal{G}} f^*(-L^*v) + g^*(v) = -\mu^* .$$

Fenchel-Rockafellar duality

Duality theorem (1)

Let \mathcal{H} and \mathcal{G} be two real Hilbert spaces.

Let $f \in \Gamma_0(\mathcal{H})$, $g \in \Gamma_0(\mathcal{G})$, and $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$.

$$\text{zer}(\partial f + L\partial gL^*) \neq \emptyset \quad \Leftrightarrow \quad \text{zer}((-L)\partial f^*(-L^*) + \partial g^*) \neq \emptyset.$$

Fenchel-Rockafellar duality

Duality theorem (1)

Let \mathcal{H} and \mathcal{G} be two real Hilbert spaces.

Let $f \in \Gamma_0(\mathcal{H})$, $g \in \Gamma_0(\mathcal{G})$, and $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$.

$$\text{zer}(\partial f + L\partial gL^*) \neq \emptyset \quad \Leftrightarrow \quad \text{zer}((-L)\partial f^*(-L^*) + \partial g^*) \neq \emptyset.$$

Proof:

$$\begin{aligned} (\exists x \in \mathcal{H}) \quad 0 \in \partial f(x) + L^* \partial g(Lx) &\Leftrightarrow (\exists x \in \mathcal{H})(\exists v \in \mathcal{G}) \quad \begin{cases} -L^*v \in \partial f(x) \\ v \in \partial g(Lx) \end{cases} \\ &\Leftrightarrow (\exists x \in \mathcal{H})(\exists v \in \mathcal{G}) \quad \begin{cases} x \in \partial f^*(-L^*v) \\ Lx \in \partial g^*(v) \end{cases} \\ &\Leftrightarrow (\exists v \in \mathcal{G}) \quad 0 \in -L\partial f^*(-L^*v) + \partial g^*(v). \end{aligned}$$

Fenchel-Rockafellar duality

Duality theorem (2)

Let \mathcal{H} and \mathcal{G} be two real Hilbert spaces.

Let $f \in \Gamma_0(\mathcal{H})$, $g \in \Gamma_0(\mathcal{G})$, and $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$.

- If there exists $\hat{x} \in \mathcal{H}$ such that $0 \in \partial f(\hat{x}) + L^* \partial g(L\hat{x})$, then \hat{x} is a solution to the primal problem. Moreover, there exists a solution \hat{v} to the dual problem such that $-L^* \hat{v} \in \partial f(\hat{x})$ and $L\hat{x} \in \partial g^*(\hat{v})$.
- If there exists $(\hat{x}, \hat{v}) \in \mathcal{H} \times \mathcal{G}$ such that $-L^* \hat{v} \in \partial f(\hat{x})$ and $L\hat{x} \in \partial g^*(\hat{v})$ then \hat{x} (resp. \hat{v}) is a solution to the primal (resp. dual) problem.

If $(\hat{x}, \hat{v}) \in \mathcal{H} \times \mathcal{G}$ is such that $-L^* \hat{v} \in \partial f(\hat{x})$ and $L\hat{x} \in \partial g^*(\hat{v})$, then (\hat{x}, \hat{v}) is called a **Kuhn-Tucker point**.

Augmented Lagrangian method

ADMM algorithm (*Alternating-direction method of multipliers*)

⇒ **Lagrangian interpretation**

$$\underset{x \in \mathcal{H}}{\text{minimize}} \quad f(x) + g(Lx) \quad \Leftrightarrow \quad \underset{\substack{x \in \mathcal{H}, y \in \mathcal{G} \\ Lx=y}}{\text{minimize}} \quad f(x) + g(y)$$

Lagrange function:

$$(\forall (x, y, v) \in \mathcal{H} \times \mathcal{G}^2) \quad \mathcal{L}(x, y, z) = f(x) + g(y) + \langle v \mid Lx - y \rangle$$

where $v \in \mathcal{G}$ denotes the Lagrange multiplier.

Alternating-direction method of multipliers

Idea: iterations for finding a saddle point $(\hat{x}, \hat{y}, \hat{z})$:

$$(\forall n \in \mathbb{N}) \quad \begin{cases} x_n \in \operatorname{Argmin} \mathcal{L}(\cdot, y_n, z_n) \\ y_{n+1} \in \operatorname{Argmin} \mathcal{L}(x_n, \cdot, z_n) \\ v_{n+1} \text{ such that } \mathcal{L}(x_n, y_{n+1}, v_{n+1}) \geq \mathcal{L}(x_n, y_{n+1}, v_n). \end{cases}$$

But the convergence is not guaranteed in general !

Alternating-direction method of multipliers

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But the convergence is not guaranteed in general !

Solution: introduce an **Augmented Lagrange function**.

Let $\gamma \in]0, +\infty[$, we define

$$(\forall (x, y, z) \in \mathcal{H} \times \mathcal{G}^2) \quad \tilde{\mathcal{L}}(x, y, z) = f(x) + g(y) + \gamma \langle z \mid Lx - y \rangle + \frac{\gamma}{2} \|Lx - y\|^2$$

The Lagrange multiplier is $v = \gamma z$.

Alternating-direction method of multipliers

Algorithm for finding a saddle point:

$$(\forall n \in \mathbb{N}) \quad \begin{cases} x_n = \operatorname{argmin}_{x \in \mathcal{H}} \tilde{\mathcal{L}}(x, y_n, z_n) \\ y_{n+1} = \operatorname{argmin}_{y \in \mathcal{G}} \tilde{\mathcal{L}}(x_n, y, z_n) \\ z_{n+1} \text{ such that } \tilde{\mathcal{L}}(x_n, y_{n+1}, z_{n+1}) \geq \tilde{\mathcal{L}}(x_n, y_{n+1}, z_n). \end{cases}$$

By performing a gradient ascent on the Lagrange multiplier,

$$\begin{aligned} (\forall n \in \mathbb{N}) \quad & \begin{cases} x_n = \operatorname{argmin}_{x \in \mathcal{H}} f(x) + \gamma \langle z_n | Lx - y_n \rangle + \frac{\gamma}{2} \|Lx - y_n\|^2 \\ y_{n+1} = \operatorname{argmin}_{y \in \mathcal{G}} g(y) + \gamma \langle z_n | Lx_n - y \rangle + \frac{\gamma}{2} \|Lx_n - y\|^2 \\ z_{n+1} = z_n + \frac{1}{\gamma} \nabla_z \tilde{\mathcal{L}}(x_n, y_{n+1}, z_n) \end{cases} \\ \Leftrightarrow (\forall n \in \mathbb{N}) \quad & \begin{cases} x_n = \operatorname{argmin}_{x \in \mathcal{H}} \frac{1}{2} \|Lx - y_n + z_n\|^2 + \frac{1}{\gamma} f(x) \\ y_{n+1} = \operatorname{prox}_{\frac{g}{\gamma}}(z_n + Lx_n) \\ z_{n+1} = z_n + Lx_n - y_{n+1}. \end{cases} \end{aligned}$$

Augmented Lagrange method

ADMM algorithm (*Alternating-direction method of multipliers*)

Let \mathcal{H} and \mathcal{G} be two Hilbert spaces. Let $f \in \Gamma_0(\mathcal{H})$ et $g \in \Gamma_0(\mathcal{G})$.

Let $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ such that L^*L is an isomorphism and let $\gamma \in]0, +\infty[$.

$$(\forall n \in \mathbb{N}) \quad \begin{cases} x_n = \operatorname{argmin}_{x \in \mathcal{H}} \frac{1}{2} \|Lx - y_n + z_n\|^2 + \frac{1}{\gamma} f(x) \\ s_n = Lx_n \\ y_{n+1} = \operatorname{prox}_{\frac{g}{\gamma}}(z_n + s_n) \\ z_{n+1} = z_n + s_n - y_{n+1}. \end{cases}$$

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Let \mathcal{H} and \mathcal{G} be two Hilbert spaces. Let $f \in \Gamma_0(\mathcal{H})$ et $g \in \Gamma_0(\mathcal{G})$.

Let $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ such that L^*L is an isomorphism and let $\gamma \in]0, +\infty[$.

We assume that $\text{int}(\text{dom } g) \cap L(\text{dom } f) \neq \emptyset$ or $\text{dom } g \cap \text{int}(L(\text{dom } f)) \neq \emptyset$ and that $\text{Argmin}(f + g \circ L) \neq \emptyset$. Let

$$(\forall n \in \mathbb{N}) \quad \begin{cases} x_n = \underset{x \in \mathcal{H}}{\text{argmin}} \frac{1}{2} \|Lx - y_n + z_n\|^2 + \frac{1}{\gamma} f(x) \\ s_n = Lx_n \\ y_{n+1} = \text{prox}_{\frac{g}{\gamma}}(z_n + s_n) \\ z_{n+1} = z_n + s_n - y_{n+1}. \end{cases}$$

We have:

- $x_n \rightarrow \hat{x}$ where $\hat{x} \in \text{Argmin}(f + g \circ L)$
- $\gamma z_n \rightarrow \hat{v}$ where $\hat{v} \in \text{Argmin}(f^* \circ (-L^*) + g^*)$.

Augmented Lagrangian method

ADMM algorithm (*Alternating-direction method of multipliers*)
 \equiv **Douglas-Rachford for the dual problem**

Let \mathcal{H} and \mathcal{G} be two Hilbert spaces. Let $f \in \Gamma_0(\mathcal{H})$ and $g \in \Gamma_0(\mathcal{G})$.
Let $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ such that L^*L is an isomorphism and let $\gamma \in]0, +\infty[$.
The Douglas-Rachford iterations to minimize $f^* \circ (-L^*) + g^*$ are

$$(\forall n \in \mathbb{N}) \quad \begin{cases} v_n = \text{prox}_{\gamma g^*} u_n \\ w_n = \text{prox}_{\gamma f^* \circ (-L^*)}(2v_n - u_n) \\ u_{n+1} = u_n + w_n - v_n \end{cases}$$

Augmented Lagrangian method

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The Douglas-Rachford iterations to minimize $f^* \circ (-L^*) + g^*$ are

$$(\forall n \in \mathbb{N}) \quad \begin{cases} v_n = u_n - \gamma \text{prox}_{\gamma^{-1}g}(\gamma^{-1}u_n) \\ 2v_n - u_n - w_n \in \gamma \partial(f^* \circ (-L^*)) w_n \\ u_{n+1} = u_n + w_n - v_n \end{cases}$$

Augmented Lagrangian method

ADMM algorithm (*Alternating-direction method of multipliers*)

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Let \mathcal{H} and \mathcal{G} be two Hilbert spaces. Let $f \in \Gamma_0(\mathcal{H})$ and $g \in \Gamma_0(\mathcal{G})$.

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The Douglas-Rachford iterations to minimize $f^* \circ (-L^*) + g^*$ are

$$(\forall n \in \mathbb{N}) \quad \begin{cases} v_n = u_n - \gamma \operatorname{prox}_{\gamma^{-1}g}(\gamma^{-1}u_n) \\ 2v_n - u_n - w_n \in -\gamma L \circ \partial f^* \circ (-L^*)w_n \\ u_{n+1} = u_n + w_n - v_n \end{cases}$$

Augmented Lagrangian method

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The Douglas-Rachford iterations to minimize $f^* \circ (-L^*) + g^*$ are

$$(\forall n \in \mathbb{N}) \quad \begin{cases} v_n = u_n - \gamma \operatorname{prox}_{\gamma^{-1}g}(\gamma^{-1}u_n) \\ 2v_n - u_n - w_n = -\gamma Lx_n \\ x_n \in \partial f^*(-L^*w_n) \\ u_{n+1} = u_n + w_n - v_n \end{cases}$$

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The Douglas-Rachford iterations to minimize $f^* \circ (-L^*) + g^*$ are

$$(\forall n \in \mathbb{N}) \quad \begin{cases} v_n = u_n - \gamma \operatorname{prox}_{\gamma^{-1}g}(\gamma^{-1}u_n) \\ L^*(u_n - 2v_n - \gamma Lx_n) \in \partial f(x_n) \\ u_{n+1} = u_n + w_n - v_n \end{cases}$$

using $y_n = \gamma^{-1}(u_n - v_n)$ and $z_n = \gamma^{-1}v_n$

Augmented Lagrangian method

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The Douglas-Rachford iterations to minimize $f^* \circ (-L^*) + g^*$ are

$$(\forall n \in \mathbb{N}) \quad \begin{cases} y_n = \text{prox}_{\gamma^{-1}g}(\gamma^{-1}u_n) \\ x_n = \underset{x \in \mathcal{H}}{\text{argmin}} \frac{1}{2} \|Lx - y_n + z_n\|^2 + \frac{1}{\gamma} f(x) \\ u_{n+1} = \gamma z_n + \gamma Lx_n \end{cases}$$

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Primal-dual optimization algorithm : $\min_x f(x) + h(x) + g(Lx)$

Condat-Vũ algorithm:

$$(\forall n \in \mathbb{N}) \quad \begin{cases} p_n = \text{prox}_{\tau f}(x_n - \tau(\nabla h(x_n) + L^* v_n)) \\ q_n = \text{prox}_{\sigma g^*}(v_n + \sigma(L(2p_n - x_n))) \\ (x_{n+1}, v_{n+1}) = (x_n, v_n) + \lambda_n((p_n, q_n) - (x_n, v_n)). \end{cases}$$

- Remark:

- * $h \in C_{\zeta}^{1,1}(\mathcal{H})$

- * No operator inversion.

- * Allow the use of proximable or/and differentiable functions.

- * Convergence when $\frac{1}{\tau} - \sigma\|L\|^2 > \frac{\zeta}{2}$.

- * $x_n \rightarrow \hat{x} \in \text{Argmin}(f + h + g \circ L)$

Primal-dual optimization algorithm : $\min_x f(x) + h(x) + g(Lx)$

Condat-Vũ algorithm: \Rightarrow **Chambolle-Pock algorithm**

$$(\forall n \in \mathbb{N}) \quad \begin{cases} p_n = \text{prox}_{\tau f}(x_n - \tau(\nabla h(x_n) + L^* v_n)) \\ q_n = \text{prox}_{\sigma g^*}(v_n + \sigma(L(2p_n - x_n))) \\ (x_{n+1}, v_{n+1}) = (x_n, v_n) + \lambda_n((p_n, q_n) - (x_n, v_n)). \end{cases}$$

- Remark:

* When $h = 0$, $\lambda_n \equiv 1$ and $\sigma\tau\|L\|^2 < 1$, this yields the Chambolle-Pock algorithm.

Primal-dual optimization algorithm : $\min_x f(x) + h(x) + g(Lx)$

Condat-Vũ algorithm: \Rightarrow **Chambolle-Pock algorithm**

$$(\forall n \in \mathbb{N}) \quad \begin{cases} x_{n+1} = \text{prox}_{\tau f}(x_n - \tau L^* v_n) \\ y_n = 2x_{n+1} - x_n \\ v_{n+1} = \text{prox}_{\sigma g^*}(v_n + \sigma L y_n). \end{cases}$$

- Remark:

* When $h = 0$, $\lambda_n \equiv 1$ and $\sigma\tau\|L\|^2 < 1$, this yields the Chambolle-Pock algorithm.

Primal-dual optimization algorithm : $\min_x f(x) + h(x) + g(Lx)$

Condat-Vũ algorithm: \Rightarrow Forward-backward algorithm

$$(\forall n \in \mathbb{N}) \quad \begin{cases} p_n = \text{prox}_{\tau f}(x_n - \tau(\nabla h(x_n) + L^* v_n)) \\ q_n = \text{prox}_{\sigma g^*}(v_n + \sigma(L(2p_n - x_n))) \\ (x_{n+1}, v_{n+1}) = (x_n, v_n) + \lambda_n((p_n, q_n) - (x_n, v_n)). \end{cases}$$

- Remark:

- * When $h = 0$, $\lambda_n \equiv 1$ and $\sigma\tau\|L\|^2 < 1$, this yields the Chambolle-Pock algorithm.

- * When $g = 0$ and $L = 0$, this yields the forward-backward algorithm.

Primal-dual optimization algorithm : $\min_x f(x) + h(x) + g(Lx)$

Condat-Vũ algorithm: \Rightarrow Forward-backward algorithm

$$(\forall n \in \mathbb{N}) \quad \begin{cases} p_n = \text{prox}_{\tau f}(x_n - \tau \nabla h(x_n)) \\ x_{n+1} = x_n + \lambda_n(p_n - x_n). \end{cases}$$

- Remark:

- * When $h = 0$, $\lambda_n \equiv 1$ and $\sigma\tau\|L\|^2 < 1$, this yields the Chambolle-Pock algorithm.

- * When $g = 0$ and $L = 0$, this yields the forward-backward algorithm.

Primal-dual optimization algorithm : $\min_x f(x) + h(x) + g(Lx)$

Condat-Vũ algorithm: \Rightarrow Douglas-Rachford algorithm

$$(\forall n \in \mathbb{N}) \quad \begin{cases} p_n = \text{prox}_{\tau f}(x_n - \tau(\nabla h(x_n) + L^* v_n)) \\ q_n = \text{prox}_{\sigma g^*}(v_n + \sigma(L(2p_n - x_n))) \\ (x_{n+1}, v_{n+1}) = (x_n, v_n) + \lambda_n((p_n, q_n) - (x_n, v_n)). \end{cases}$$

- Remark:

- * When $h = 0$, $\lambda_n \equiv 1$ and $\sigma\tau\|L\|^2 < 1$, this yields the Chambolle-Pock algorithm.

- * When $g = 0$ and $L = 0$, this yields the forward-backward algorithm.

- * In the limit case when $h = 0$, $\lambda_n \equiv 1$, $L = \text{Id}$ and $\sigma = 1/\tau$, this yields the Douglas-Rachford algorithm.

Primal-dual optimization algorithm : $\min_x f(x) + h(x) + g(Lx)$

Condat-Vũ algorithm: \Rightarrow Douglas-Rachford algorithm

$$(\forall n \in \mathbb{N}) \quad \begin{cases} x_{n+1} = \text{prox}_{\tau f}(x_n - \tau v_n) \\ s_n = \text{prox}_{\tau g}(2x_{n+1} - (x_n - \tau v_n)) \\ x_{n+1} - \tau v_{n+1} = (x_n - \tau v_n) + s_n - x_{n+1} \end{cases}$$

- Remark:

- * When $h = 0$, $\lambda_n \equiv 1$ and $\sigma\tau\|L\|^2 < 1$, this yields the Chambolle-Pock algorithm.

- * When $g = 0$ and $L = 0$, this yields the forward-backward algorithm.

- * In the limit case when $h = 0$, $\lambda_n \equiv 1$, $L = \text{Id}$ and $\sigma = 1/\tau$, this yields the Douglas-Rachford algorithm.

Optimization algorithms

Forward-Backward	$f_1 + f_2$	f_1 grad. Lipschitz prox_{f_2}	[Combettes,Wajs,2005]
ISTA	$f_1 + f_2$	f_1 grad. Lipschitz $f_2 = \lambda \ \cdot \ _1$	[Daubechies et al, 2003]
Douglas-Rachford	$f_1 + f_2$	prox_{f_1} prox_{f_2}	[Combettes,Pesquet, 2007]
PPXA	$\sum_i f_i$	prox_{f_i}	[Combettes,Pesquet, 2008]
ADMM	$\sum_i f_i \circ L_i$	prox_{f_i} $(\sum_{i=1}^m L_i^* L_i)^{-1}$	[Eckstein, Yao, 2015]
Chambolle-Pock	$f_1 + f_2 \circ L_2$	prox_{f_1} prox_{f_2}	[Chambolle, Pock, 2011]
Condat-Vũ	$f_1 + f_2 \circ L_2 + f_3$	prox_{f_1} prox_{f_2} f_3 grad. Lipschitz	[Condat, 2013][Vũ, 2013]