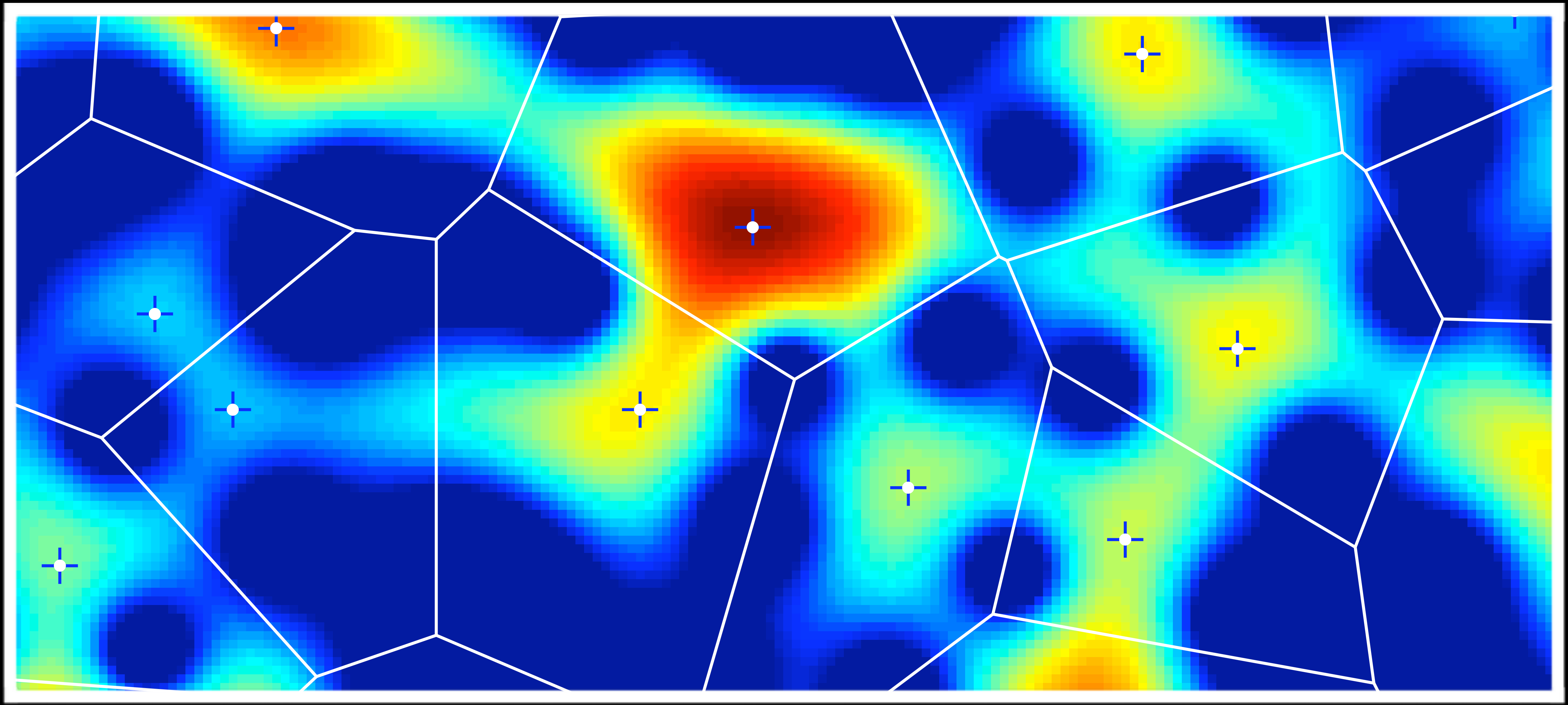


Time-frequency filtering based on spectrogram zeros

Patrick Flandrin

Lab PHYS
ENS de LYON

Spectrogram extrema?

- Built-in redundancy in any spectrogram (Heisenberg uncertainty)
  [Daubechies, '92]

- Full characterization by zeros in the case of Gaussian windows (Weierstrass-Hadamard factorization)
  [Korsch *et al.*, '97]

- Reassignment vector field driven by local maxima
  [Chassande-Mottin *et al.*, '97][F. *et al.*, '03]

Spectrogram

- Definition from Short-Time Fourier Transform (STFT)

$$S_x^{(h)}(t,\omega) = \left| F_x^{(h)}(t,\omega) \right|^2$$

$$F_x^{(h)}(t,\omega) = \int_{-\infty}^{+\infty} x(s)\overline{h(s-t)}\exp\left\{-i\omega\left(s-\frac{t}{2}\right)\right\}ds$$

- Reproducing kernel

$$F_x^{(h)}(t',\omega') = \iint_{-\infty}^{+\infty} K(t',\omega';t,\omega)F_x^{(h)}(t,\omega)dt\frac{d\omega}{2\pi}$$

$$K(t',\omega';t,\omega) = C\,F_h^{(h)}(t'-t,\omega'-\omega)$$

Uncertainty

- Reproducing kernel $F_h^{(h)}$ = ambiguity function A_h

- Volume inequalities
  [Lieb, '90]

$$\begin{cases} \|A_x\|_p \geq B_p \|x\|_2^2 & \text{for} \quad p < 2 \\ \|A_x\|_p \leq B_p \|x\|_2^2 & \text{for} \quad p > 2 \end{cases}$$

with $B_p = (2/p)^{1/p}$ and equality for Gaussians (Gabor logons)

- Support inequalities
  [Gröchenig, '99]

$$\iint_\Omega |A_x(\xi,\tau)|^2 \, d\tau \, \frac{d\xi}{2\pi} \geq (1-\epsilon)\|x\|_2^2 \Rightarrow |\Omega| \geq 1-\epsilon$$

wGn as a generic example

- Circular white Gaussian noise (wGn)

$$\mathbb{E}\left\{n(t)\overline{n(t')}\right\} = \gamma_0\,\delta(t-t') \quad ; \quad \mathbb{E}\{n(t)n(t')\} = 0$$

- Circular Gaussian window

$$g(t) = \pi^{-1/4}\exp\{-t^2/2\} \Rightarrow A_g(\xi,\tau) = \exp\left\{-\frac{1}{4}(\xi^2+\tau^2)\right\}$$

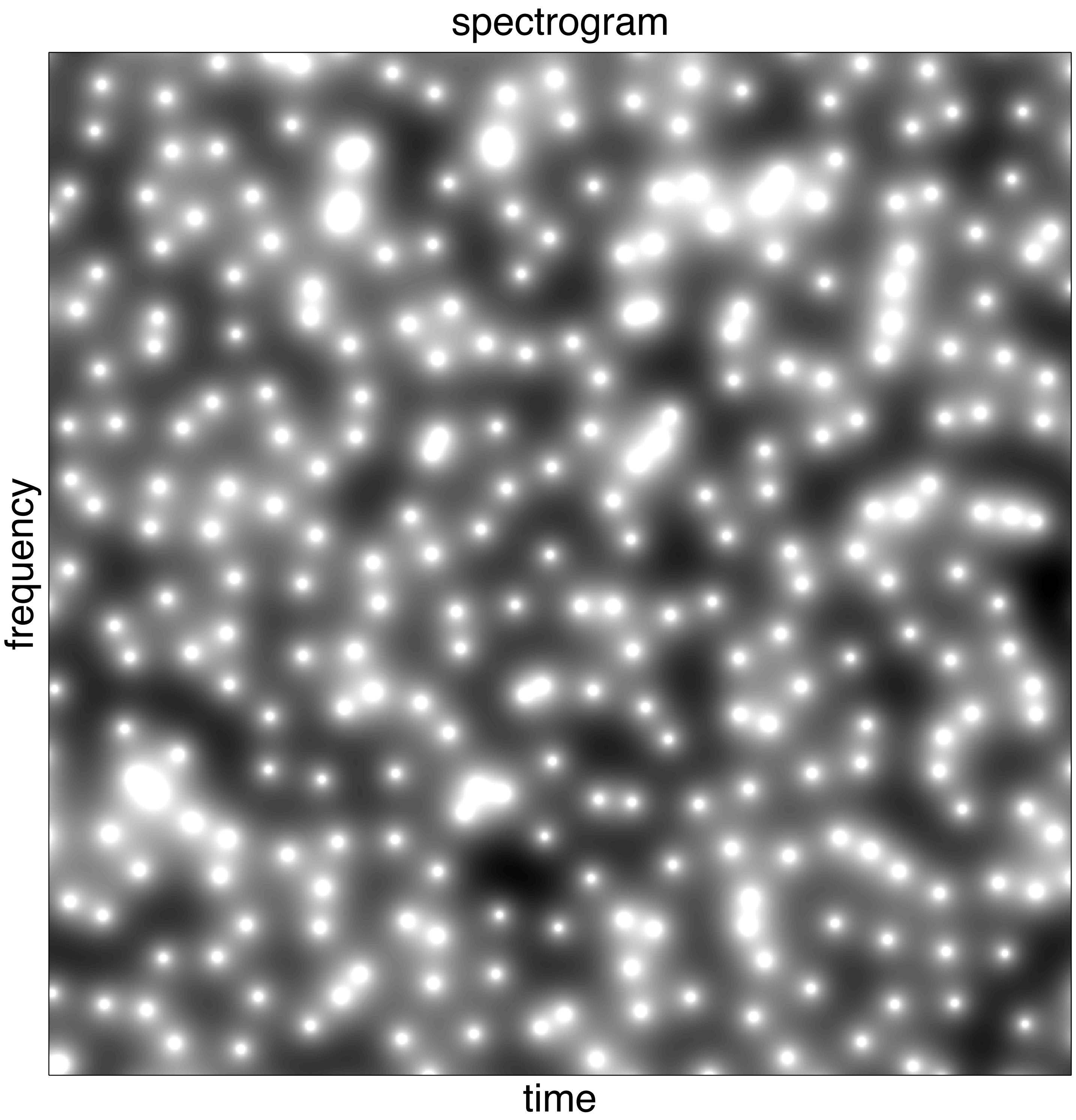
- Spectrogram = stationary 2D random field
  [F., '99]

$$\mathrm{cov}\left\{S_n^{(g)}(t,\omega), S_n^{(g)}(t',\omega')\right\} = \gamma_0^2\,\exp\left\{-\frac{1}{2}d^2((t,\omega),(t',\omega'))\right\}$$

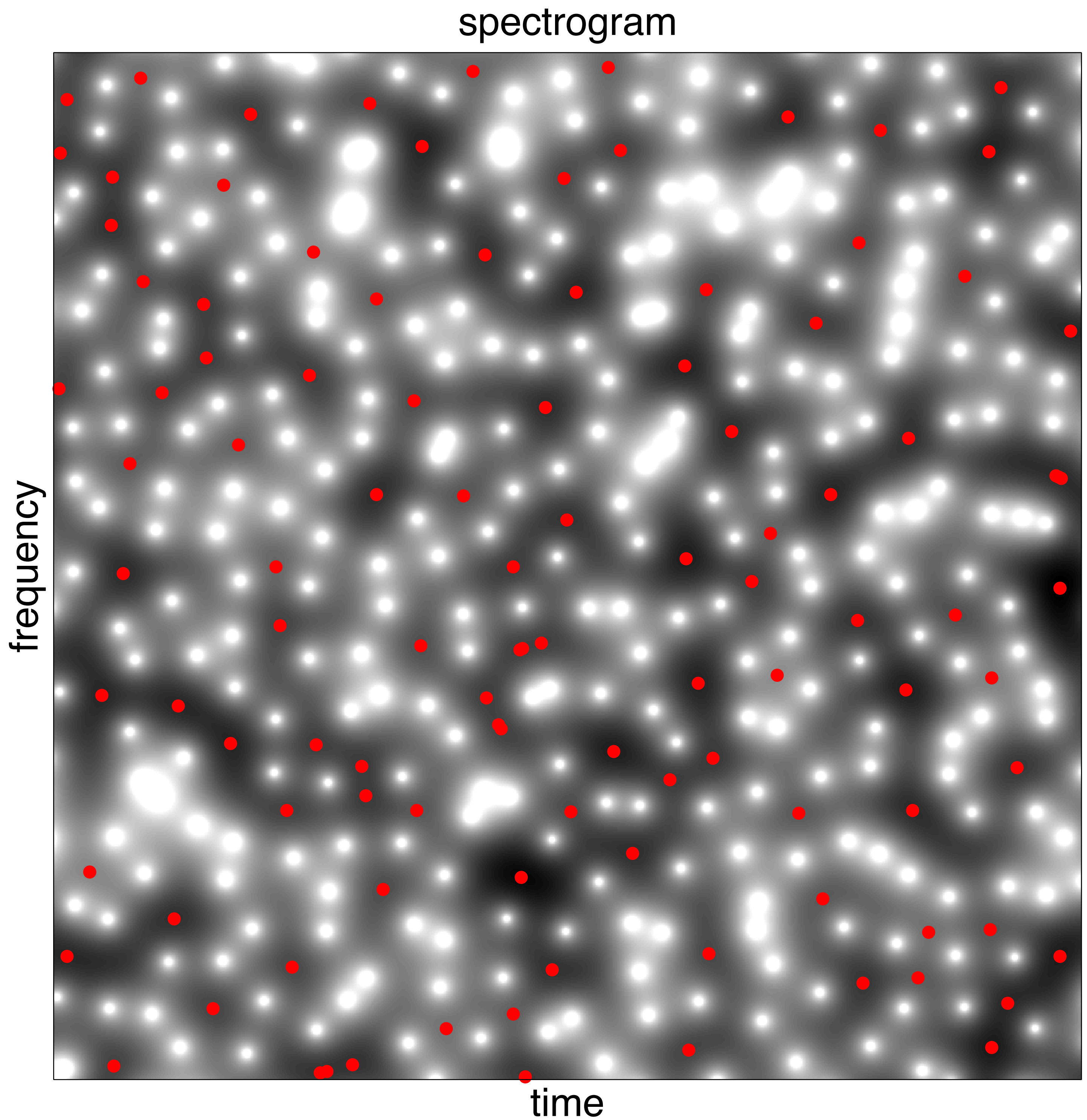
$$d((t,\omega),(t',\omega')) = \sqrt{(t-t')^2+(\omega-\omega')^2}$$

Homogeneous distribution of random energy « patches »

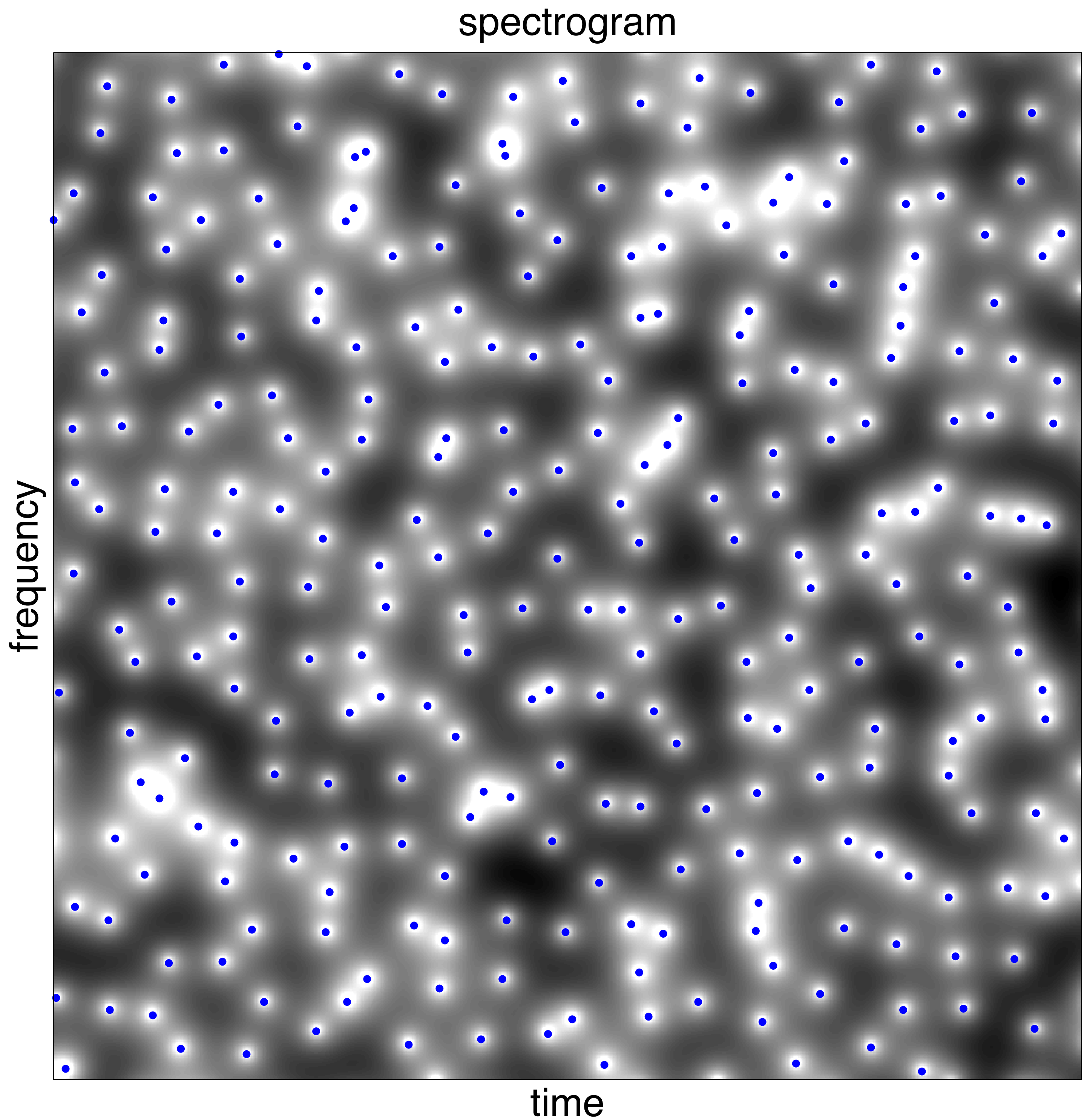
Spectrogram of wGn



Spectrogram of wGn + maxima



Spectrogram of wGn + zeros



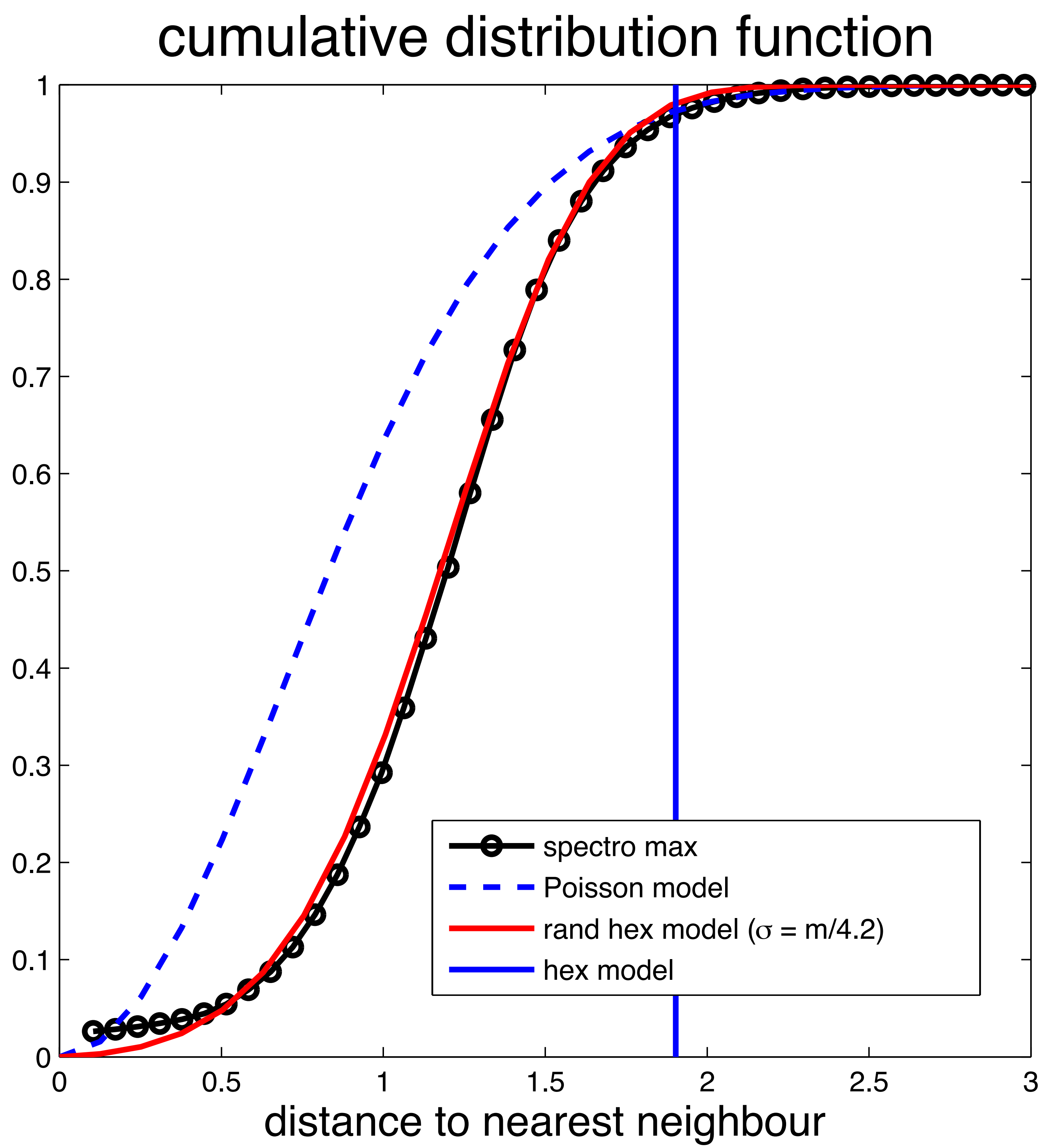
Local maxima as a 2D point process

- Gabor logons attached to an hexagonal lattice (maximum packing wrt uncertainty)
- Random phases and Gaussian random shifts (deviation from a Poisson distribution)
- Nearest-neighbour cumulative distribution function [F., FoCM '14]

$$\mathrm{Prob}(D \leq d) = 1 - \left(1 - \int_0^d F(r; m, 2\sigma^2) dr\right)^6$$

$$F(r; m, \sigma^2) = \frac{r}{\sigma^2} \exp\left\{-\frac{1}{2\sigma^2}\left(r^2 + m^2\right)\right\} I_0\left(\frac{rm}{\sigma^2}\right)$$

Data, Poisson and model



The Bargmann connection

- Bargmann transform
  [Bargmann, ’61]

$$\mathcal{F}_x(z) = \int_{-\infty}^{+\infty} \pi^{-\frac{1}{4}} \exp\left\{ -\frac{1}{2}s^2 - isz + \frac{1}{4}z^2 \right\} x(s)\, ds$$

- From time-frequency to the complex plane

$$z = \omega + it$$

- Factorization

$$F_x^{(g)}(t, \omega) = \exp\left\{ -\frac{1}{4}|z|^2 \right\} \mathcal{F}_x(z)$$

Zeros

- Entire function of order 2

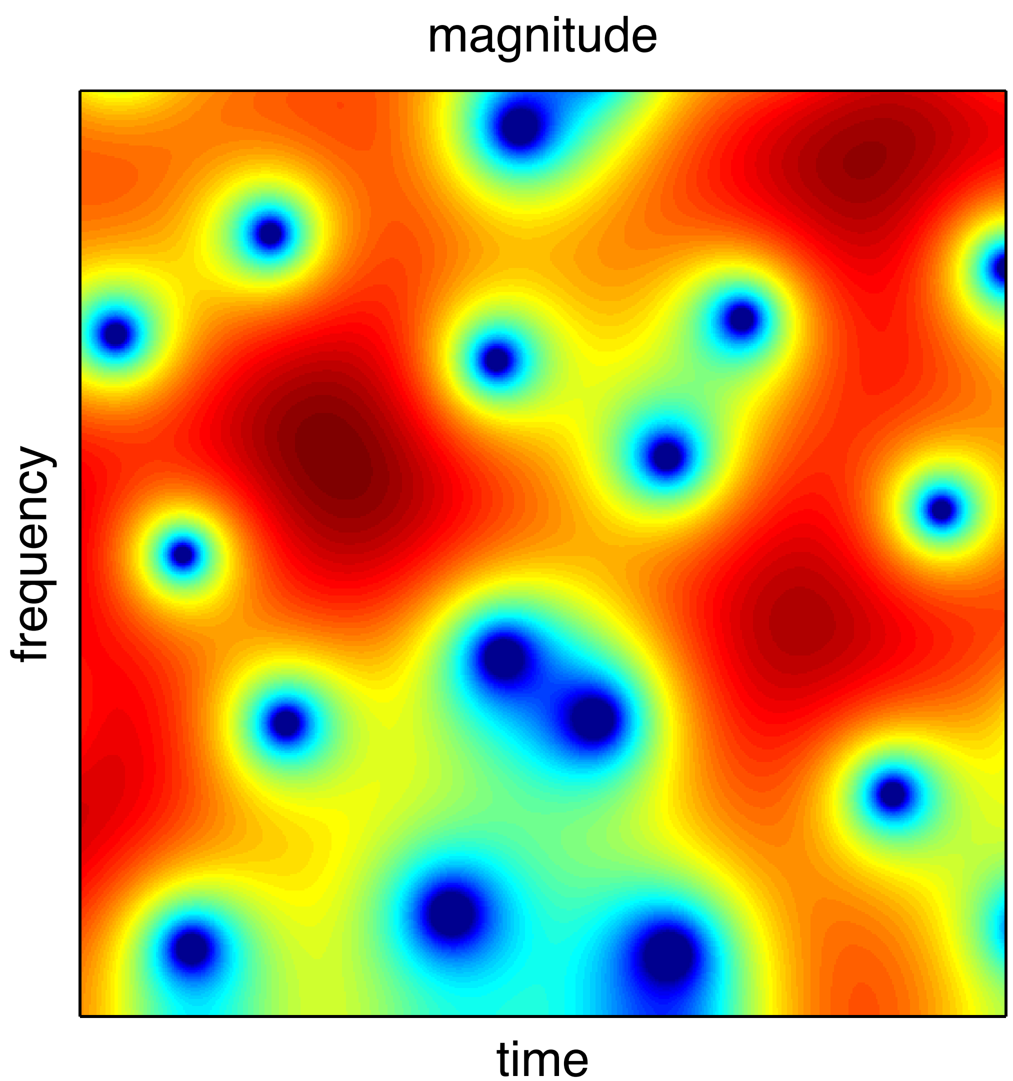
$$|\mathcal{F}_x(z)| \leq \|x\| \exp\left\{\frac{1}{4}|z|^2\right\}$$

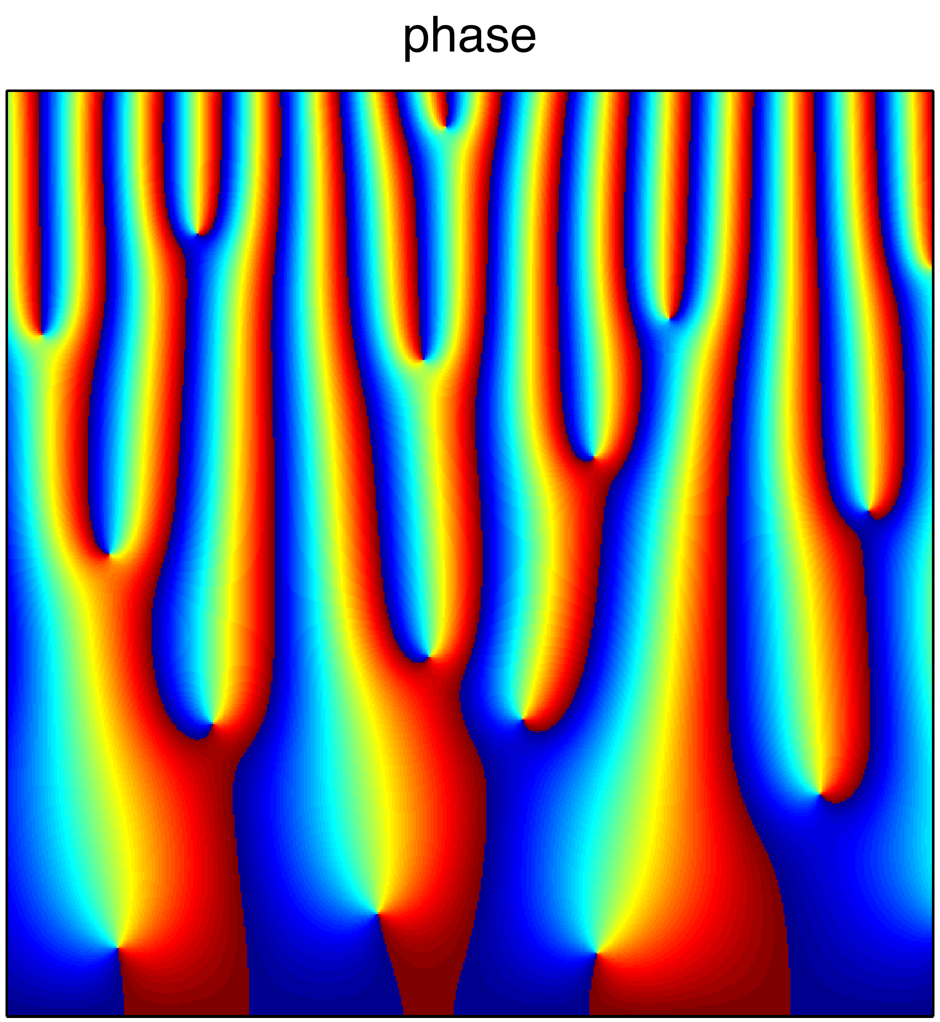
- Weierstrass-Hadamard factorization
  [Korsch *et al.*, '97]

$$\mathcal{F}_x(z) = z^m \exp\{P_2(z)\} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left\{\frac{z}{z_n} + \frac{1}{2}\left(\frac{z}{z_n}\right)^2\right\}$$

Full characterization of STFT (and spectrogram) by the only zeros of the transform in the complex plane

Patterns around zeros





Phase-magnitude

- Factorization

$$\mathcal{F}_x(t,\omega) = \mathcal{M}_x(t,\omega)\ \exp\{i\Phi_x(t,\omega)\}$$

- Phase-magnitude as a Cauchy pair
[Chassande-Mottin *et al.*, '97]

$$\frac{\partial \Phi_x}{\partial t}(t,\omega) = \frac{\partial}{\partial \omega} \log \mathcal{M}_x(t,\omega)$$

$$\frac{\partial \Phi_x}{\partial \omega}(t,\omega) = -\frac{\partial}{\partial t} \log \mathcal{M}_x(t,\omega)$$

Universal behavior

- From Weierstrass-Hadamard factorization
  [Auger *et al.*, '12]

  - magnitude

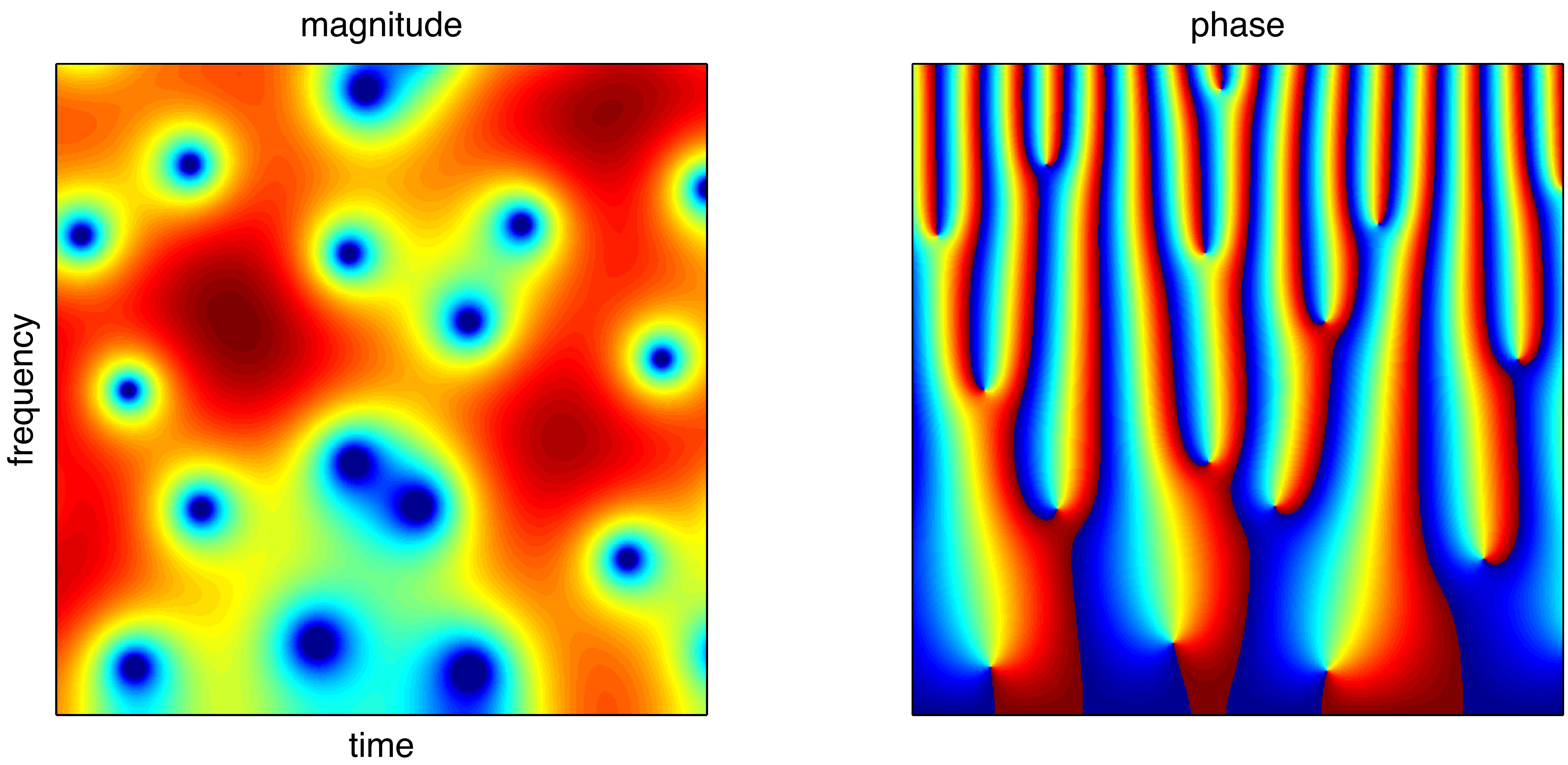
$$\mathcal{M}_x(t,\omega)\big|_{t\sim t_n,\omega\sim\omega_n} \propto \sqrt{(\omega-\omega_n)^2+(t-t_n)^2}$$

  - phase gradient
    [Balasz *et al.*, '11]

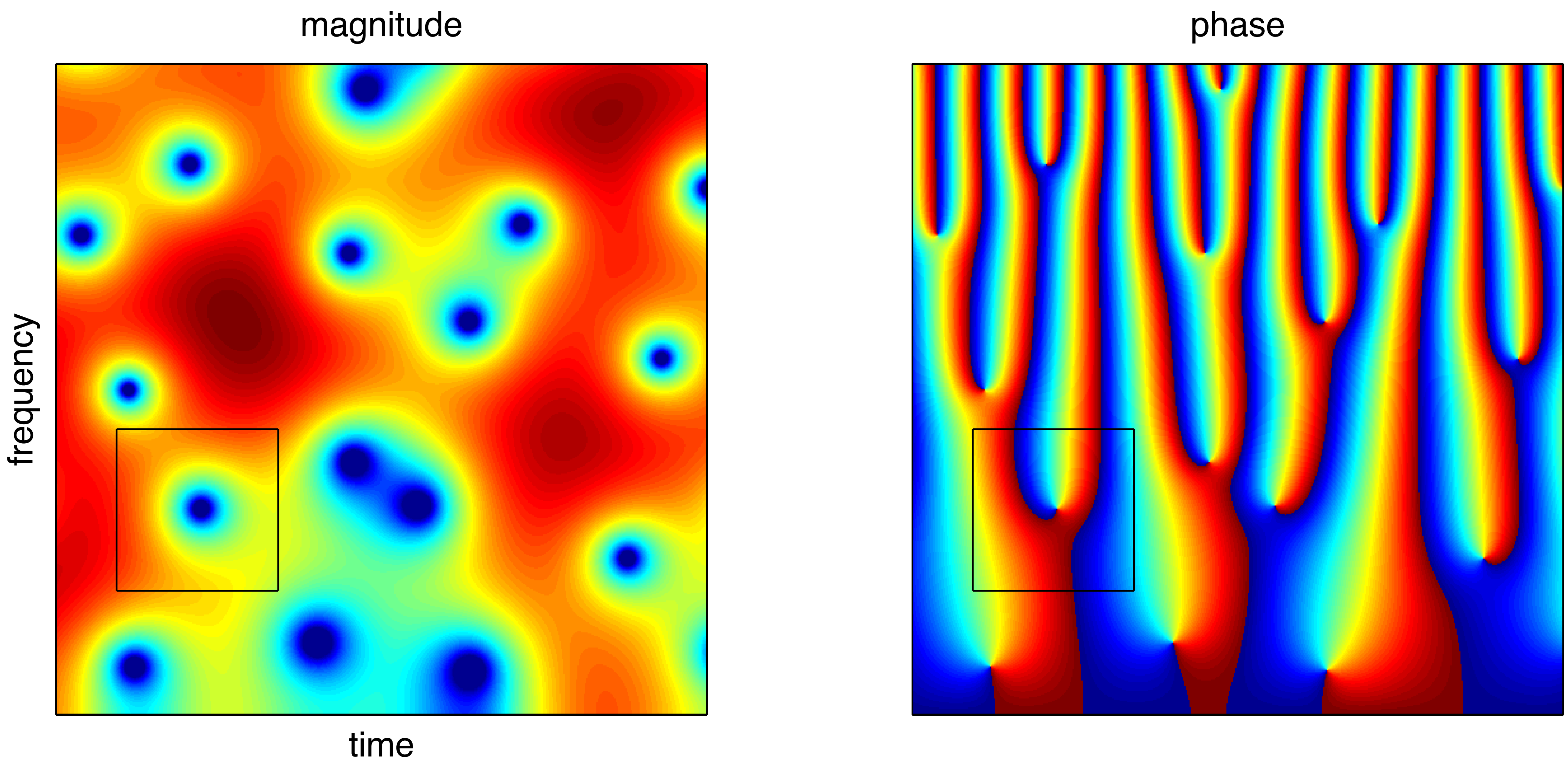
$$\frac{\partial \Phi_x}{\partial t}(t_n,\omega)\bigg|_{\omega\sim\omega_n} \sim \frac{1}{\omega-\omega_n}$$

$$\frac{\partial \Phi_x}{\partial \omega}(t,\omega_n)\bigg|_{t\sim t_n} \sim \frac{1}{t_n-t}$$

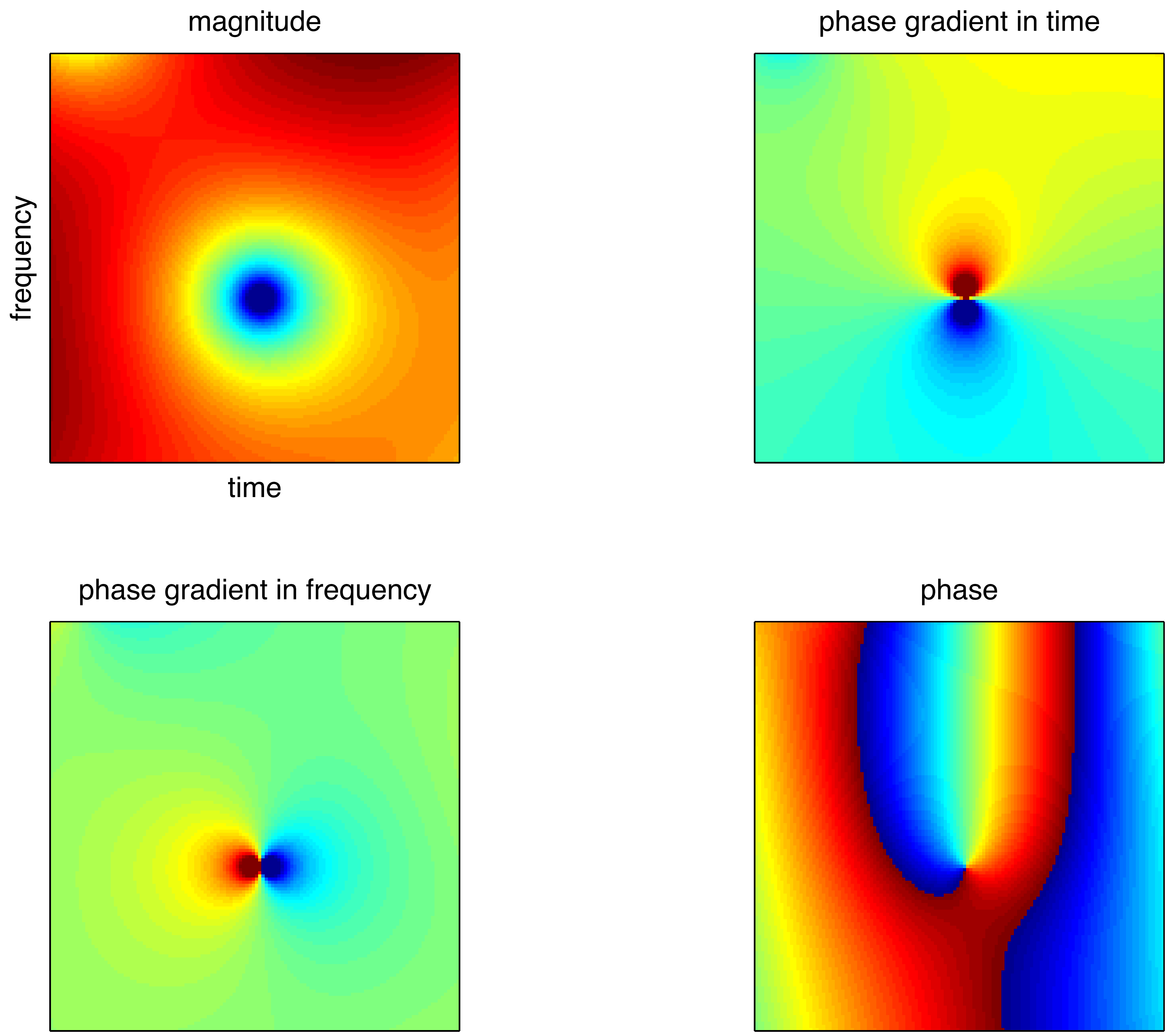
Patterns around zeros



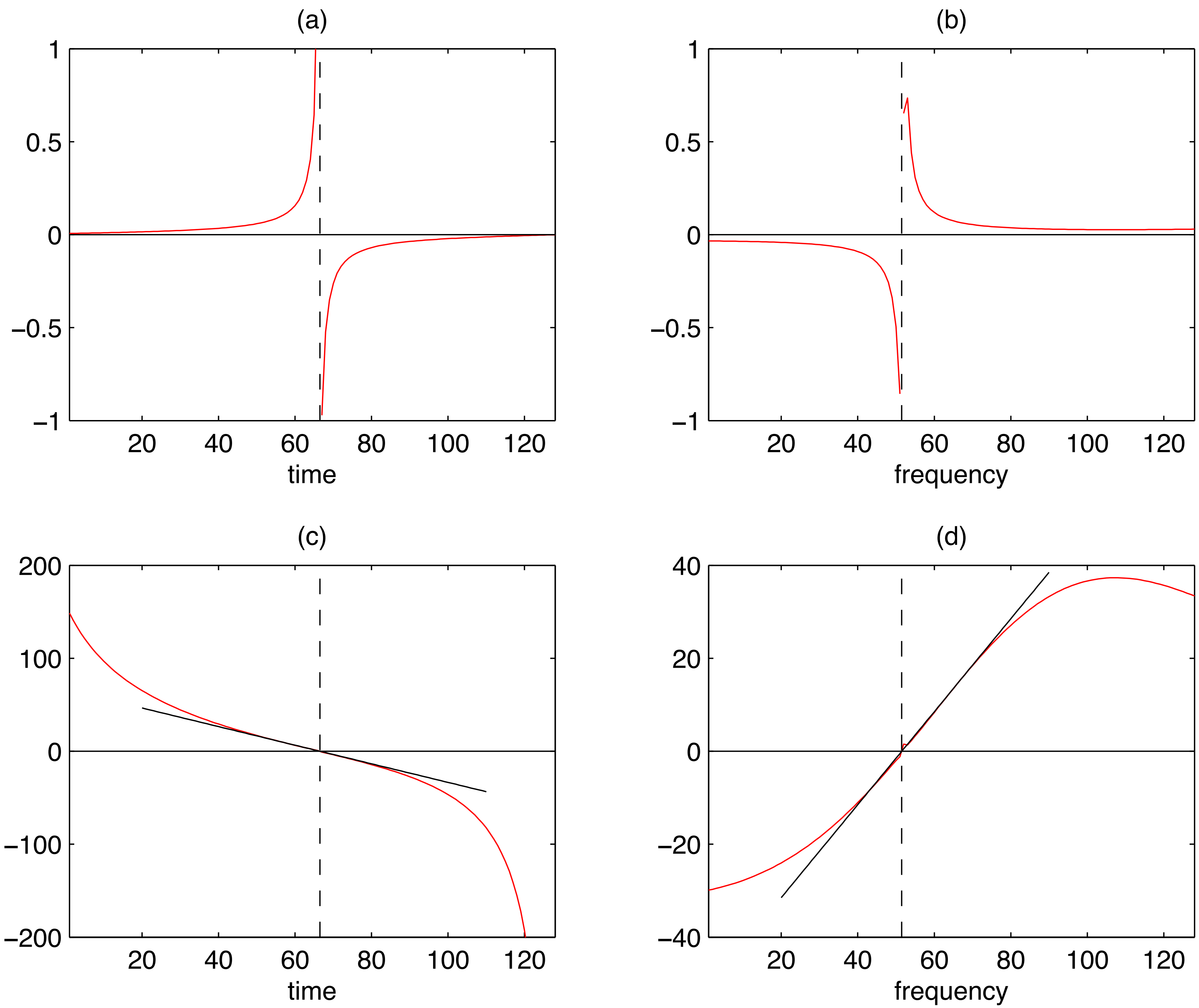
Patterns around zeros



Patterns around zeros

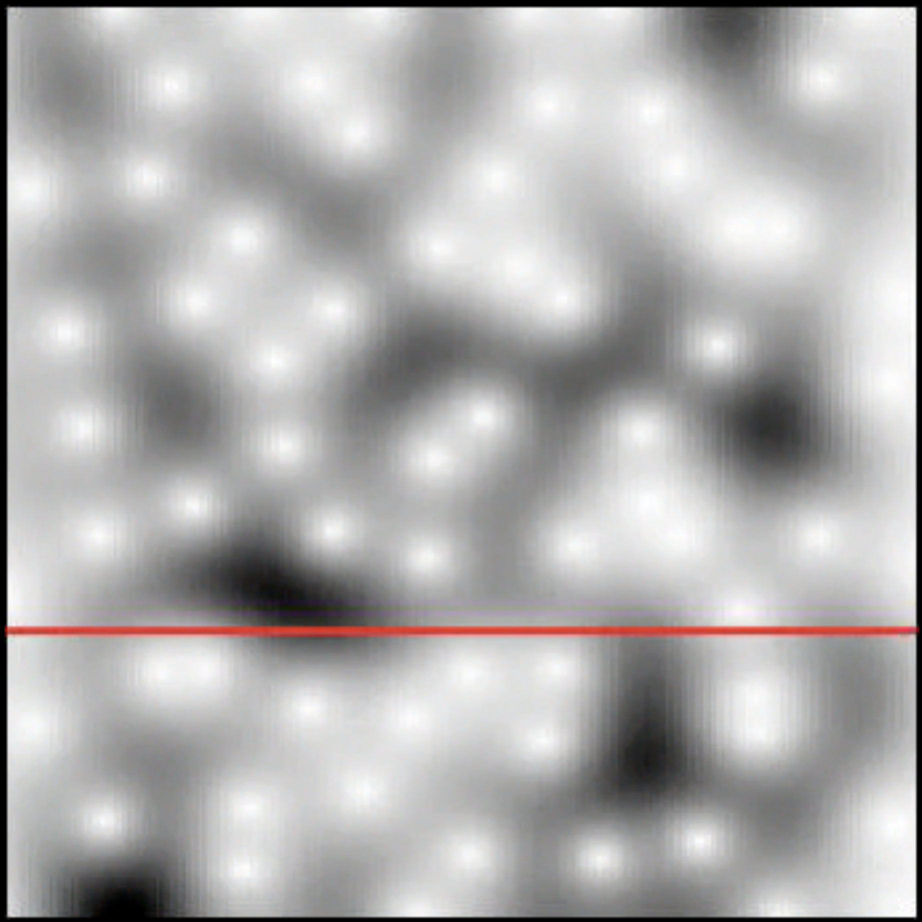


Patterns around zeros



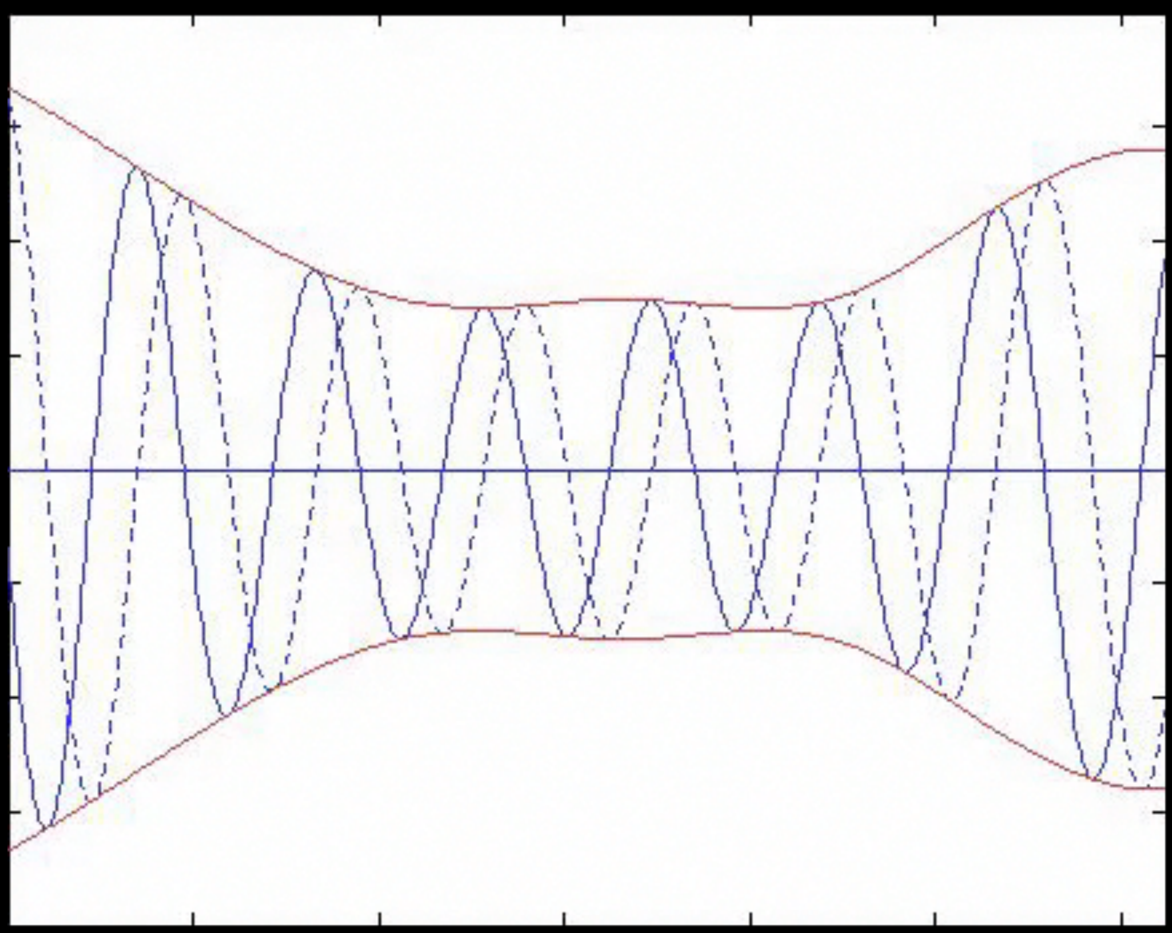
Passing through zeros

spectrogram

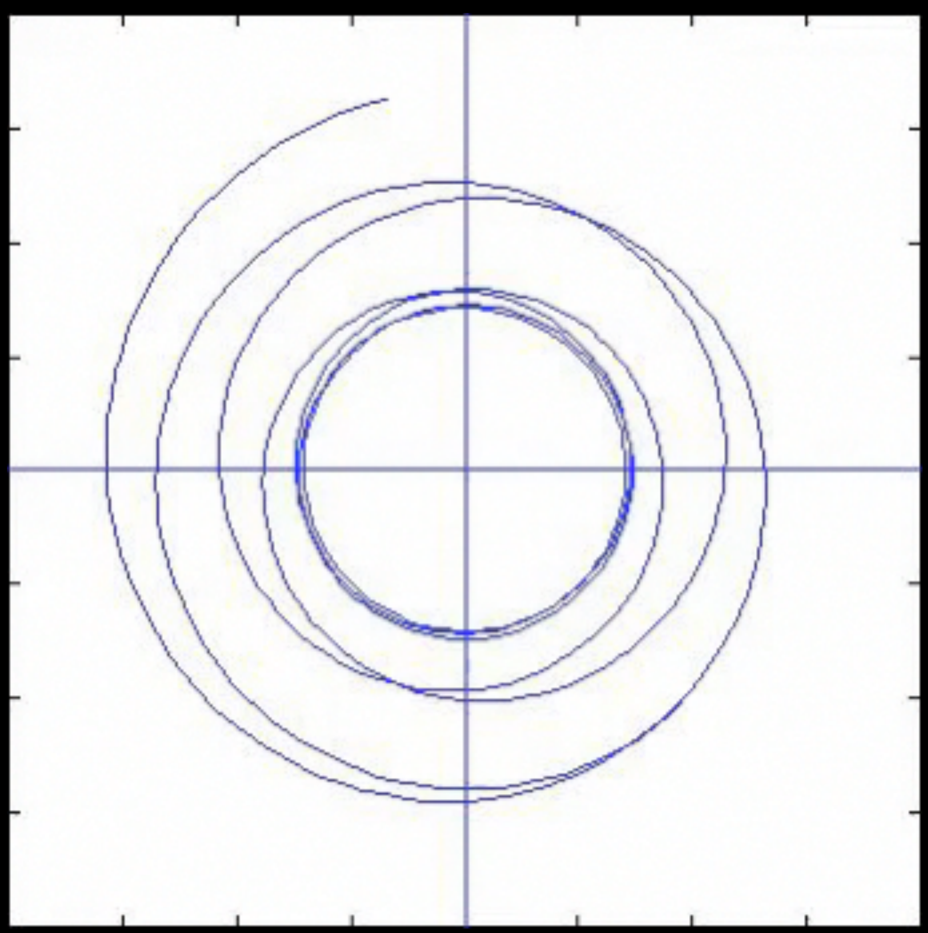


Passing through zeros

STFT real/imaginary parts

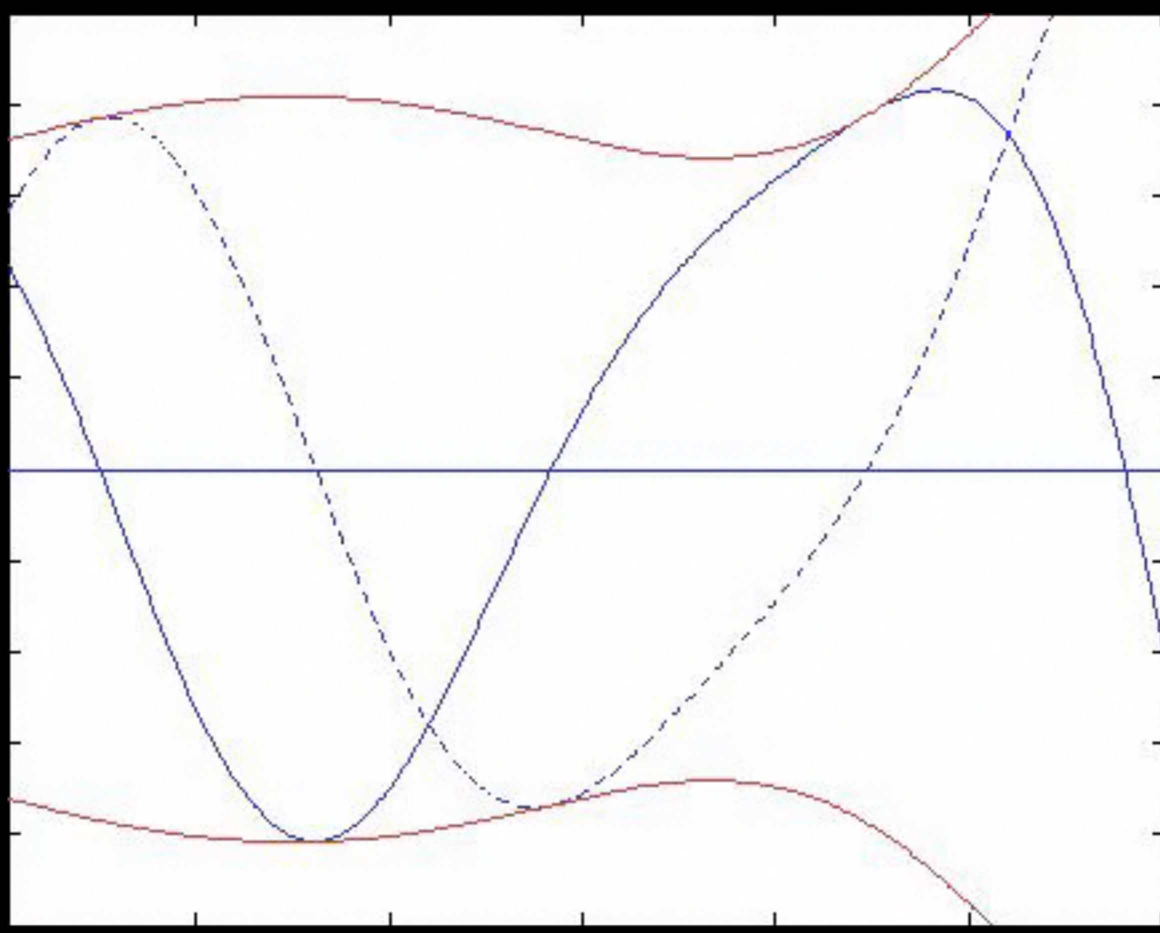


Argand diagram

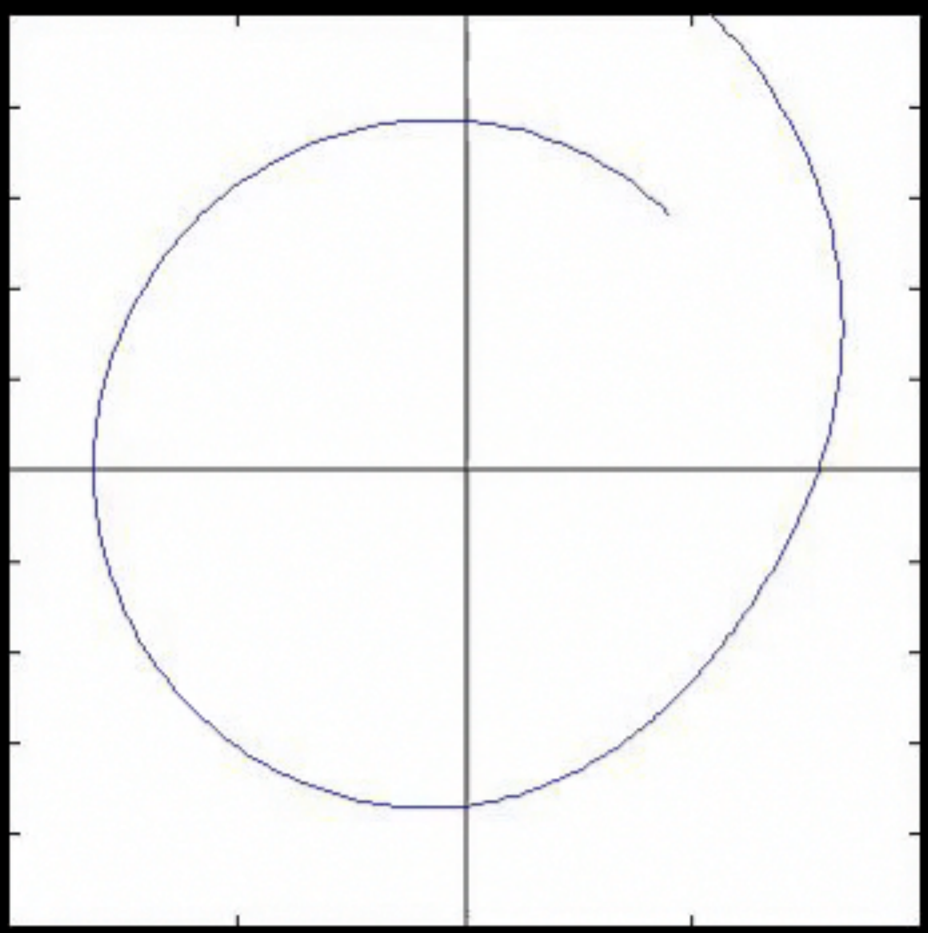


Passing through zeros

STFT real/imaginary parts



Argand diagram



Interpretation

- Undeterminacy of phase when magnitude vanishes

$$\Phi_x(t,\omega) = \tan^{-1}\left(\frac{\mathrm{Im}\{F_x^{(g)}(t,\omega]\}}{\mathrm{Re}\{F_x^{(g)}(t,\omega]\}}\right)$$

- Winding number increased by 1 when a frequency line intersects a zero
- Situation similar to a phase dislocation in crystals or wave trains
  [Nye & Berry, '74]
- Built-in superposition of tones with slightly different frequencies within the reproducing kernel

Voronoi and Delaunay

- Voronoi tessellation based on spectrogram local maxima (« *Voronoi cell attached to a given maximum = set of all points closer to this maximum than to any other one* »)
  [Voronoi, '08]

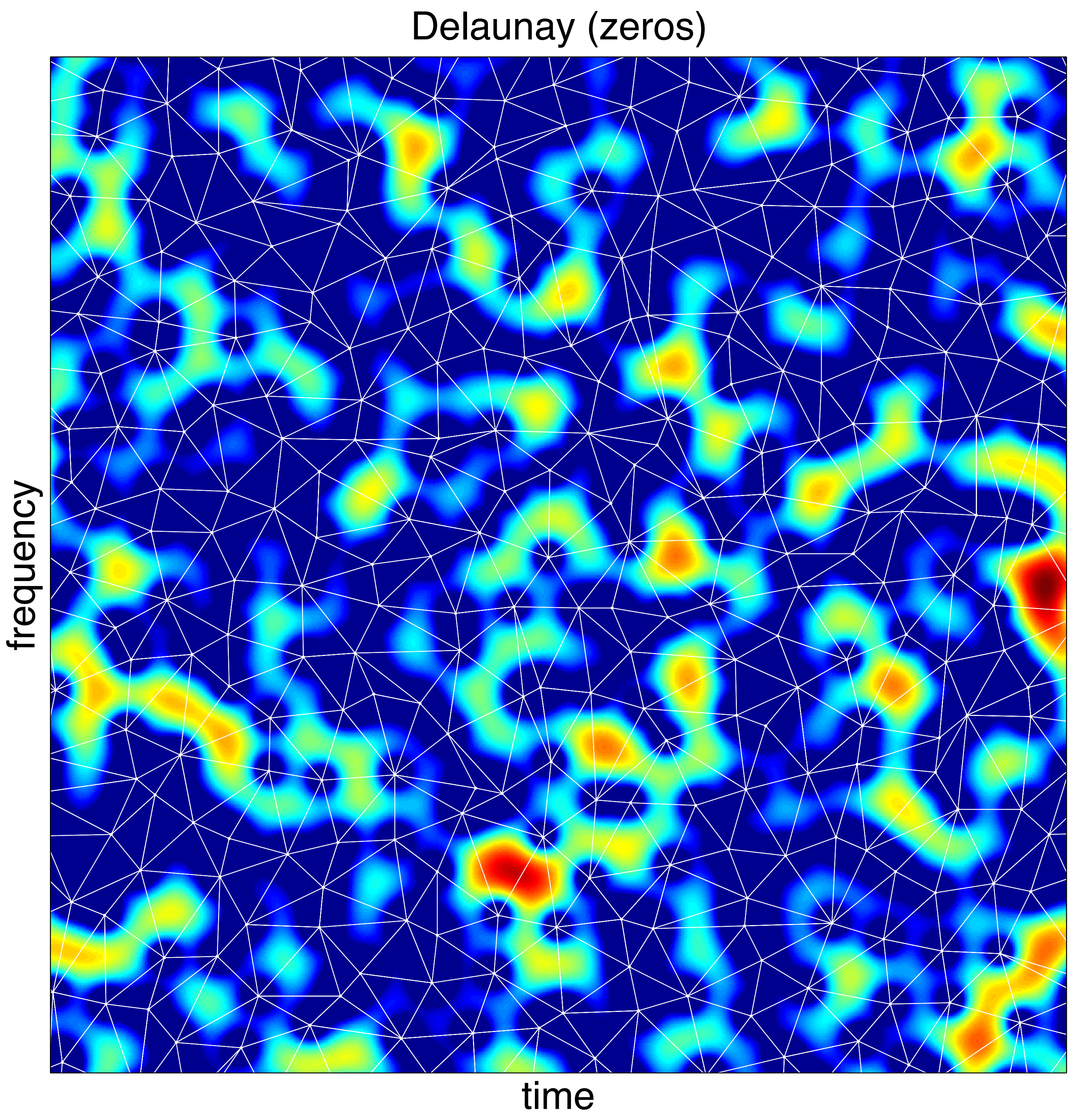
- Paving of the time-frequency plane by polygonal cells

  - local energy patches

  - reassignment basins of attraction

- Duality with Delaunay triangulation connecting all local maxima
  [Delaunay, '34]

Delaunay triangulation



From Boris to Sonia (*interlude*)

Boris Delaunay
mathematician
(1890-1980)

From Boris to Sonia (*interlude*)

Sonia Delaunay
painter
(1885-1979)

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From Boris to Sonia (*interlude*)

Sonia Delaunay
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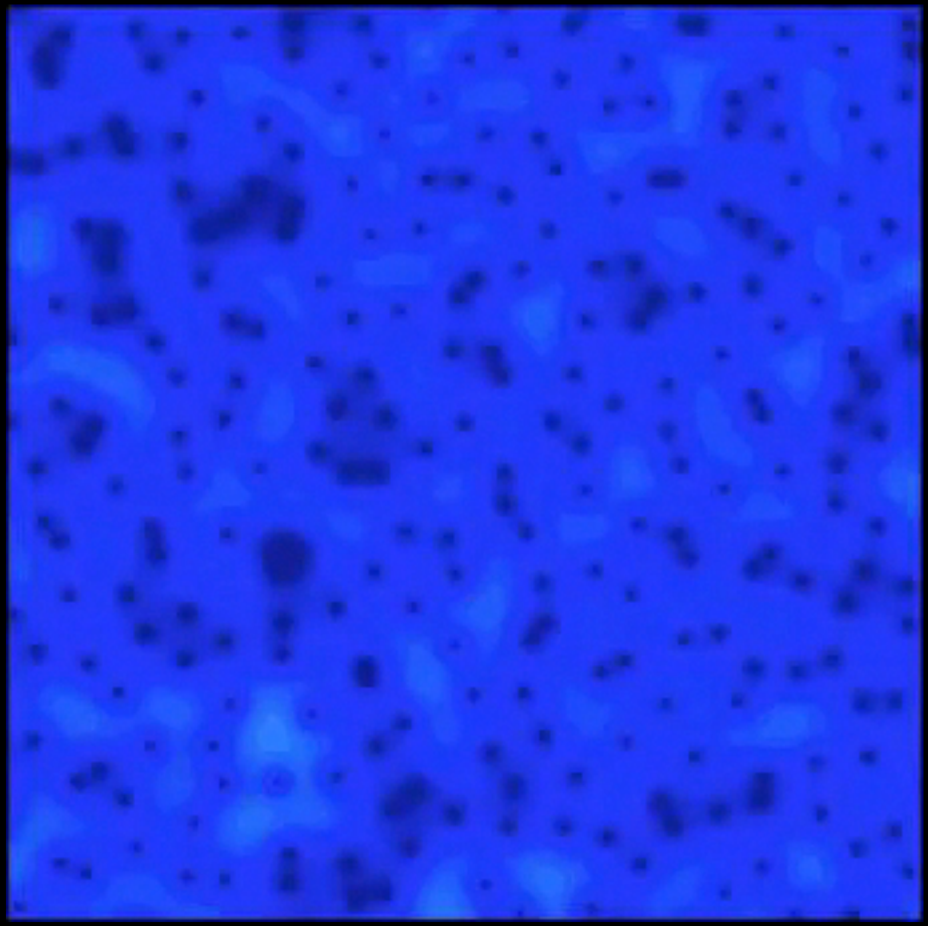
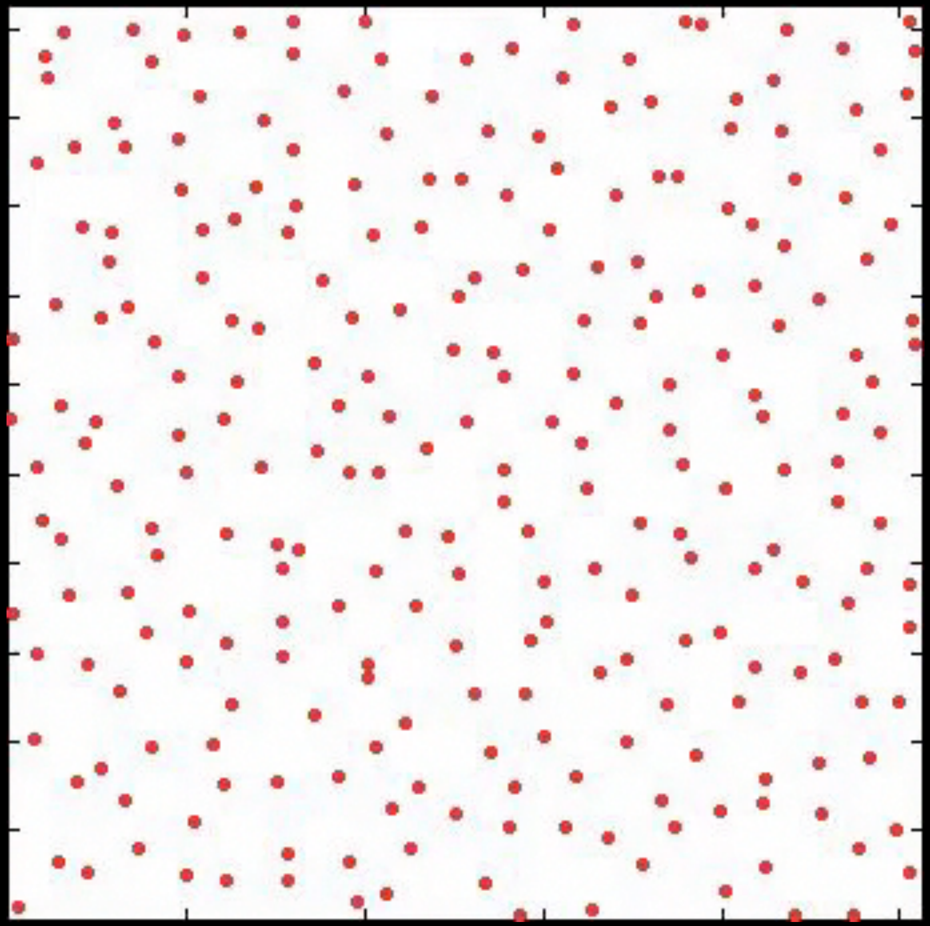
Tissu simultané, 1928

From noise to signal

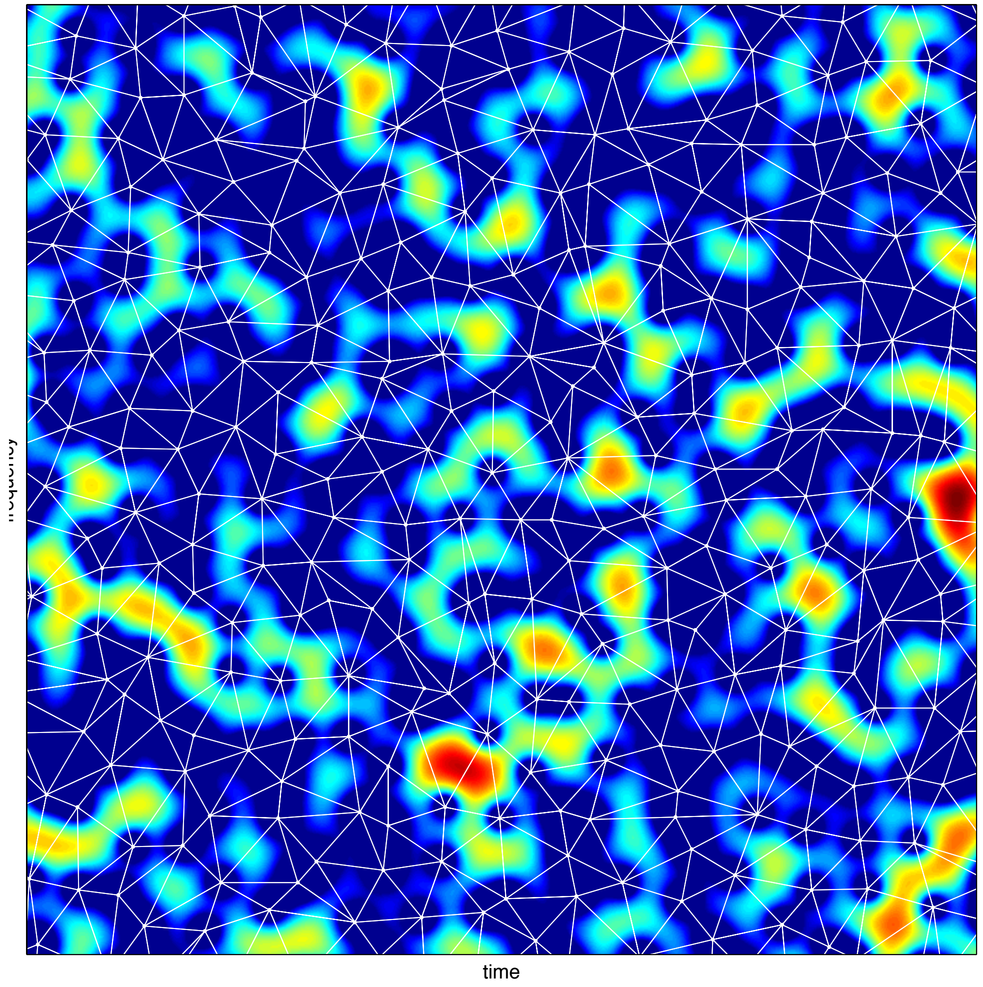
spectrogram

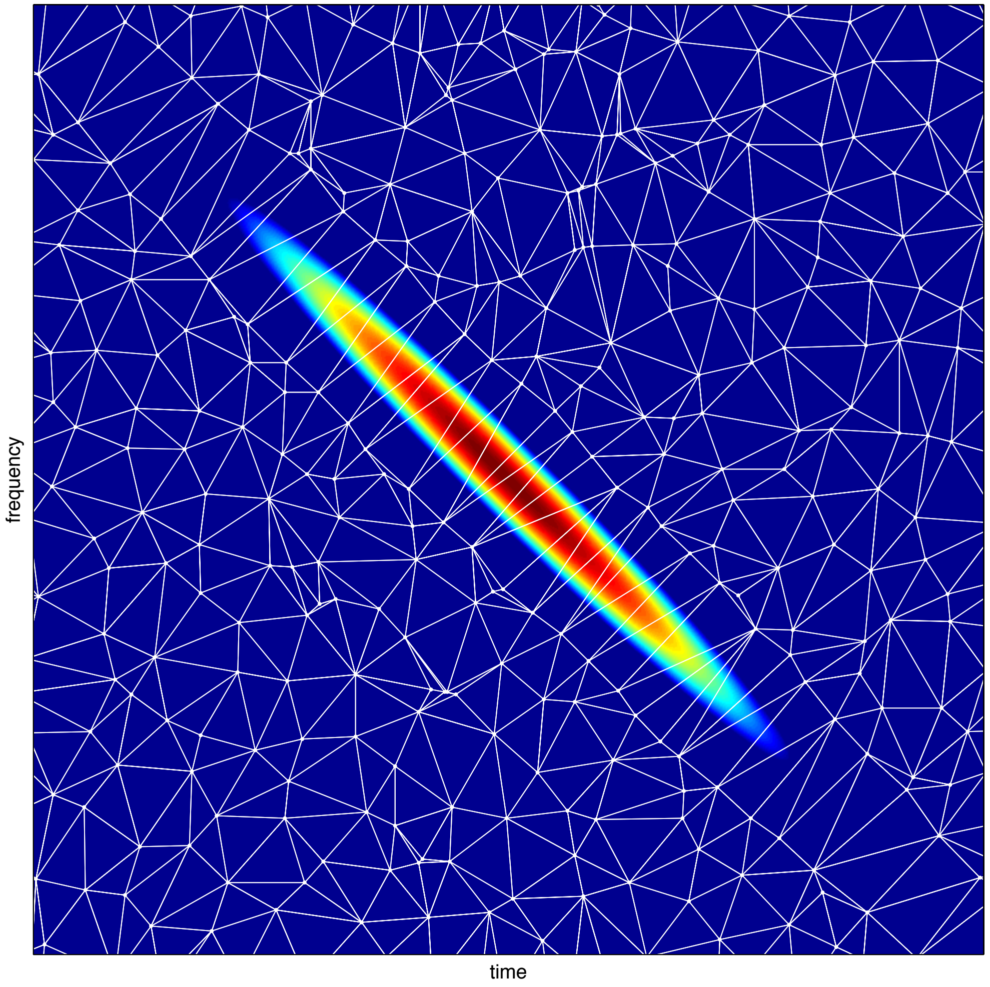


zeros

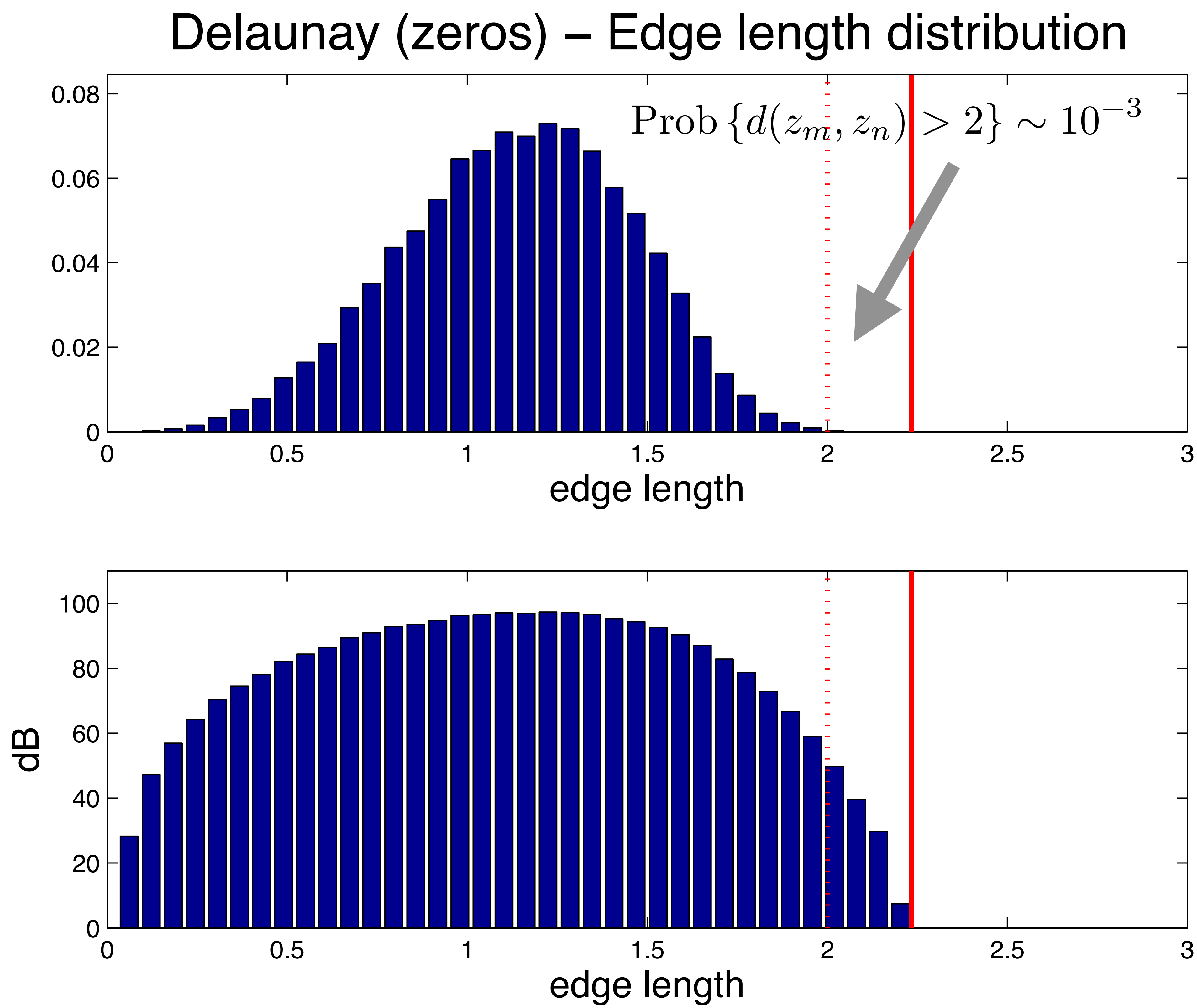


From noise to signal





Edge length distribution

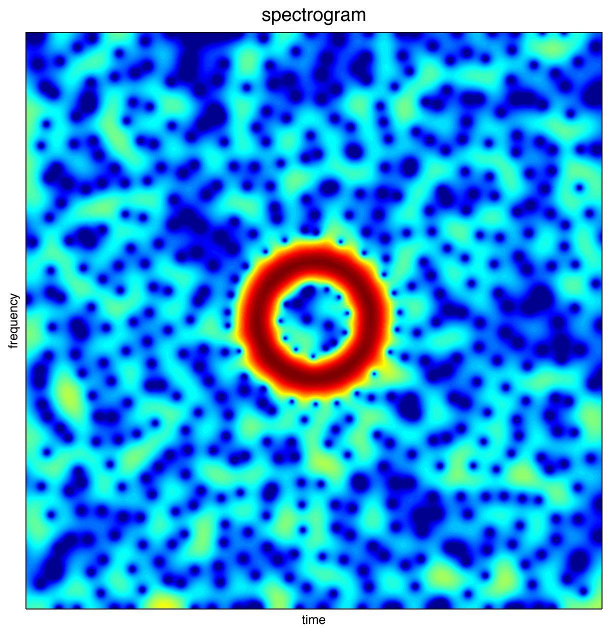


« Filtering » based on zeros

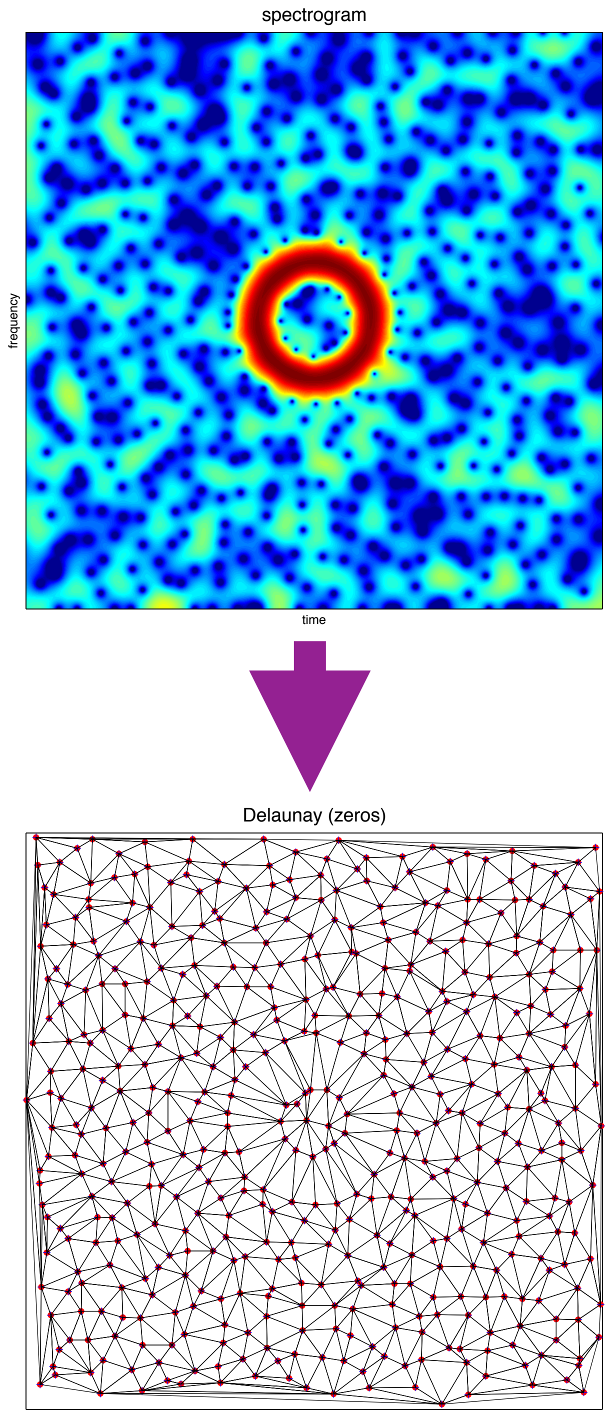
- Perform Delaunay triangulation over STFT zeros z_m
- Identify outlier edges such that $|e_{mn}| = d(z_m, z_n) > 2$
- Keep triangles with at least one outlier edge
- Group adjacent triangles in connected, disjoint domains $\mathcal{D}_j$
- Reconstruct disentangled components, domain by domain

$$x_j(t) = \frac{1}{h^*(0)} \int_{(t,\omega)\in\mathcal{D}_j} F_x^{(h)}(t,\omega)\, \frac{d\omega}{2\pi}$$

An example (Hermite function)



An example (Hermite function)



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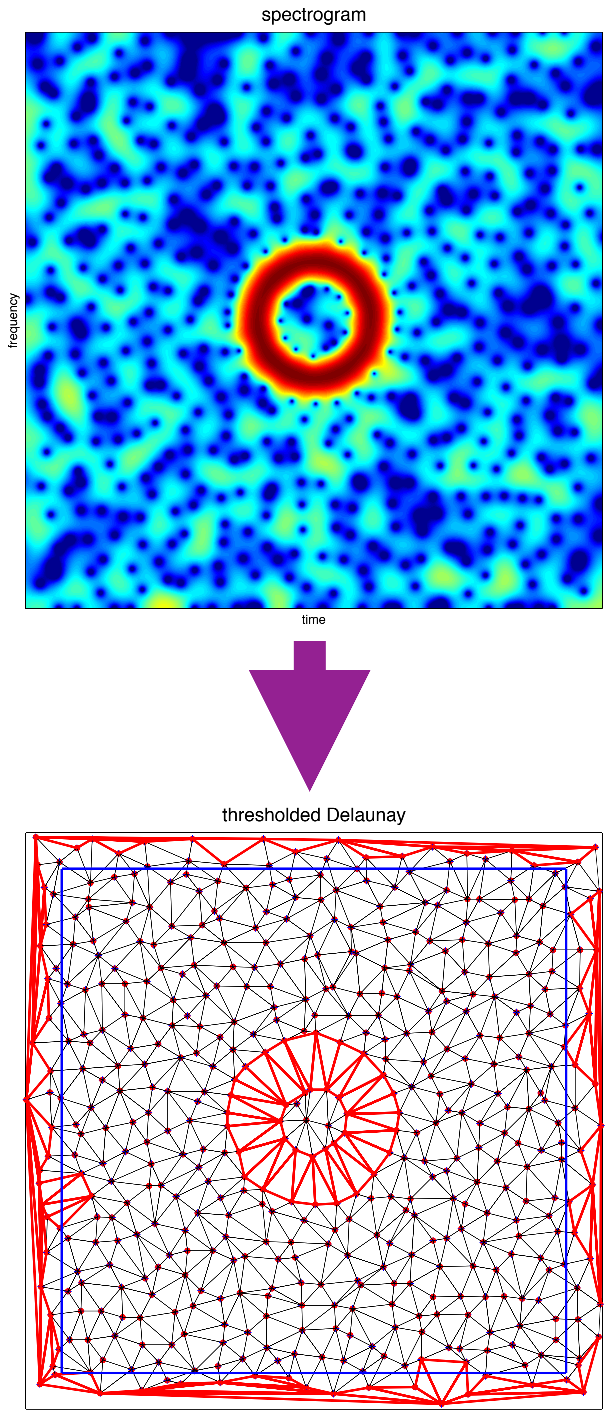

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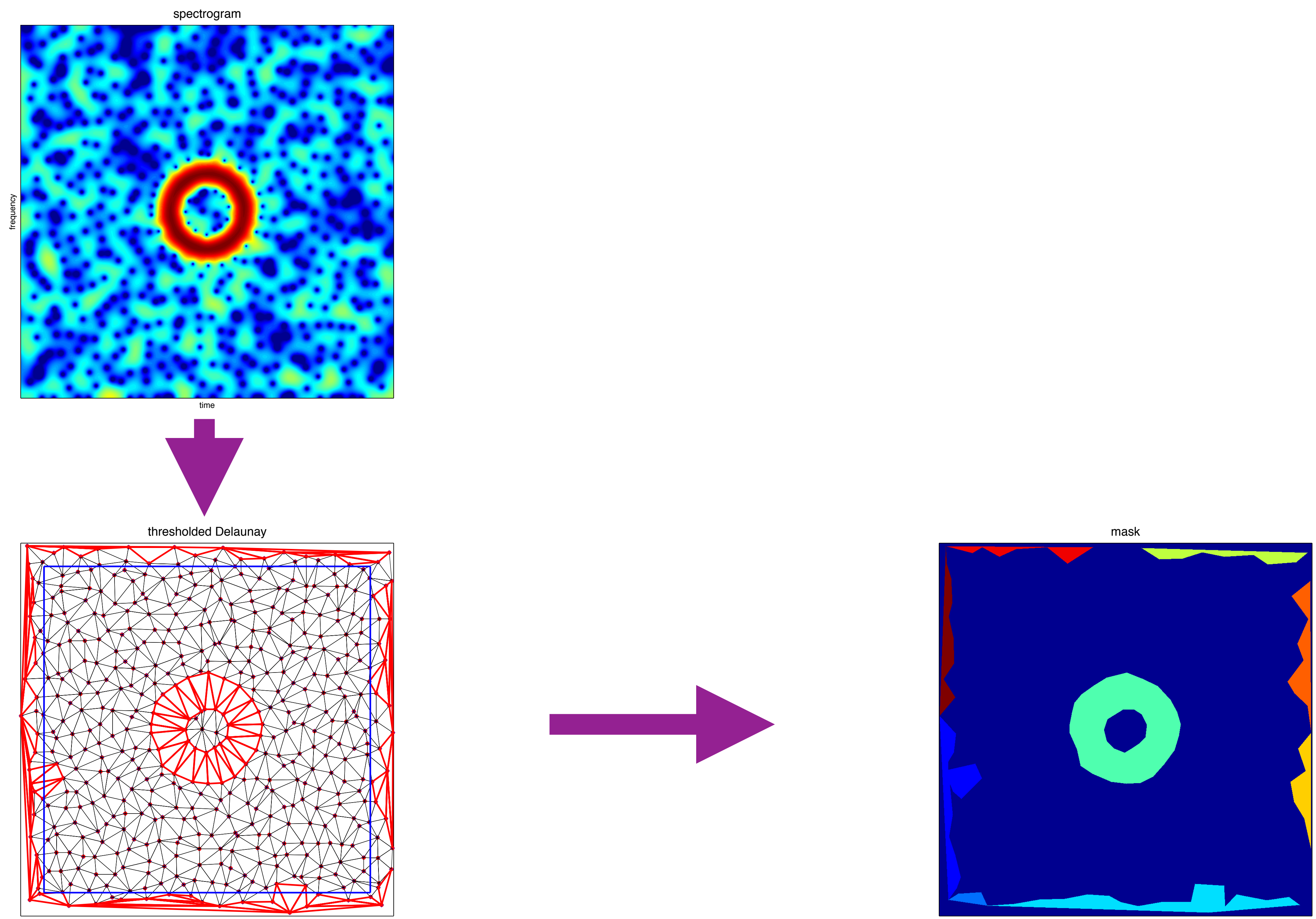

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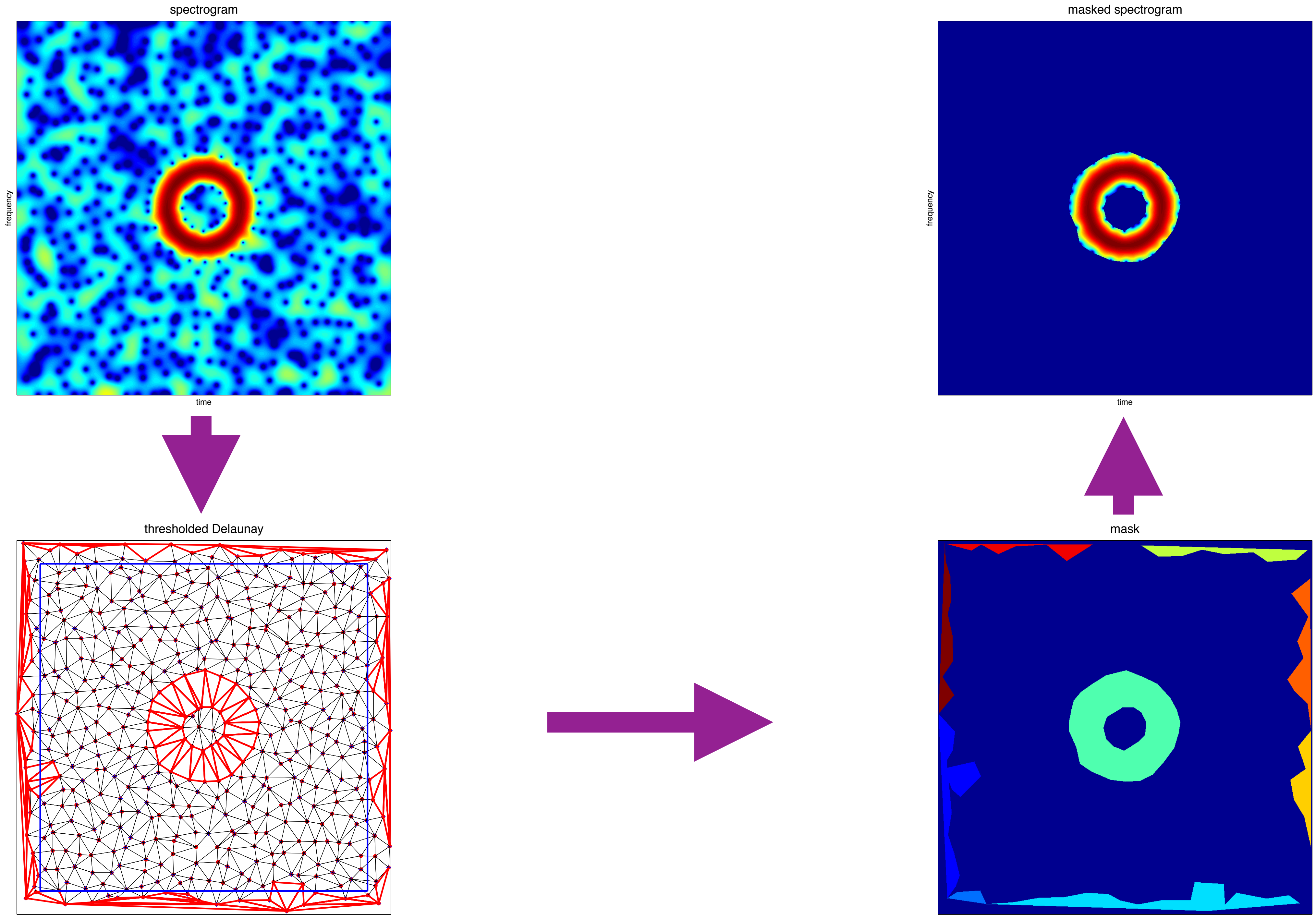

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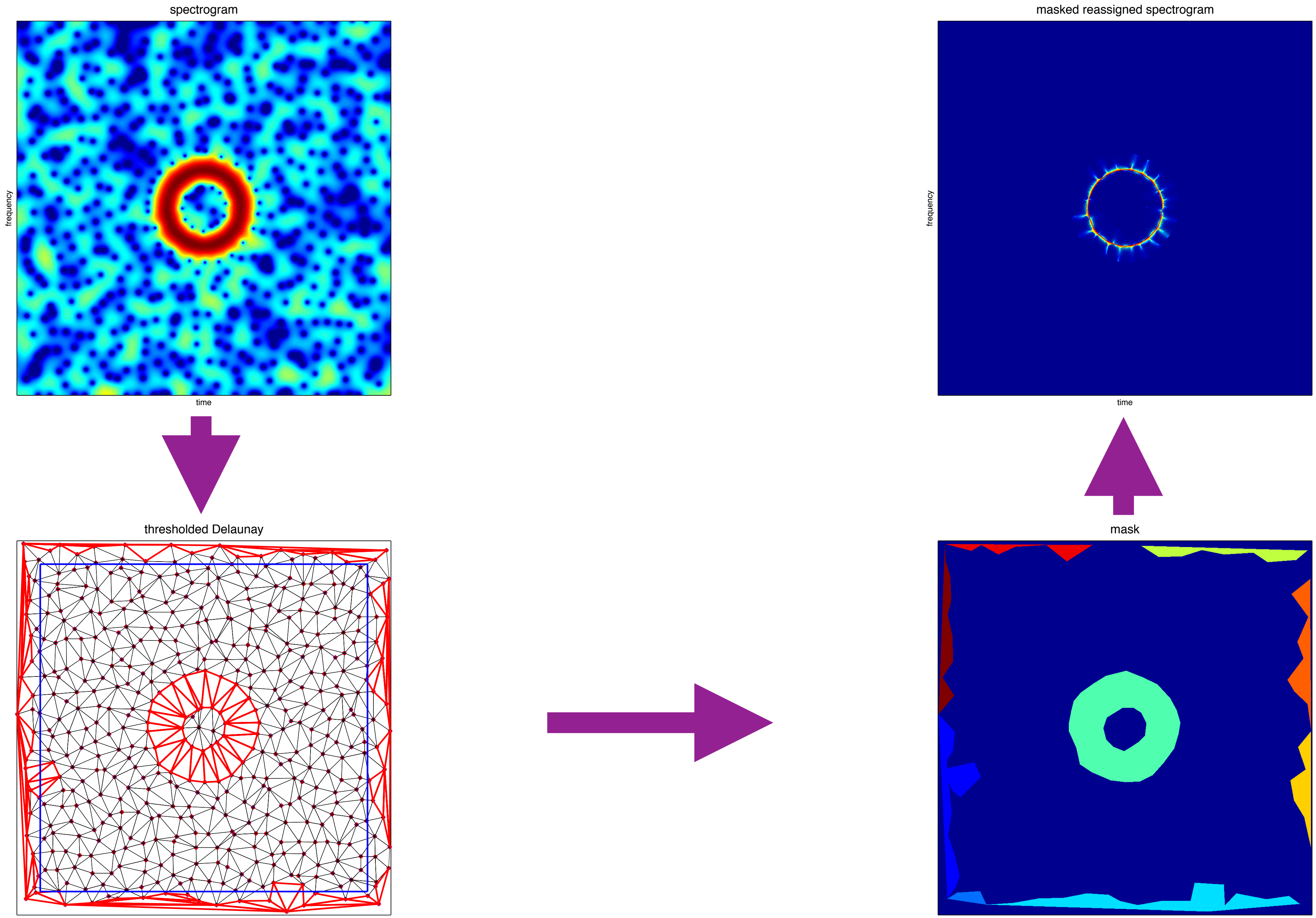
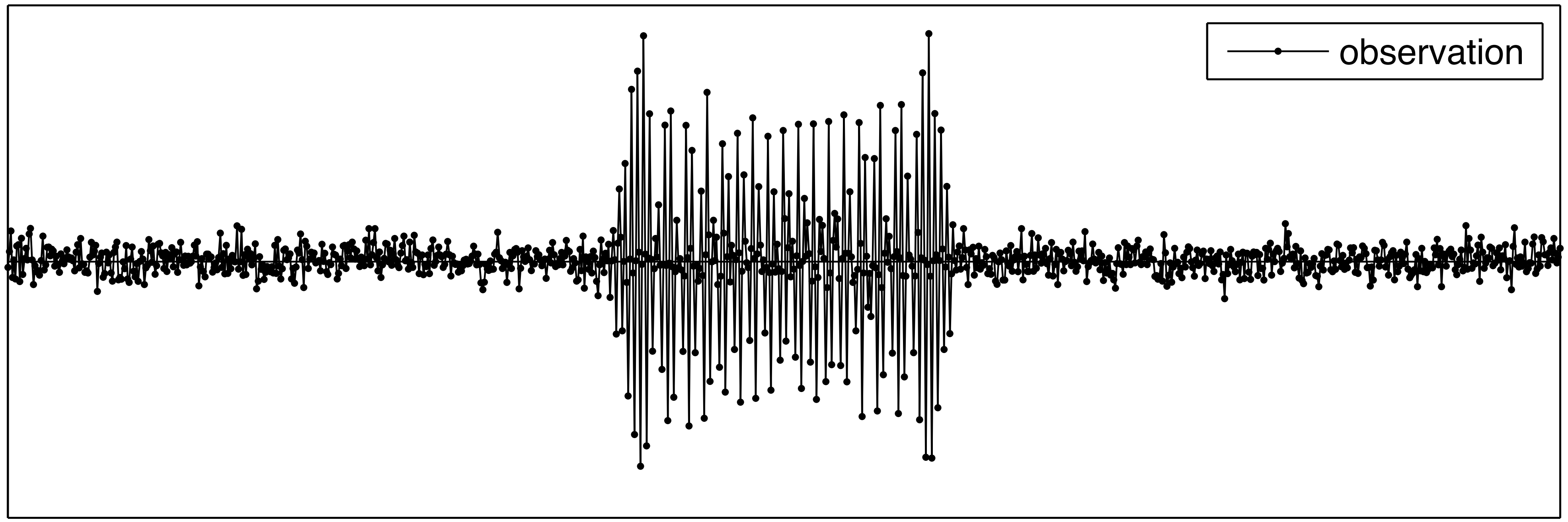
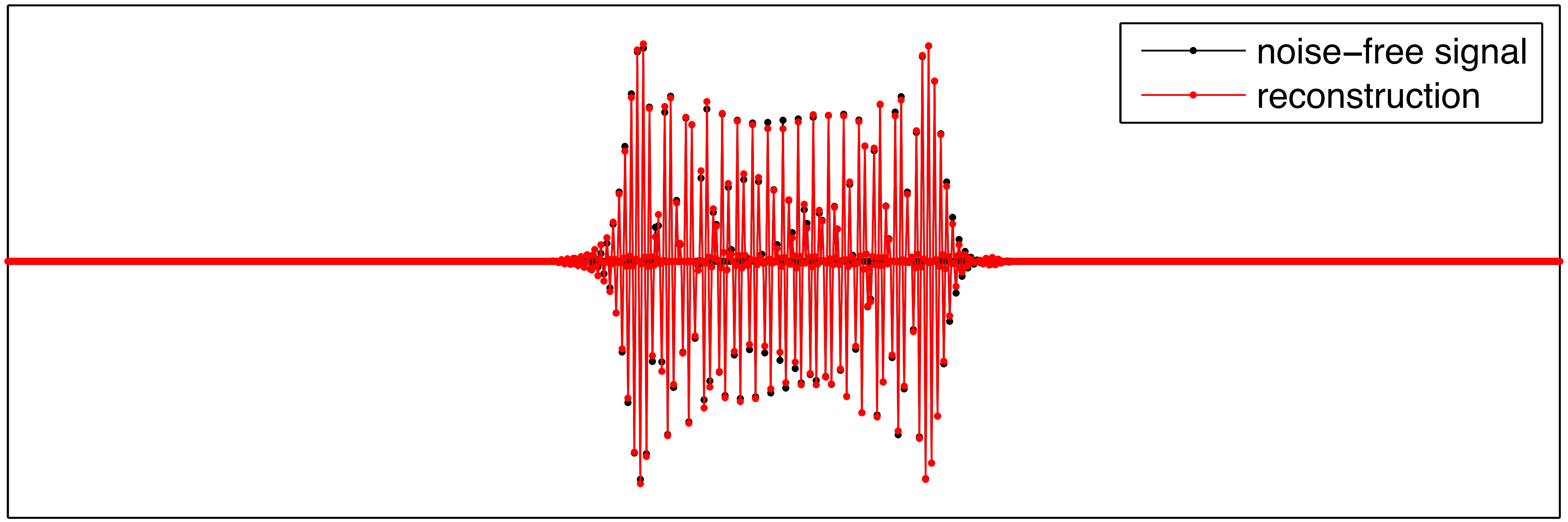


An example (Hermite function)

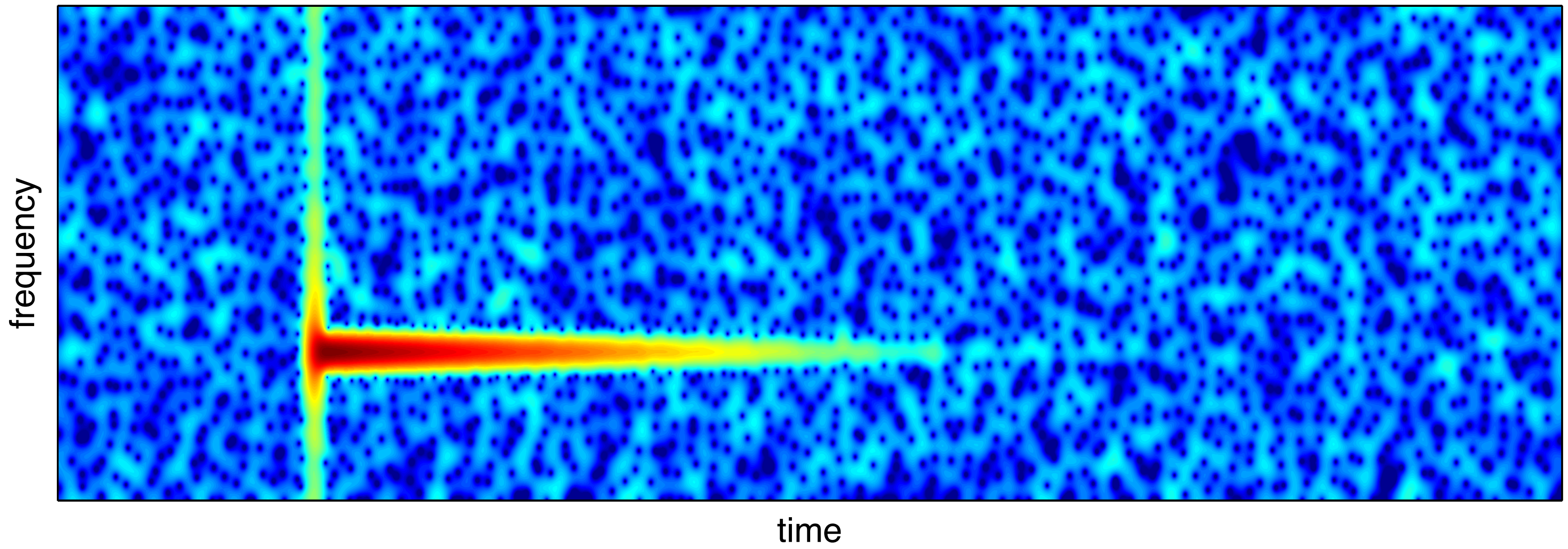
input SNR = 10 dB

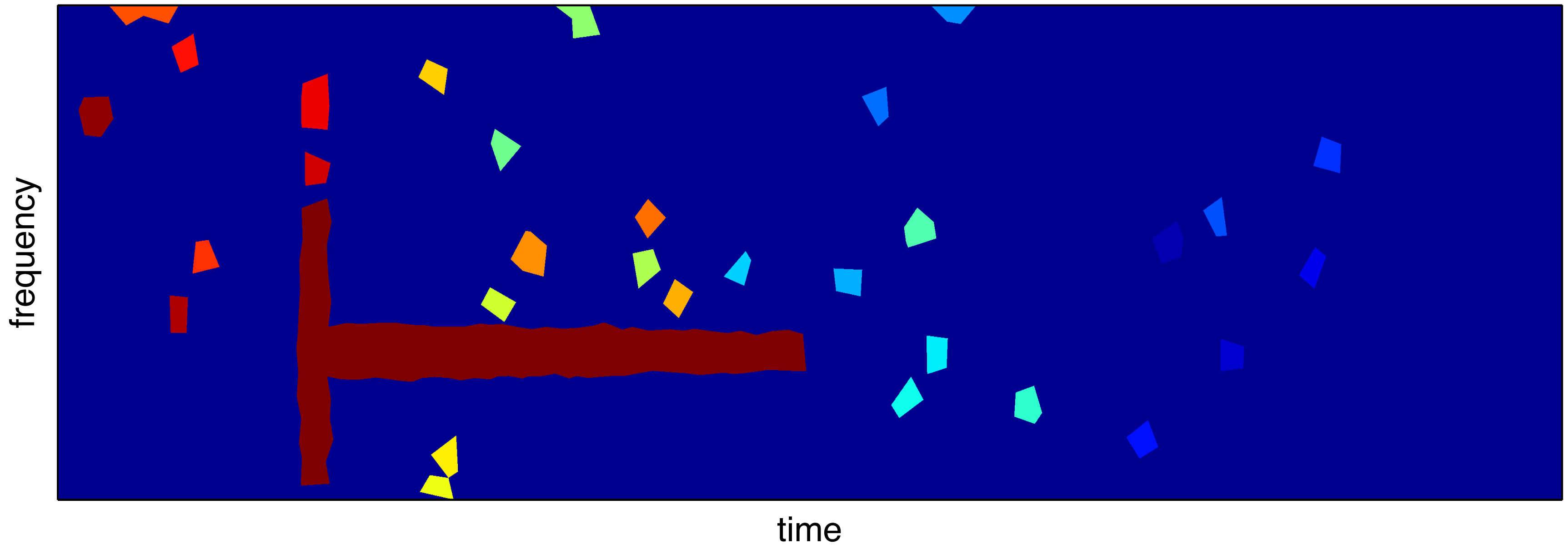


output SNR = 25.7 dB



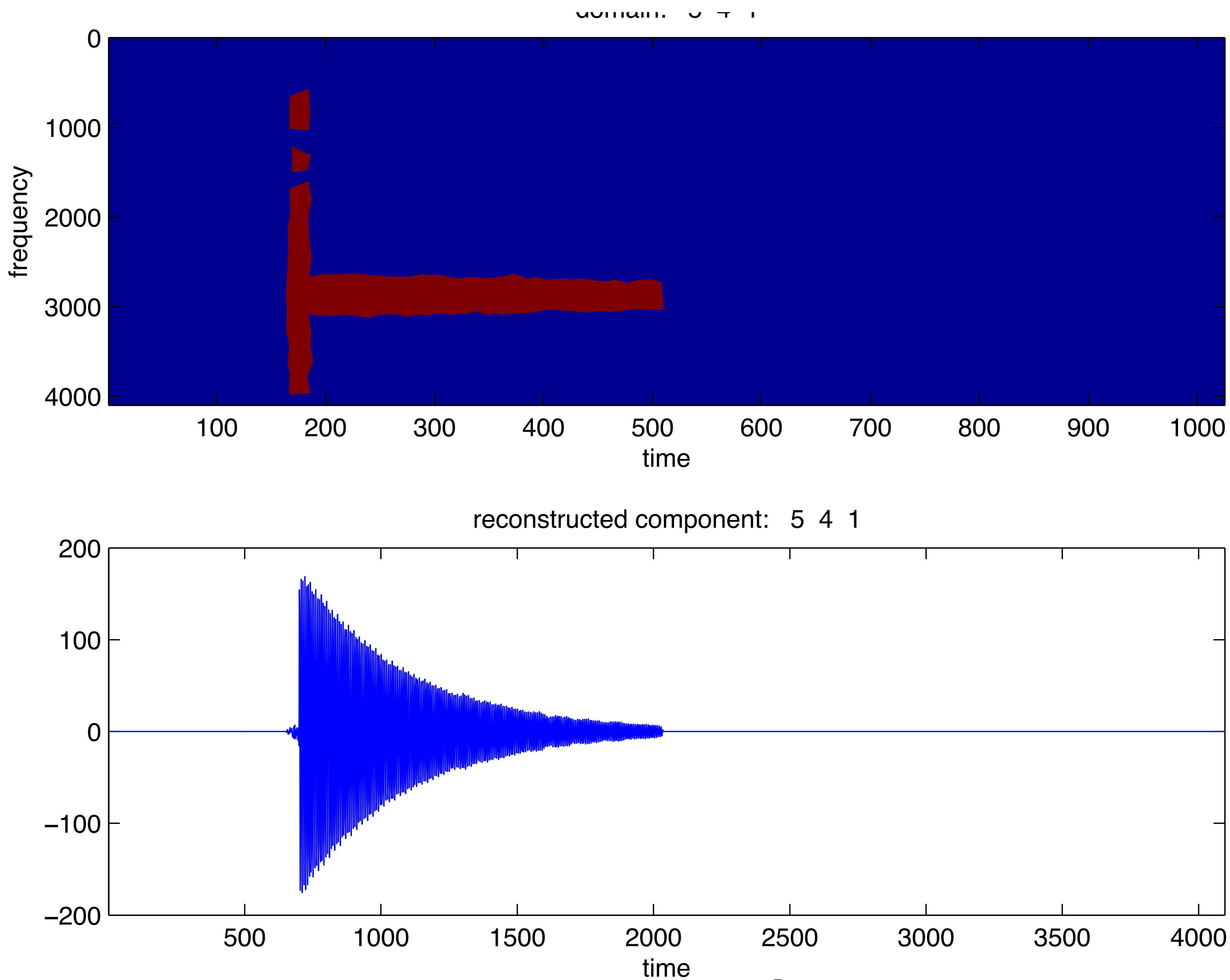
Another example





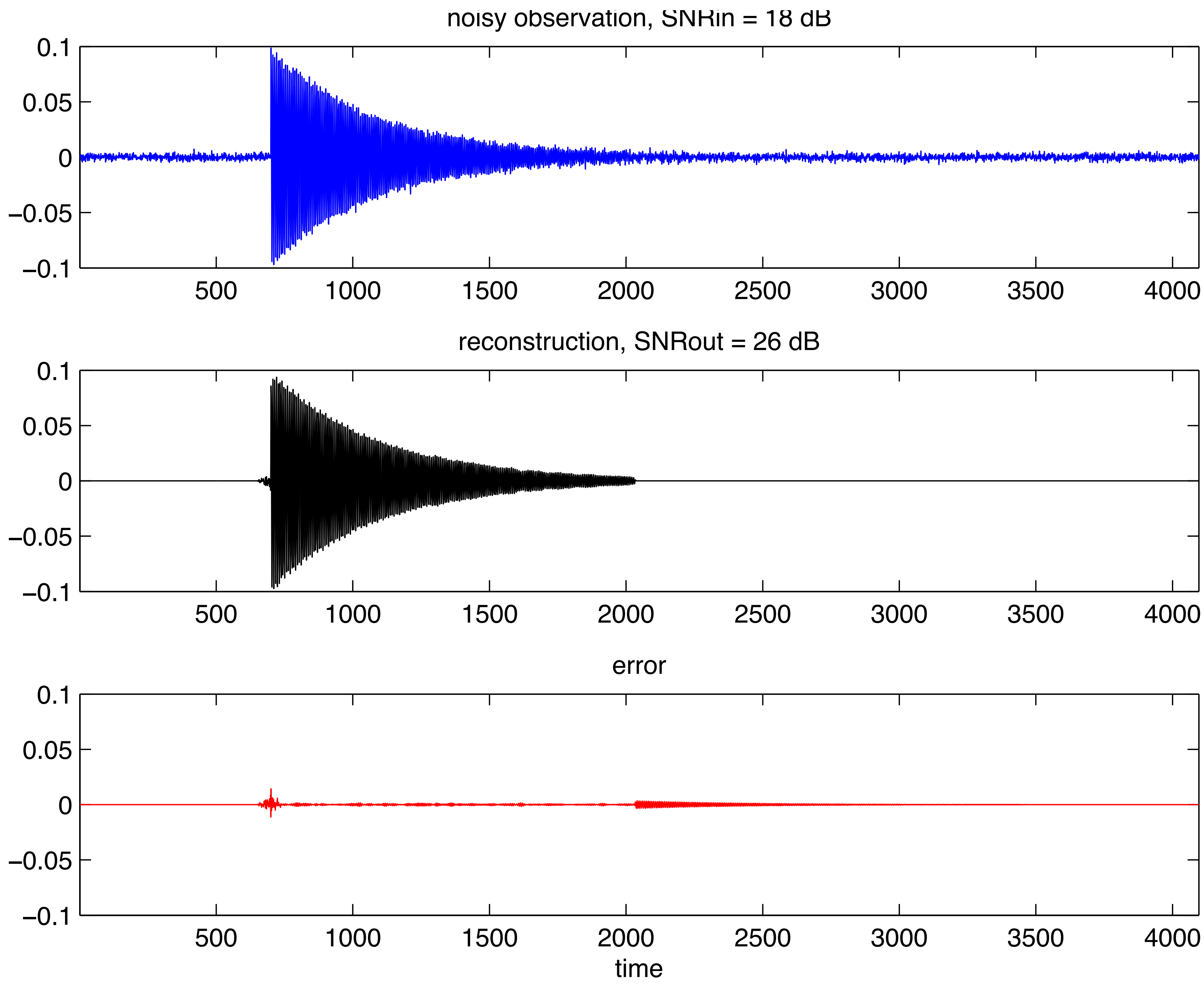
[joint work with Ph. Depalle (McGill)]

Another example (cont'd)



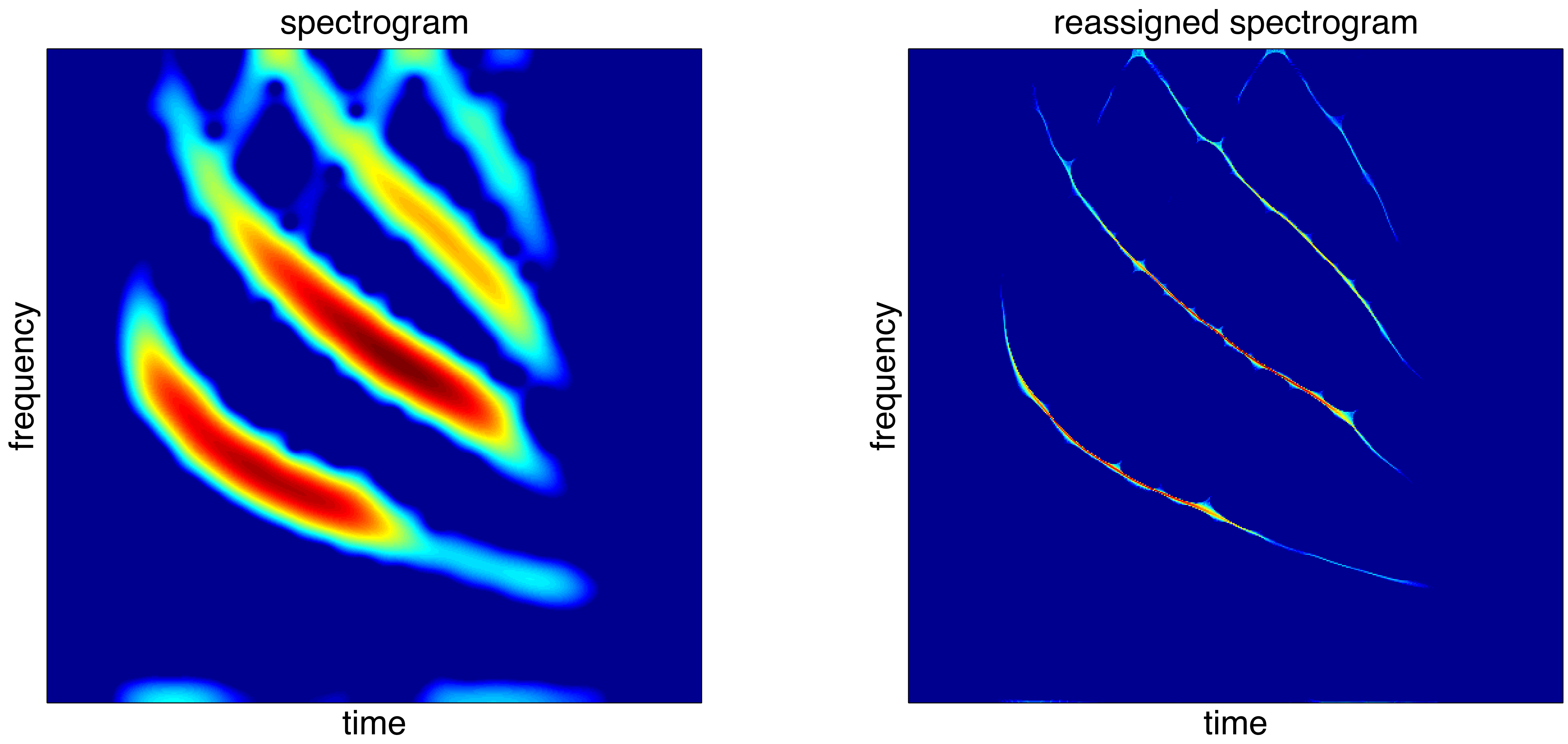
[joint work with Ph. Depalle (McGill)]

Another example (cont'd)

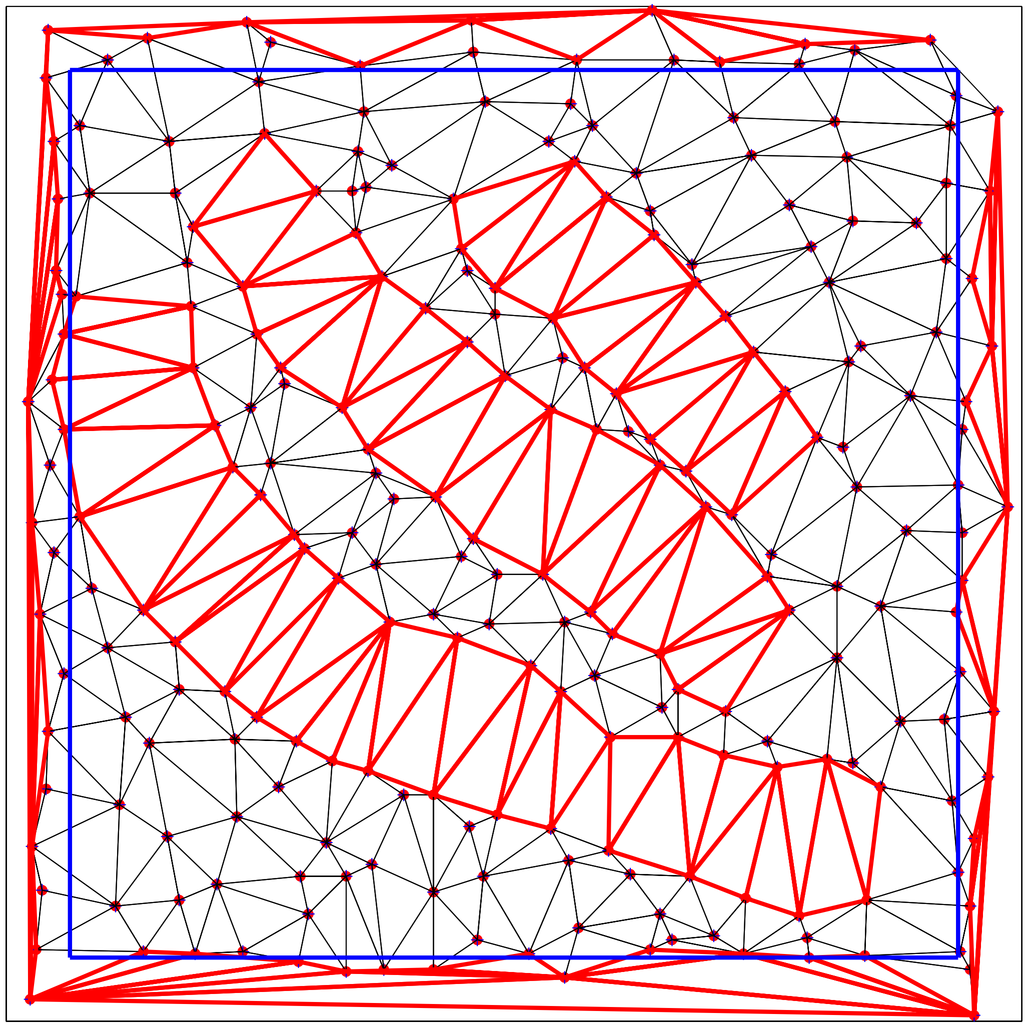


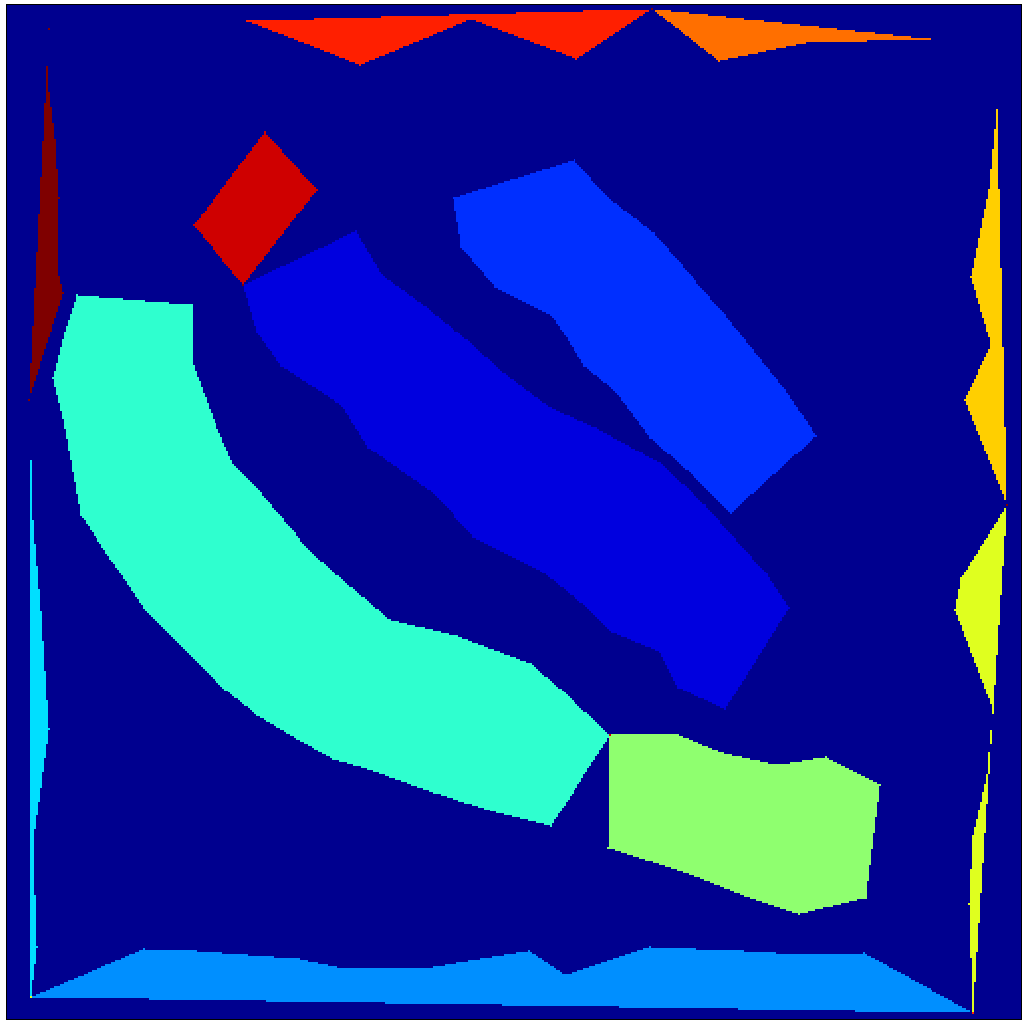
[joint work with Ph. Depalle (McGill)]

A multicomponent signal

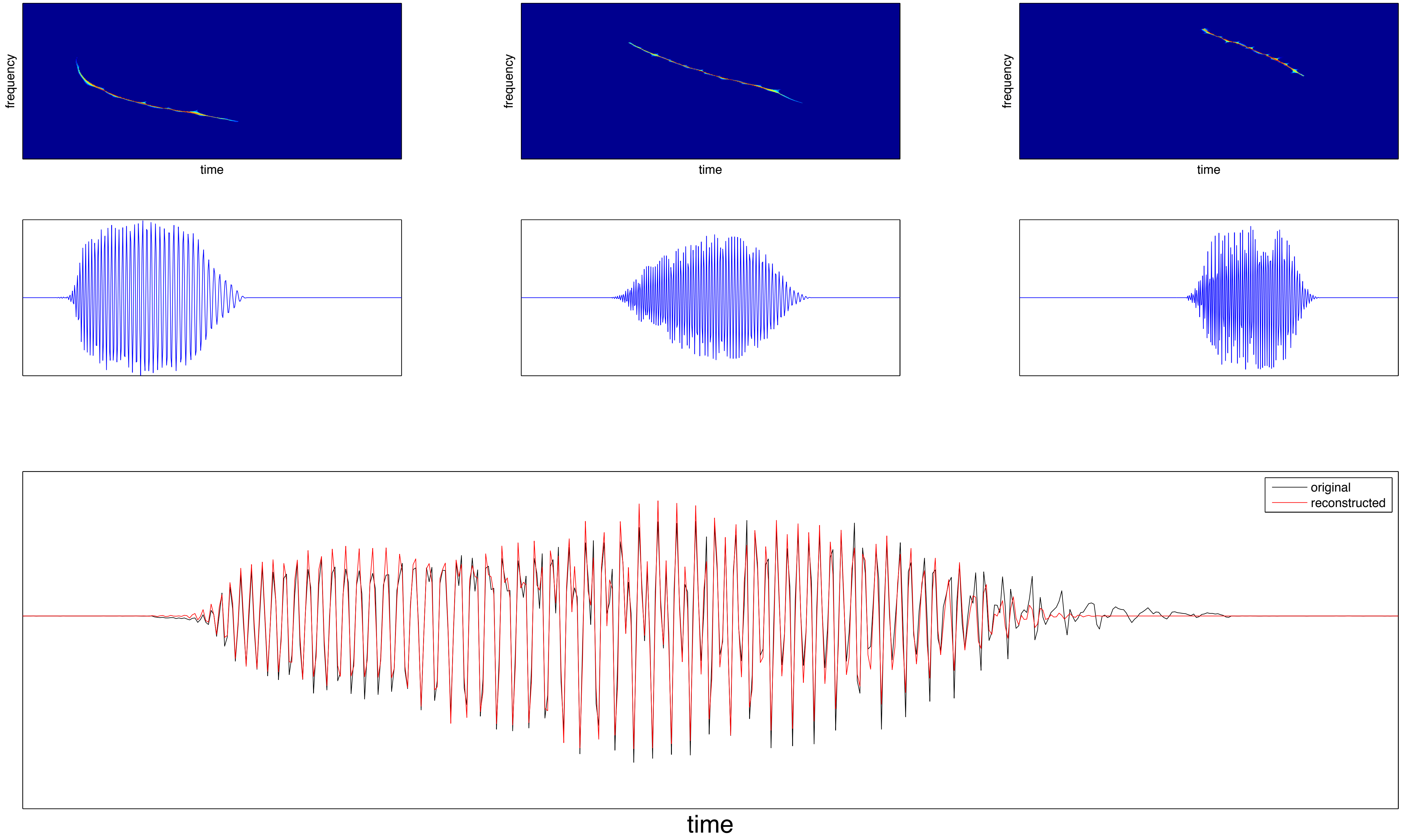


Delaunay domains





Disentangling & reconstructing



Concluding remarks

- Simplified description of a spectrogram by local extrema
- Characterization by zeros as a complement/alternative to ridges or contours
  [Delprat *et al.*, '92, Gardner *et al.*, '12]
- Geometrical (length, area, etc.) thresholds?

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