Analysis of superimposed oriented patterns via synchrosqueezing.

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Some oriented patterns synthetized by G. Peyré....

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- $A > 0$: amplitude, φ : phase
- f highly oscillating pattern : $|\nabla A|, |\nabla^2 \varphi| \ll |\nabla \varphi|$.

Introduction

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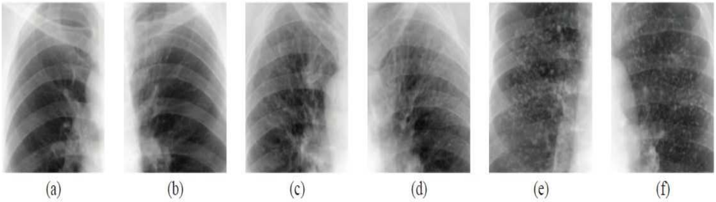
- AM–FM models = superimposition of locally parallel textures :

$$f(x_1,x_2) = \sum_{\ell} A_\ell(x_1,x_2)\cos(\varphi_\ell(x_1,x_2)),\ (x_1,x_2)\in\mathbb{R}^2\ .$$

Introduction

Applications in medical imaging for computer aided diagnotics : (for e.g. detection of pneumocosis)



Radiographs of chests ([V.Murray et al.,2009]).

Introduction

Our goal :

- Consider multi–component images.
- Extract and analyze simulteanously the different components of the image.
- Our tool : an extension to the bidimensional context of the 1D–SynchroSqueezed wavelet Transform (SST) introduced by [Daubechies et al.,2011].

The 1D–SST [Daubechies et al.,2011]

- Multicomponent signal s :

$$s(t) = \sum_{\ell=1}^{L} A_\ell(t) \cos(\varphi_\ell(t)) .$$

- Finding A_ℓ, φ_ℓ = ill–posed problem in general.

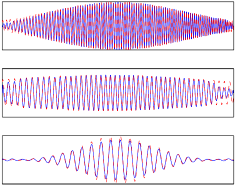
The 1D–SST [Daubechies et al.,2011]

Several methods based on wavelet analysis developed in the 90's for decomposition and demodulation of multicomponent signals :

- Reassignment method [Auger-Flandrin 1995],
- Wavelet ridges [Carmona-Hwang-Torrésani 1997, 1999],
- Squeezing method [Daubechies-Maes 1996].

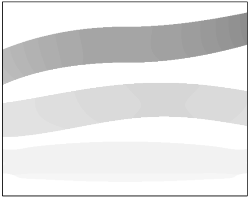
Another point of view : Empirical Mode Decomposition [Huang *et al* 1998], [Flandrin *et al* 2004].....

The 1D–SST [Daubechies et al.,2011]

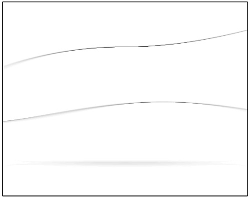


Multicomponent AM-FM signal f

Wavelet Transform $W_f(a, b)$



$\omega_f(a, b)$ (to approximate $\varphi'_\ell(b)$)



SST$= S_f(k, b)$

Principles of 1D–SST [Daubechies et al.,2011]

Based on :

- The 1D continuous wavelet transform (CWT) : ψ = wavelet, $f \in L^2(\mathbb{R})$,

$$W_f(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} f(x) \overline{\psi\left(\frac{x-b}{a}\right)} dx \ .$$

- The pointwise (or Morlet) reconstruction formula :
For f analytic (i.e. $\hat{f}$ vanishes on $\mathbb{R}_-$),

$$f(b) = \frac{1}{c_\psi} \int_0^{+\infty} W_f(a,b) \frac{da}{a^{3/2}}$$

with $c_\psi = \int_{-\infty}^{+\infty} \frac{\overline{\hat{\psi}(\xi)}}{\xi} d\xi$.

Principles of 1D–SST [Daubechies et al.,2011]

Example : $f(x) = A\cos(\omega x)$.

- CWT of its analytic signal $F(x) = A\ e^{i\omega x}$:

$$W_F(a,b) = \frac{1}{\sqrt{a}}\int_{-\infty}^{+\infty} F(x)\overline{\psi\left(\frac{x-b}{a}\right)}dx = A\sqrt{a}\ \overline{\hat{\psi}(a\omega)}\ e^{i\omega b}$$

- Then

$$\partial_b W_F(a,b) = i\omega W_F(a,b) \to \omega = -i\frac{\partial_b W_F(a;b)}{W_F(a;b)}\ ,$$

and $\quad W_F(a_0,b) = \lambda_\psi\sqrt{a_0}F(b)\quad$ where $a_0 = \dfrac{k_0}{\omega}$,

(k_0 peak wavenumber of ψ).

Principles of 1D–SST [Daubechies et al.,2011]

Extension to the general case :

- **Monocomponent complex signal** : $f(x) = A(x)\exp(i\varphi(x))$, with slowly varying A, φ (IMF).
- Candidate instantaneous frequency :

$$\omega_F(a,b) = -i\frac{\partial_b W_F(a,b)}{W_F(a,b)}, \quad \text{when} \quad |W_F(a;b)| > \varepsilon$$

- Estimate :

$$|\omega_F(a,b) - \varphi'(b)| < \varepsilon \ ,$$

with suitable conditions (C_ε) on A, φ.

- For $a\phi'(b) \equiv k_0$

$$W_F(a,b) \equiv \lambda_\psi \sqrt{a} F(b) \ ,$$

(k_0 peak wavenumber of ψ).

Principles of 1D–SST [Daubechies et al.,2011]

- **Multicomponent complex signal** $f(x)$: superposition of several IMFs assumed to be slowly varying and well separated in time-frequency domain :

$$f(x) = \sum_{\ell=1}^{L} A_\ell(x)\ e^{i\varphi_\ell(x)}$$

- SynchroSqueezed wavelet Transform (SST) [h unit window] :

$$S^\delta_{f,\varepsilon}(b,k) = \int_{|W_f(a,b)|>\varepsilon} W_f(a,b)\frac{1}{\delta}h\left(\frac{k-\mathrm{Re}(\omega_f(a,b))}{\delta}\right)\frac{da}{a^{3/2}}\ .$$

- Estimate :

$$\lim_{\delta\to 0}\frac{1}{C_\psi}\int_{\{k;\,|k-\varphi'_\ell(b)|\le\varepsilon\}} S^\delta_{f,\varepsilon}(b,k)dk = A_\ell(b)e^{i\varphi_\ell(b)} + O(\varepsilon)$$

2D extension of the analytic signal $\longrightarrow$ monogenic signal

The 2D–SST

The Hilbert Transform for unidimensional signals

- Hilbert transform associated to f, $\mathcal{H}f$

$$\mathcal{H}f(t) = \lim_{\varepsilon \to 0} \left(\frac{1}{\pi} \int_{|t-s|>\varepsilon} \frac{f(s)}{t-s} \, \mathrm{d}s \right) .$$

- Analytic (complex) signal : $F(x) = f(x) + \mathrm{i}\, \mathcal{H}\mathrm{f}(\mathrm{x})$ $\quad (\hat{F} = 0$ on $\mathbb{R}_-)$
- AM–FM analysis : $F(x) = A(x)e^{i\varphi(x)}$
  - Instantaneous amplitude : $A(x) = |F(x)|$
  - Instantaneous frequency : $\omega(x) = \varphi'(x)$

The 2D–SST

The Riesz Transform and the monogenic signal [Felsberg-Sommer 2001]

- Riesz transform associated to f, $\mathcal{R}f = \begin{pmatrix} \mathcal{R}_1 f \\ \mathcal{R}_2 f \end{pmatrix}$ with for $i = 1, 2$

$$\mathcal{R}_i f(x) = \lim_{\varepsilon \to 0} \left(\frac{1}{\pi} \int_{|x-y|>\varepsilon} \frac{(x_i - y_i)}{|x-y|^3} f(y)\,\mathrm{d}y \right) .$$

- Monogenic signal associated to f :
$\mathcal{M}f = \begin{pmatrix} f \\ \mathcal{R}f \end{pmatrix} = f + \mathrm{i}\ \mathcal{R}_1\mathrm{f} + \mathrm{j}\ \mathcal{R}_2\mathrm{f}.$
- AM–FM analysis : $\mathcal{M}f = A(x)\ \mathrm{e}^{\varphi(\mathrm{x})\mathrm{n}_\theta(\mathrm{x})}$.
  - Instantaneous amplitude : $A(x) = |\mathcal{M}f(x)|$
  - Instantaneous frequency : $\omega(x) = \nabla\varphi(x)$
  - Local orientation : θ.

Polar form of the monogenic signal

- Quaternion :

$$q = q_0 + q_1\ \mathrm{i} + q_2\ \mathrm{j} + q_3\ \mathrm{k},\ (q_0, q_1, q_2, q_3) \in \mathbb{R}^4\ ,$$

  (with
  $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1, \mathrm{ij} = -\mathrm{ji} = \mathrm{k},\ \mathrm{jk} = -\mathrm{kj} = \mathrm{i},\ \mathrm{ki} = -\mathrm{ik} = \mathrm{j}$)
- *Modulus* : $|q| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$,
- *Conjugate* : $\bar{q} = q_0 - q_1\ \mathrm{i} - q_2\ \mathrm{j} - q_3\ \mathrm{k}\ (q\bar{q} = \bar{q}q = |q|^2)$.

Polar form of the monogenic signal

- Polar form of a unit quaternion q ($|q| = 1$) s.t. $(q_1, q_2, q_3) \neq (0, 0, 0)$:

$$q = (\cos\varphi + n\ \sin\varphi) = \mathrm{e}^{\varphi n}\ ,$$

with

$$n = \frac{q_1\ \mathrm{i} + q_2\ \mathrm{j} + q_3\ \mathrm{k}}{|q_1\ \mathrm{i} + q_2\ \mathrm{j} + q_3\ \mathrm{k}|},\ \cos(\varphi) = \mathrm{Re}(q),\ \sin\varphi = |q_1\ \mathrm{i} + q_2\ \mathrm{j} + q_3\ \mathrm{k}|\ .$$

- General quaternions :

$$q = |q|\ (\cos\varphi + n\ \sin\varphi) = |q|\ \mathrm{e}^{\varphi n}\ .$$

- Monogenic signal associated to f :

$$\mathcal{M}f = \begin{pmatrix} f \\ \mathcal{R}f \end{pmatrix} = f + \mathrm{i}\ \mathcal{R}_1\mathrm{f} + \mathrm{j}\ \mathcal{R}_2\mathrm{f} = \mathrm{A(x)}\ \mathrm{e}^{\varphi \mathrm{n}_\theta}\ ,$$

with $n_\theta = \cos\theta\ \mathrm{i} + \sin\theta\ \mathrm{j}$.

Example : $f(x) = A_0 \cos(k \cdot x)$

- Riesz transform of f :

$$\mathcal{R}f(x) = A_0 \begin{pmatrix} \sin(k \cdot x)\cos\theta_0 \\ \sin(k \cdot x)\sin\theta_0 \end{pmatrix} = A_0 \frac{k}{|k|} \sin(k \cdot x) \, ,$$

with $\theta_0 = \mathrm{Arctan}(\frac{k_2}{k_1})$.

- Monogenic signal associated to f :

$$\begin{aligned} \mathcal{M}f(x) = \begin{pmatrix} f(x) \\ \mathcal{R}f(x) \end{pmatrix} &= A_0 \begin{pmatrix} \cos(k \cdot x) \\ \sin(k \cdot x)\cos\theta_0 \\ \sin(k \cdot x)\sin\theta_0 \end{pmatrix} \\ &= A_0 \, \mathrm{e}^{(k \cdot x)(\cos\theta_0 \ \mathrm{i} + \sin\theta_0 \ \mathrm{j})} \, . \end{aligned}$$

- Amplitude, phase and local orientation of f :

$$A(x) = A_0, \ \varphi(x) = k \cdot x, \ \theta(x) = \theta_0 = \mathrm{Arctan}(k_2/k_1) \, .$$

Monogenic Wavelet Transform [Olhede-Metikas 2009], [Unser-Van De Ville 2009]

- 2D directional CWT [Antoine et al., 2004]

$$c_f(a,b,\alpha) = \int_{\mathbb{R}^2} f(x)\,\overline{\psi_{a,b,\alpha}(x)}\,\mathrm{d}x\,, \;\; \psi_{a,b,\alpha}(x) = \frac{1}{a}\psi\left(r_{-\alpha}\frac{x-b}{a}\right)\,.$$

If ψ isotropic, $\psi_{a,b,\alpha} = \psi_{a,b,0} = \psi_{a,b}$ $c_f(a,b) = c_f(a,b,0)$.

- Pointwise reconstruction formula, in the isotropic case :

$$f(x) = \frac{1}{\tilde{C}_\psi}\int_0^{+\infty} c_f(a,x)\,\frac{\mathrm{d}a}{a^2} \quad \text{with} \quad \tilde{C}_\psi = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{\overline{\widehat{\psi}(\xi)}}{|\xi|^2}\,\mathrm{d}\xi\,.$$

Monogenic Wavelet Transform [Olhede-Metikas 2009] , [Unser-Van De Ville 2009]

- Isotropic CWT of the monogenic signal $F = \mathcal{M}f = f + \mathcal{R}_1 f\ \mathrm{i} + \mathcal{R}_2 f\ \mathrm{j}$:

$$c_F(a,b) = (c_f + c_{\mathcal{R}_1 f}\ \mathrm{i} + c_{\mathcal{R}_2 f}\ \mathrm{j})\,(a,b)\ .$$

- Monogenic Wavelet Transform of f if ψ real isotropic :

$$c_f^{(M)}(a,b,\alpha) = \int_{\mathbb{R}^2} f(x)\ (\mathcal{M}\psi)_{a,b,\alpha}(x)\ \mathrm{d}x\ .$$

- Link between $c_F(a,b)$ and $c_f^{(M)}(a,b,\alpha)$:

$$c_F(a,b) = \begin{pmatrix} 1 & 0 \\ 0 & -r_\alpha \end{pmatrix} c_f^{(M)}(a,b,\alpha)\ .$$

Example

- $f(x) = A_0 \cos(k \cdot x)$.
- $F(x) = A\ e^{(k \cdot x) n_\theta} \qquad k = (k_1, k_2)$.
- Isotropic wavelet transform of F :

$$c_F(a,b) = a\widehat{\psi}(ak)\left(Ae^{(k \cdot b) n_\theta}\right) .$$

Example

- For $i = 1, 2$:

$$\partial_{b_i} c_F(a,b) = k_i n_\theta \left(a\widehat{\psi}(ak) \right) \left(Ae^{(k\cdot b)n_\theta} \right) \ .$$

- Instantaneous frequency k and orientation n_θ :

$$k_2 n_\theta = \partial_{b_2} c_F(a,b) \times (c_F(a,b))^{-1}$$

- On the "ridge" $a = a_0 = \frac{|k_0|}{|k|}$

$$F(b) = \lambda_\psi a_0^{-1} \ c_F(a_0, b) \ .$$

Intrinsic Monogenic Mode Function (IMMF)

- Intrinsic Monogenic Mode Function (IMMF) with accuracy $\varepsilon > 0$:

$$F(x) = A(x)\mathrm{e}^{\varphi(x) n_{\theta(x)}} \text{ with } n_{\theta(x)} = \cos(\theta(x))\,\mathrm{i} + \sin(\theta(x))\,\mathrm{j}\;.$$

- A, φ, n_θ slowly varying functions :

$$|\nabla A(x)|, |\nabla \theta(x)|, |\nabla^2 \varphi(x)| < \varepsilon |\nabla \varphi(x)|\;.$$

  - A : local amplitude of F.
  - φ, n_θ : local scalar phase and orientation of F.
  - $\nabla\varphi$: instantaneous frequency.

Intrinsic Monogenic Mode Function (IMMF)

- Candidate to approximate the instantaneous frequency :

$$\begin{aligned} \Lambda_1(a,b) &= \partial_{b_1} c_F(a,b) \times (c_F(a,b))^{-1} \, , \\ \Lambda_2(a,b) &= \partial_{b_2} c_F(a,b) \times (c_F(a,b))^{-1} \, . \end{aligned}$$

- Estimate : for $i = 1,2$,

$$|\Lambda_i(a,b) - \partial_{b_i}\varphi(b) n_{\theta(b)}| \leq \varepsilon \quad where \quad |c_F(a,b)| > \varepsilon \, .$$

Monogenic Synchrosqueezed Wavelet Transform (MSST)

- Multicomponent signal $F(x)$ superposition of IMMFs of accuracy ε, well separated in the space-frequency domain :

$$F(x) = \sum_{\ell=1}^{L} A_\ell(x)\ \mathrm{e}^{\varphi_\ell(x) n_{\theta_\ell(x)}}$$

- MSST = local CWT-reconstruction at fixed point b, in the ε-vicinity of the estimated instantaneous frequencies (Λ_1, Λ_2) :

$$S^{\delta}_{F,\varepsilon}(b, k, n) = \int_{|c_F(a,b)|>\varepsilon} c_F(a, b) \frac{1}{\delta^2} \prod_{i=1}^{2} h\left(\frac{k_i - \mathrm{Re}(\overline{n}\ \Lambda_i(a, b))}{\delta}\right) \frac{\mathrm{d}a}{a^2}$$

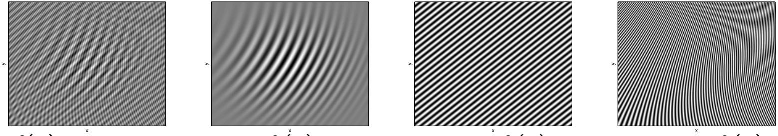
$(h \in C_c^\infty$ s.t. $\int h = 1)$.

Monogenic Synchrosqueezed Wavelet Transform (MSST)

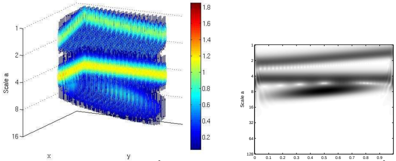
ℓ^{th}-IMMF estimate ($\hat{\psi}$ compactly supported) :

$$\lim_{\delta\to 0} \frac{2\pi}{\tilde{C}_\psi} \int_{\mathbb{S}^1} \int_{\{k;\, \max_i |k_i n - \partial_{b_i}\varphi_\ell(b) n_{\theta_\ell(b)}| \le \varepsilon\}} S^\delta_{f,\varepsilon}(b,k,n)\,\mathrm{d}k\,\mathrm{d}n$$
$$= \quad A_\ell(b)\mathrm{e}^{\varphi_\ell(b) n_{\theta_\ell(b)}} + O(\varepsilon)\ .$$

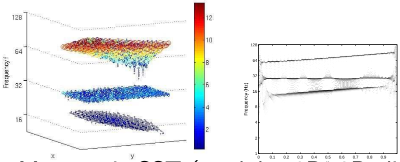
Multicomponent signal



$$f(x) = f_1(x) + f_2(x) + f_3(x)$$



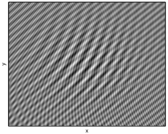
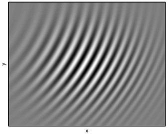
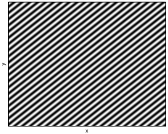
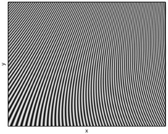
Monogenic WT (modulus 3D, 2D slice)



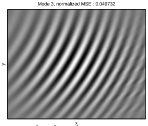
Monogenic SST (modulus 3D, 2D slice)

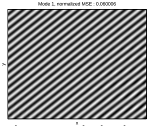
Monogenic Synchrosqueezed Wavelet Transform of Images

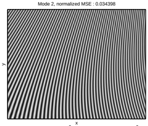
Multicomponent signal : reconstruction of modes

$f(x) \quad = \quad f_1(x) \quad + \quad f_2(x) \quad + \quad f_3(x)$



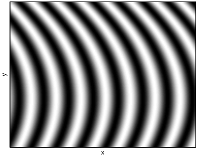




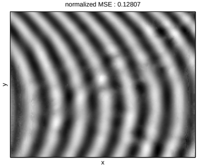
Reconstructed modes $= f_1(x)$ (MSE=0.05) $+ f_2(x)$ (MSE= 0.06) $+ f_3(x)$ (MSE=0.034)

$$\left\{ \begin{array}{lcl} f_1(x_1,x_2) & = & e^{-10((x_1-0.5)^2+(x_2-0.5)^2))}\sin(10\pi(x_1^2+x_2^2+2(x_1+0.2x_2)) \\ f_2(x_1,x_2) & = & 1.2\sin(40\pi(x_1+x_2)) \\ f_3(x_1,x_2) & = & \cos(2\pi(70x_1+20x_1^2+50x_2-20x_2^2-41x_1x_2)) \end{array} \right.$$

Extraction of AM-FM modes from a real image



$$f(x) \quad = \quad f_1(x) \quad + \quad f_2(x)$$



Reconstructed modes $= f_1(x)$ (MSE=0.31) $+ f_2(x)$ (MSE= 0.13)

Conclusion

- 2D generalization of the SynchroSqueezed wavelet Method in the monogenic Signal framework.
- MSST allows to link local orientations to instantaneous frequency : new tool for characterization of oriented textures.
- Applications on real datas, comparison with other Mode Decompositions (EMD).

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