Currents and normal cycles models for curve or surface registration. Applications to brain image analysis.

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Workshop on Image processing and applications - Marseille.

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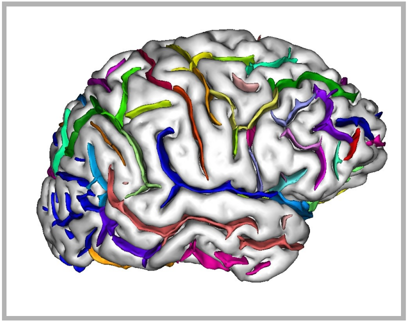
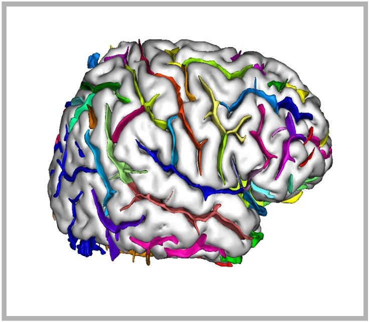
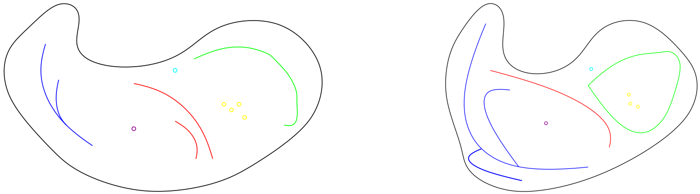
Brain has a complex and variable geometry





Comparison and matching of geometrical structures

- Let $\mathtt{s}_1, \dots, \mathtt{s}_m$ and $\mathtt{t}_1, \dots, \mathtt{t}_m$ be two lists of geometric features living in the ambient space $\Omega \subset \mathbf{R}^d$. We aim at comparing these objects by detecting "similar" features located at "similar" positions in both objects.



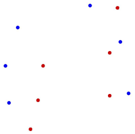
- One way is to estimate a deformation map $\phi : \Omega \to \Omega$ which will try to find spatially coherent matchings between parts of the objects. This can be formulated as a variational problem:

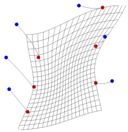
$$J(\phi) = \gamma E(\phi) + A_1(\phi.\mathtt{s}_1, \mathtt{t}_1) + \cdots + A_m(\phi.\mathtt{s}_m, \mathtt{t}_m),$$

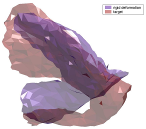
where $E(\phi)$ is the deformation cost (which typically evaluates the regularity of the deformation map),

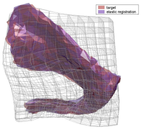
Comparison and matching of geometrical structures

$\phi.\mathtt{s}_i$ is the object $\mathtt{s}_i$ transported via the deformation, and $A(\phi.\mathtt{s}_i, \mathtt{t}_i)$ is a measure of dissimilarity between the matched objects.









Groups of elastic deformations

- Deformations are obtained by **integrating** a family $(v_t), t \in [0,1]$ **of vector fields**:
$$\phi_t^v(x) = \int_0^t v_s \circ \phi_s^v(x) ds.$$
- $(v_t)_{t\in[0,1]} \in L^2([0,1], V)$, where V is a Hilbert space of vector fields continuously embedded in the space of C^1 vector fields.
- $\mathcal{A}_V = \{\phi_1^v, v \in L^2([0,1], V)\}$ **group of diffeomorphisms**, with the metric $d_V(\mathrm{id}, \phi) = \inf_v \left\{ \sqrt{\int_0^1 \|v_t\|_V^2 dt}, \phi_1^v = \phi \right\}$, with the right-invariance rule : $d_V(\phi, \psi) = d_V(\mathrm{id}, \psi \circ \phi^{-1})$.

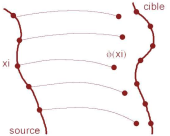
Reproducing kernel and reduction theorem

- When evaluation functionals $\delta_x^\alpha : v \mapsto \alpha \cdot v(x)$ are continuous in V, V has a reproducing kernel $k_V : (\mathbf{R}^d)^2 \to \mathcal{L}(\mathbf{R}^d)$ defined by

$$\langle v, k_V(x, \cdot)\alpha \rangle_V = \delta_x^\alpha(v) = \alpha \cdot v(x).$$

- **Reduction theorem** For a given matching problem, if the data attachment term A depends only on the images of a finite number of points x^i, then the optimal solution will take the form

$$v_t = \sum_{i=1}^{n} k_V(x_t^i, \cdot)\alpha_t^i$$



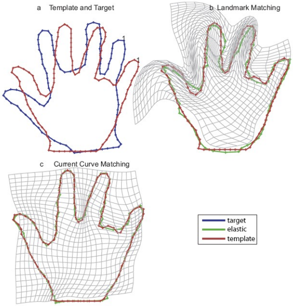
where $x_t^i = \phi_t^v(x^i)$ (points trajectories).
The **momentum vecteurs** α_t^i become the minimization variables.
- Then we have

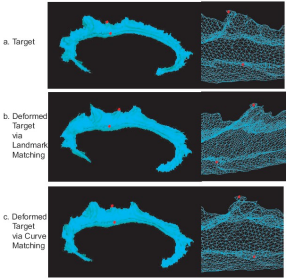
$$\|v_t\|_V^2 = \sum_{i,j=1}^{n} \alpha_t^j \cdot k_V(x_t^i, x_t^j)\alpha_t^i.$$

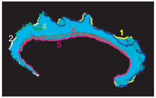
Labeled points (landmark matching)

$$\mathrm{s} = (x_i)_{1 \leq i \leq n}, \quad \mathrm{t} = (y_i)_{1 \leq i \leq n}, \qquad A(\phi.\mathrm{s}, \mathrm{t}) = \sum_{i=1}^{n} |y_i - \phi(x_i)|^2.$$

Not suitable as soon as points are unlabeled and/or are sampled from curves or surfaces.







Unlabelled point sets

- Model a point set $\{x_i\}_{i=1}^n$ as the measure
$$\mu = \sum_{i=1}^n \delta_{x_i}.$$
- Define a metric between measures by choosing a functional space H such that $\forall x, \delta_x \in H^*$ and taking the dual norm:
$$\|\mu\|_{H^*} = \mathrm{Sup}\{\mu(f), \|f\|_H \leq 1\} \qquad \text{with } \mu(f) = \sum_{i=1}^n f(x_i).$$
Then define $A(\phi.\mathtt{s}, \mathtt{t}) := \|\mu_{\mathtt{s}} - \mu_{\mathtt{t}}\|_{H^*}^2$.
- When H is a Hilbert space, the metric writes in terms of the **reproducing kernel** of H:
$$\|\mu\|_{H^*}^2 = \left\| \sum_{i=1}^n \delta_{x_i} \right\|_{H^*}^2 = \sum_{i=1}^n \sum_{j=1}^n k_H(x_i, x_j)$$

Unlabelled weighted point sets

- Model a weighted point set $\{(a_i, x_i)\}_{i=1}^n \subset (\mathbf{R} \times \Omega)^n$ as the measure

$$\mu = \sum_{i=1}^n a_i \delta_{x_i}.$$

- Define a metric between measures by choosing a functional space H such that $\forall x, \delta_x \in H^*$ and taking the dual norm:

$$\|\mu\|_{H^*} = \text{Sup}\{\mu(f), \|f\|_H \leq 1\} \qquad \text{with } \mu(f) = \sum_{i=1}^n a_i f(x_i).$$

  Then define $A(\phi.\mathtt{s}, \mathtt{t}) := \|\mu_\mathtt{s} - \mu_\mathtt{t}\|_{H^*}^2$.
- When H is a Hilbert space, the metric writes in terms of the **reproducing kernel** of H:

$$\|\mu\|_{H^*}^2 \quad = \quad \Big\| \sum_{i=1}^n a_i \delta_{x_i} \Big\|_{H^*}^2 = \sum_{i=1}^n \sum_{j=1}^n a_i a_j k_H(x_i, x_j)$$

Unlabelled vector-weighted point sets

- For any finite-dimensional vector space E, model a vector-weighted point set $\{(\eta_i, x_i)\}_{i=1}^n \subset (E \times \Omega)^n$ as the vector-valued measure

$$\vec{\mu} = \sum_{i=1}^{n} \eta_i \delta_{x_i}.$$

- Choose a functional space W of E^*-valued vector fields such that $\forall x, \forall \eta, \quad \eta\delta_x \in W^*$ and take the dual norm:

$$\|\vec{\mu}\|_{W^*} = \mathrm{Sup}\{\vec{\mu}(f), \|f\|_W \leq 1\} \qquad \text{with } \vec{\mu}(w) = \sum_{i=1}^{n} \langle\, f(x_i) \mid \eta_i \,\rangle\,.$$

  Then define $A(\phi.\mathtt{s}, \mathtt{t}) := \|\vec{\mu_{\mathtt{s}}} - \vec{\mu_{\mathtt{t}}}\|_{W^*}^2$.
- When W is a Hilbert space, the metric writes in terms of the **reproducing kernel** of W:

$$\|\vec{\mu}\|_{W^*}^2 = \left\| \sum_{i=1}^{n} \eta_i \delta_{x_i} \right\|_{W^*}^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \langle\, k_W(x_i, x_j)\eta_j \mid \eta_i \,\rangle$$

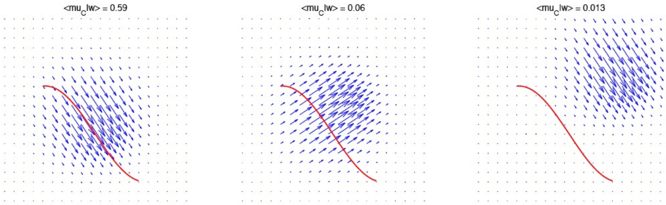
Curves as measure or currents

- Let C be a curve in $\Omega \subset \mathbf{R}^d$, parametrized by $\gamma_C : [0,1] \to \Omega$. The **uniform measure** associated to C is the following linear form, defined by its action on test **functions** $f : \Omega \to \mathbf{R}$:

$$\mu_C(f) = \int_0^1 f(\gamma_C(s)) \|\gamma_C'(s)\| ds.$$

- The **current** associated to C is the following linear form, defined by its action on test **1-forms** $\omega : \Omega \to (\mathbf{R}^d)^*$:

$$\vec{\mu}_C(\omega) = \int_0^1 \left\langle\, \omega(\gamma_C(s)) \;\middle|\; \gamma_C'(s) \,\right\rangle ds.$$



Submanifolds as currents

Let S be an oriented and bounded m-submanifold in Ω, and (U, ψ) a local map of S.

- The **uniform measure** μ_S is defined for every function f which support is included in $\psi(U)$ by:

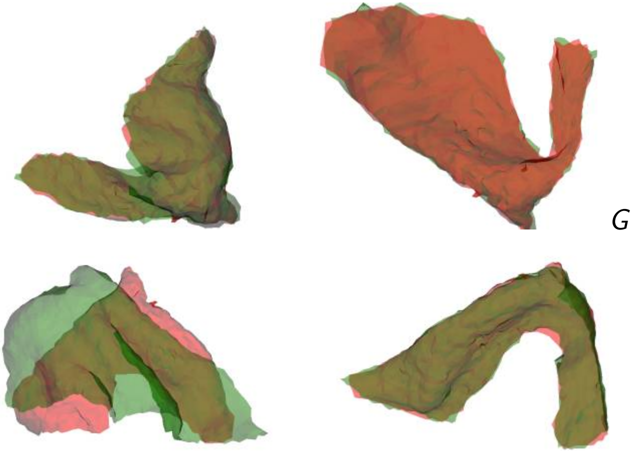
$$\mu_S(f) = \int_U f(\psi(x)) \left\| \frac{\partial \psi}{\partial x^1} \wedge \cdots \wedge \frac{\partial \psi}{\partial x^m} \right\| dx.$$

- The **current** $\vec{\mu}_S$ is defined for every m-form ω which support is included in $\psi(U)$ by:

$$\vec{\mu}_S(\omega) = \int_U \left\langle \omega(\psi(x)) \,\middle|\, \frac{\partial \psi}{\partial x^1} \wedge \cdots \wedge \frac{\partial \psi}{\partial x^m} \right\rangle dx.$$

- We extend these definitions to global test functions or m-forms with the use of a partition of unity.

Submanifolds as currents

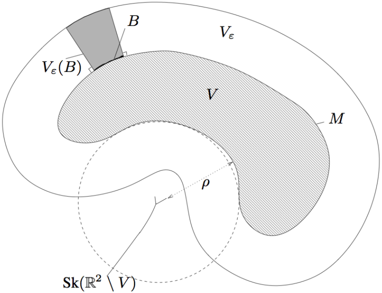


Curves as measure or currents — Properties

- Both models can handle changes in topology between shapes (e.g. one can compare and match a closed curve to an open one)
- The currents model is a priori more complete since it encodes both location and tangential information of the curves. One may think about it as a first-order model, while the measure model is zero-order.
- As a counterpart currents require to define an orientation on each curve, and on each subpart of the curve when one has to deal with disconnected or branching curves.
- Due to this orientation sensitivity, specific parts like spikes in curves are filtered out in the currents model. Depending on the application this can be seen as a good or bad property.

Curves as measure or currents — Properties

Tube formula and curvature measures



- For a set $V \in \mathbf{R}^d$ such that $M = \partial V$ is smooth, the volume of the ε-offset V_ε is a polynomial in ε which coefficients give integrals of curvatures of $M = \partial V$ when ∂V is smooth.
  - ex: in $\mathbf{R}^3$,

$$Vol(V_\varepsilon) = Vol(V) + Area(M)\varepsilon + H(M)\frac{\varepsilon^2}{2} + G(M)\frac{\varepsilon^3}{3},$$

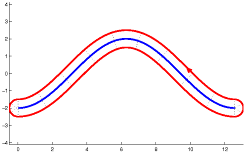
where $H(M)$ and $G(M)$ are the integrals of mean and Gauss curvatures.

Tube formula and curvature measures

- This formula can be localized so that we get integrals of curvatures restricted to any Borel subset.
- If V is only assumed to be of positive reach, $Vol(V_\varepsilon)$ (and its localized version) is still a polynomial in ε; hence its coefficients define curvature measures in this general setting.

Definitions

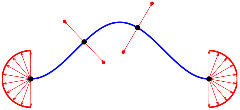
- ε**-offset** around a compact set $C \subset \mathbf{R}^d$: $C_\varepsilon = \{x \in \mathbf{R}^d, d(x, C) \leq \varepsilon\}$.

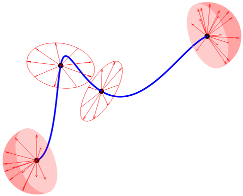


- **Normal cone** at $x \in C$:

$$\hat{\mathcal{N}}(C, x) = \{u \in \mathbf{R}^d, \exists\varepsilon > 0, \forall y \in C \cap B(x, \varepsilon), \langle x - y \, , u\rangle \leq 0\}.$$

- **Unit normal vectors** at $x \in C$: $\mathcal{N}(C, x) = \hat{\mathcal{N}}(C, x) \cap S^{d-1}$.



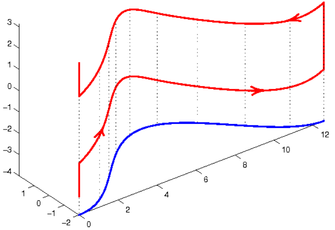


Definitions

- **Unit Normal bundle** associated to a set:

$$\mathcal{N}(C) = \{(x, \xi) \in C \times S^{d-1}, \xi \in \mathcal{N}(C, x)\}.$$

- Formally, we can see $\mathcal{N}(C)$ as the "derivative" of C_ε at $\varepsilon = 0$.
- $\mathcal{N}(C)$ is a closed sub-manifold of dimension $d-1$ in $\mathbf{R}^d \times S^{d-1}$.
- The **normal cycle** associated to C is the current $\vec{\mu}_{\mathcal{N}(C)}$ associated to $\mathcal{N}(C)$ (which is canonically oriented).

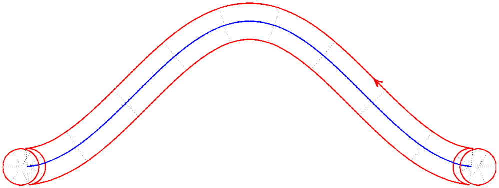


The addition formula

- For any subsets C_1, C_2, whenever it has sense,

$$\vec{\mu}_{\mathcal{N}(C_1 \cup C_2)} = \vec{\mu}_{\mathcal{N}(C_1)} + \vec{\mu}_{\mathcal{N}(C_2)} - \vec{\mu}_{\mathcal{N}(C_1 \cap C_2)}.$$

- This allows to extend the definition of normal cycles to any finite union of smooth curves (in fact to any finite union of sets of "positive reach")
- We can even define the normal cycle of a curve deprived of its end-points by simply substracting the normal cycles associated to them - which correspond to circles.



Properties

- The normal cycle is a second-order model; it encodes curvature information of the set. By computing specific integrals of the normal cycle over a small area, one gets the exact integrated values of the curvature of C on this area.
- The normal cycle does not depend on any choice of orientation on the curve, and there is no need to specify any,
- Since "spikes" are parts of high curvature; they get highly weighted in the model.
- Normal cycles are in fact a model for subsets of $\mathbf{R}^d$ and not for submanifolds of a specific dimension. Hence one can think about comparing a curve with a surface, or to model "hybrid" objects.

Designing Hilbert norms for normal cycles

- Since $\vec{\mu}_{\mathcal{N}(C)}$ is a current in the product space $\mathbf{R}^d \times S^{d-1}$, we need to define a kernel in $\mathbf{R}^d \times S^{d-1}$. This can be done by considering a product of two kernels:

$$k(\mathbf{x}, \mathbf{y}) = k((x, u), (y, v)) = k_p(x, y) k_n(u, v),$$

  where $k_p(x, y)$ is a reproducing kernel in $\mathbf{R}^d$ (e.g. $k_p(x, y) = \frac{1}{1+\|x-y\|^2/\sigma^2}$), and $k_n(u, v)$ is a reproducing kernel in S^{d-1} (e.g. the kernel given by a Sobolev metric on S^{d-1})
- Let $T(\mathbf{x}) = \tau_1(\mathbf{x}) \wedge \cdots \wedge \tau_{d-1}(\mathbf{x})$, where $(\tau_i(\mathbf{x}))_{1 \leq i \leq d-1}$ is an orthonormal basis of the tangent space $T_{\mathbf{x}}\mathcal{N}(C)$ for any $\mathbf{x} \in \mathcal{N}(C)$. Then we have

$$\|\vec{\mu}_{\mathcal{N}(C)}\|^2_{W^*} = \int_{\mathcal{N}(C)} \int_{\mathcal{N}(C)} k(\mathbf{x}, \mathbf{y}) \langle T(\mathbf{x}) \, , \, T(\mathbf{y}) \rangle \; d\sigma_{\mathcal{N}(C)}(\mathbf{x}) \; d\sigma_{\mathcal{N}(C)}(\mathbf{y}),$$

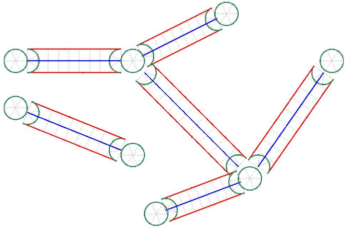
  where $d\sigma_{\mathcal{N}(C)}(\mathbf{x})$ is the volume element on the submanifold $\mathcal{N}(C)(\mathbf{x})$

Implementation for piecewise linear curves

- Let C be a piecewise linear curve, which we look at as a collection of segments which may be connected at their end-points.
- We can further decompose C as the disjoint union of open segments S_i and points P_j. The additive property for normal cycles then writes

$$\vec{\mu}_{\mathcal{N}(C)} = \sum_i \vec{\mu}_{\mathcal{N}(S_i)} + \sum_j \vec{\mu}_{\mathcal{N}(P_j)}.$$

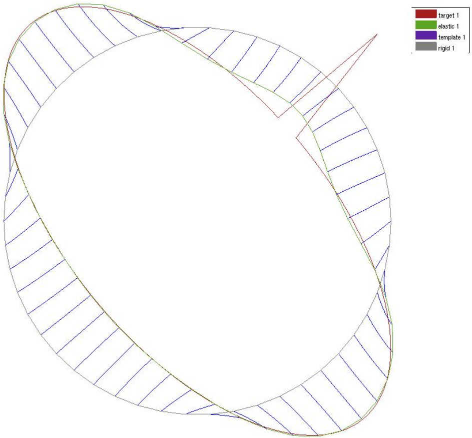
- We decompose further again into space and angular components by writing each $\vec{\mu}_{\mathcal{N}(S_i)}$ as a sum of three terms. The tangent spaces of these space and angular components are orthogonal.



Implementation for piecewise linear curves

- Hence the whole squared dual norm of $\vec{\mu}_{\mathcal{N}(C)}$ can be computed as a sum of two parts, one involving only scalar products between "space" elements (located on edges) and the other involving only scalar products between "angular" elements (located on vertices).
- The "space" part of the metric is very similar to the usual metric on currents, except that it is an orientation-free representation of curves. To compute the scalar product between two such elements we use the same approximation by vector-valued Dirac located at the center of each edge.
- For the angular part computations comes down to computing double integrals of k_n over half-spheres in S^{d-1}; which can be computed either analytically (for $d = 2$) or via precomputing look-up tables.

Experiments



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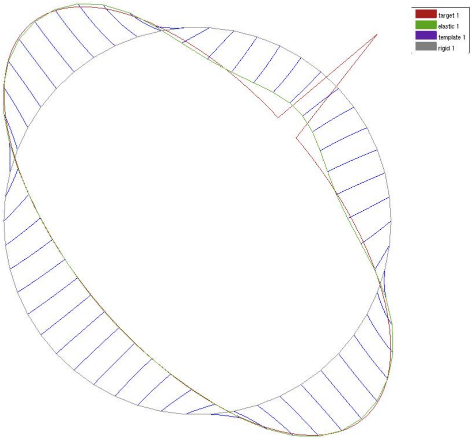

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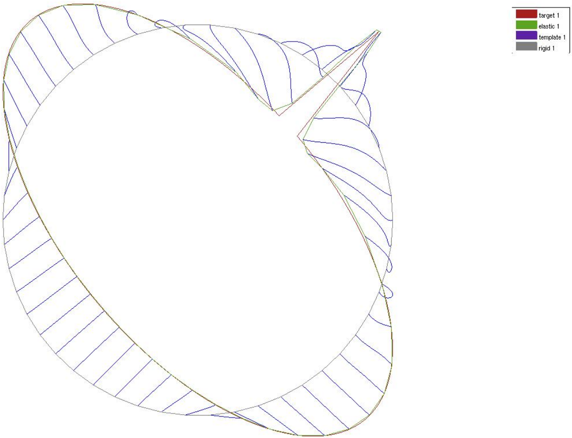

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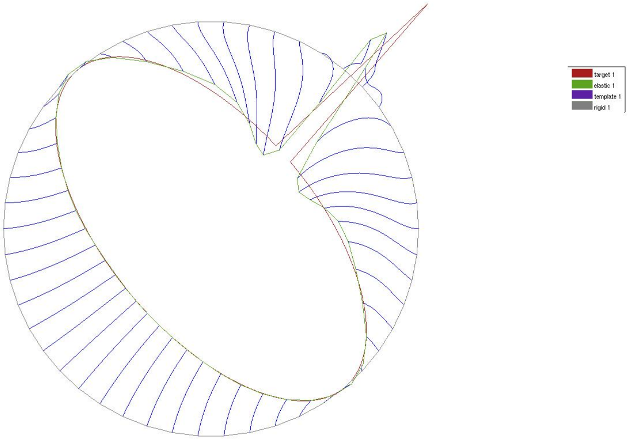
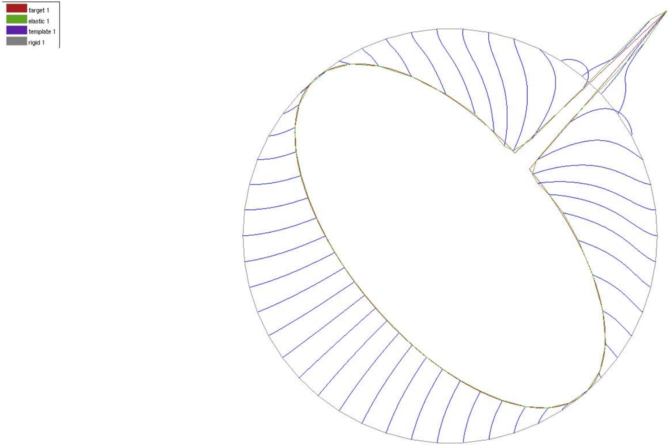
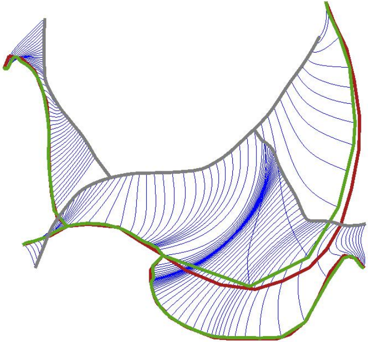


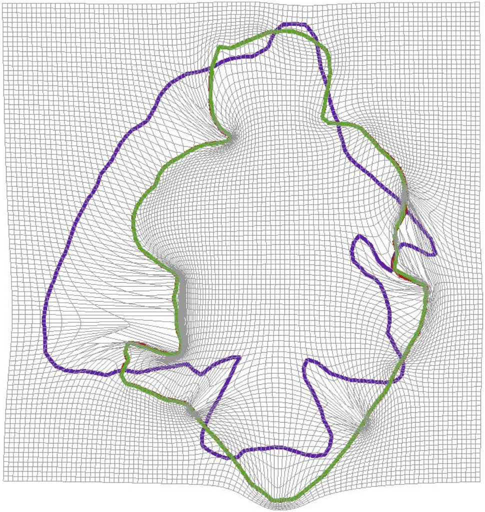
Experiments



Experiments



Experiments



Template estimation

For the analysis of a population of N individuals, we aim at computing a mean shape. How can we define such a template ?

- Forward model:

$$\vec{\mu}_{S_k} = (\phi_k)_\sharp \vec{\mu} + \vec{\varepsilon}_k$$

where $S_1, \ldots, S_N$ are the observed curves/surfaces, ϕ_k are unknown deformations and $\vec{\mu}$ the unknown template in W^*.

- Backward model:

$$\vec{\mu} = (\phi_k)_\sharp \vec{\mu}_{S_k} + \vec{\varepsilon}_k$$

- Metamorphoses (in the framework of large deformations): distribute the noise along the time-dependant diffeomorphism:

$$\vec{\mu}_{S_k} = \vec{\mu}_k(t),$$

where $\vec{\mu}_k : [0,1] \to W^*$ is a path such that $\vec{\mu}_k(0) = \vec{\mu}$, and

$$\frac{\partial \vec{\mu}_k(t)}{\partial t} = (v_k(t))_\sharp \vec{\mu}_k(t) + \vec{\varepsilon}_k(t).$$

Template estimation (with Sarang Joshi)

From a dataset of N shapes, we want to define a mean shape which will be used as template for any group analysis. How can we define such a template ?

- for images: one can define a co-registration model to estimate a template image and the n deformations of this individual to the template [Lorenzen, Davis, Joshi, MICCAI 2005]:

$$J\{(\phi_i), I\} = \gamma \sum_{k=1}^{n} E(\phi_k) + \left\| I_k \circ \phi_k^{-1} - I \right\|_2^2 .$$

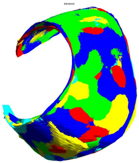
- for curves or surfaces, one can write a co-registration model by averaging in the space of measures or currents:

$$J\{(\phi_i), \hat{\vec{\mu}}\} = \gamma \sum_{k=1}^{n} E(\phi_k) + \|\phi_{k,\sharp}\vec{\mu}_k - \vec{\mu}\|_{W^*}^2 .$$

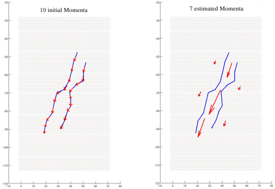
$$J\{(\phi_i)\} = \gamma \sum_{k=1}^{n} E(\phi_k) + \left\| \phi_{k,\sharp}\vec{\mu}_k - \frac{1}{n}\sum_{k=1}^{n} \phi_{k,\sharp}\vec{\mu}_k \right\|_{W^*}^2 .$$

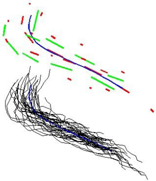
Template estimation (with Sarang Joshi)

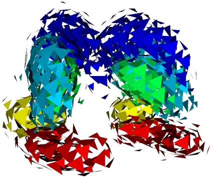
- The estimated template $\frac{1}{n}\sum_{k=1}^{n}\hat{\phi}_k\mu_k$ is not a curve nor a surface but it can be used as is to perform registrations of new individuals to the template.



- if needed, one can further approximate the template via matching pursuit (Stanley Durrlemann):

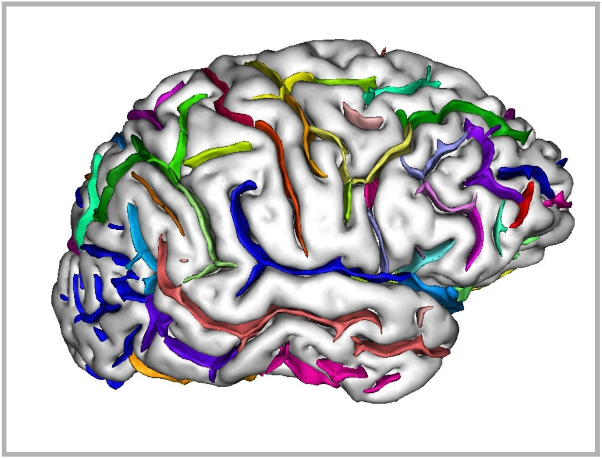




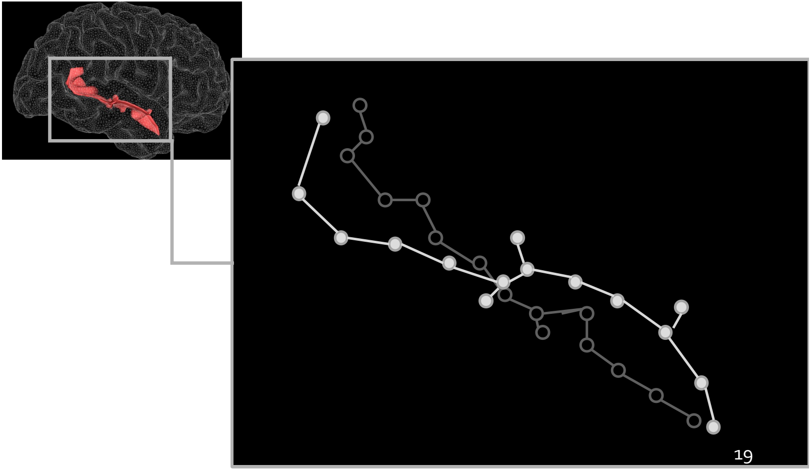


DIffeomorphic Sulcus-based COrtical registration

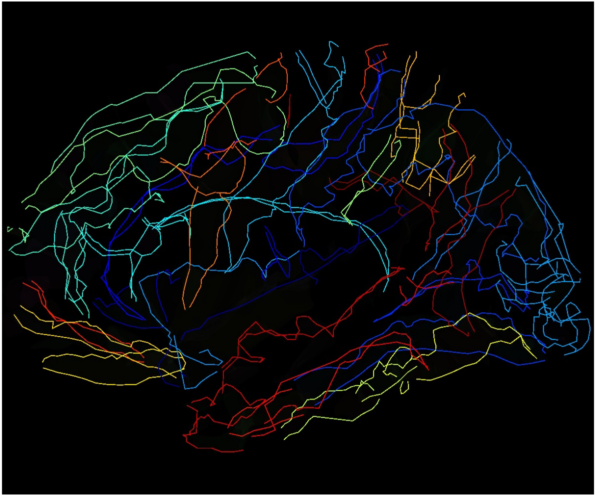
DISCO: automatic sulcal extraction and labelling



DISCO: Preliminary extraction of sulcal features



DISCO: individual sulcal imprint



DISCO : diffeomorphic group averaging