L^2 Optimal Transport via Dual Convex Programming

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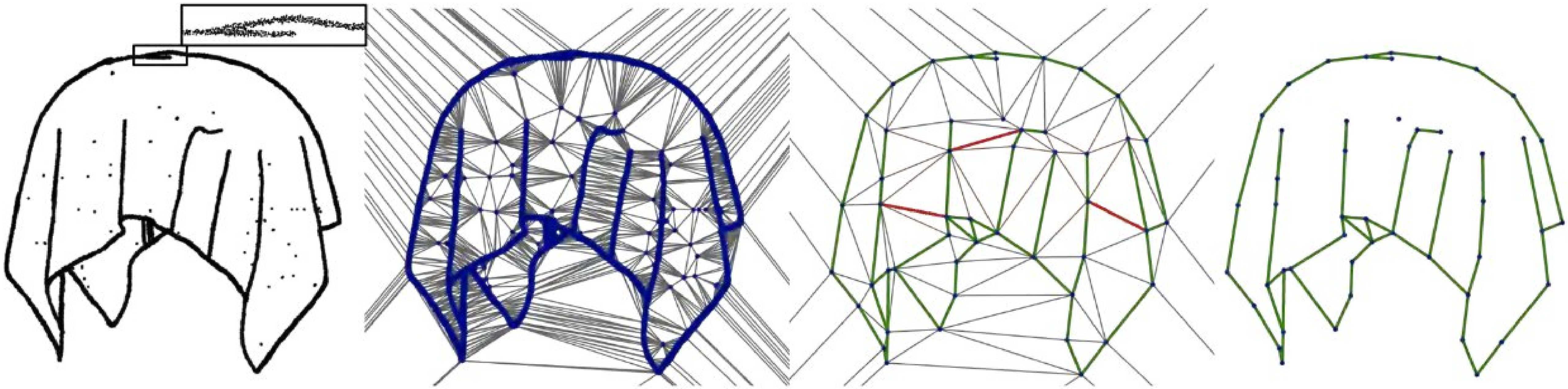
Journées de traitement d'image
Marseille
24 décembre 2011

Motivations

- Reconstructions with sharp corners and boundaries.

De Goes-Cohen-Steiner-Alliez-Desbrun (SGP '11)



Given a point sample, reconstruct the underlying object as a sub-graph of a triangulation minimizing an energy related to OT.

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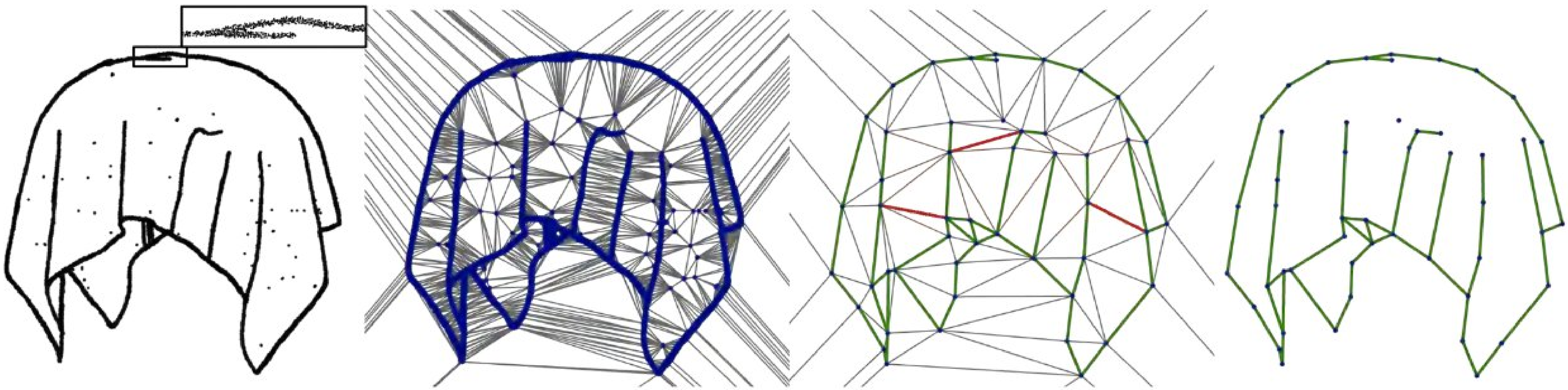
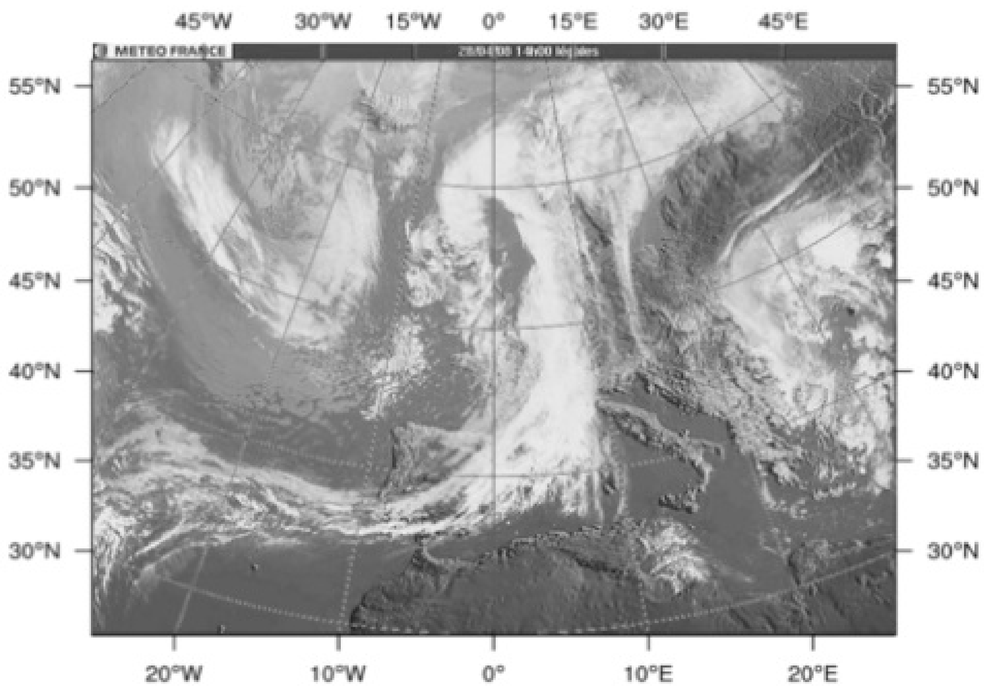


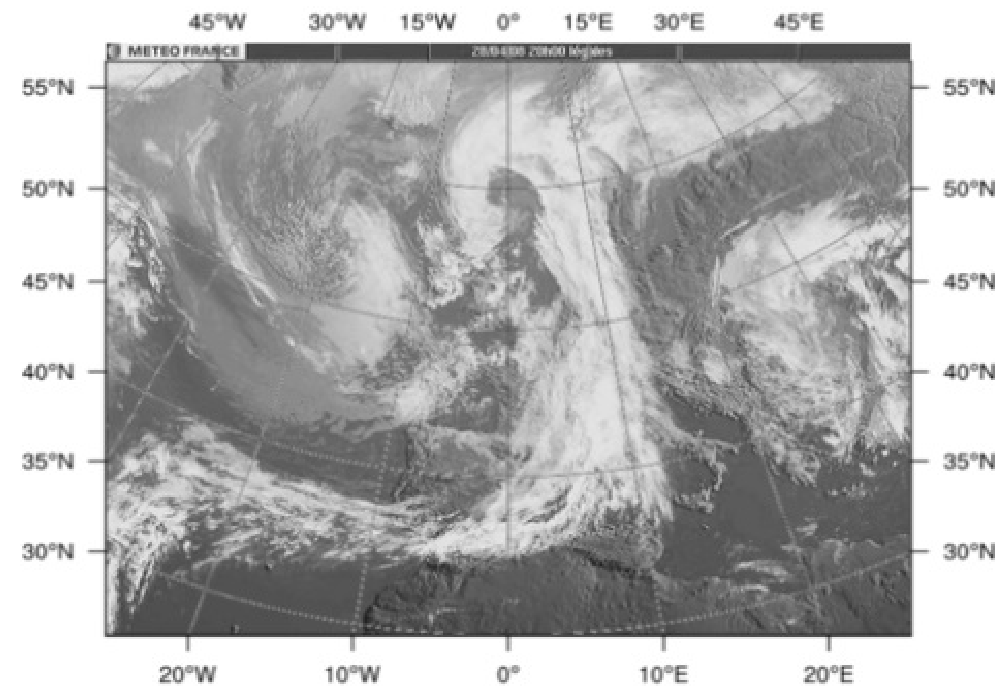
Given a point sample, reconstruct the underlying object as a sub-graph of a triangulation minimizing an energy related to OT.

Main ingredient for a 3D version: compute the L^2 optimal transport between the uniform measures ν on N points in the plane and μ on a triangle.

Motivations

- Distance between grayscale images representing a density.





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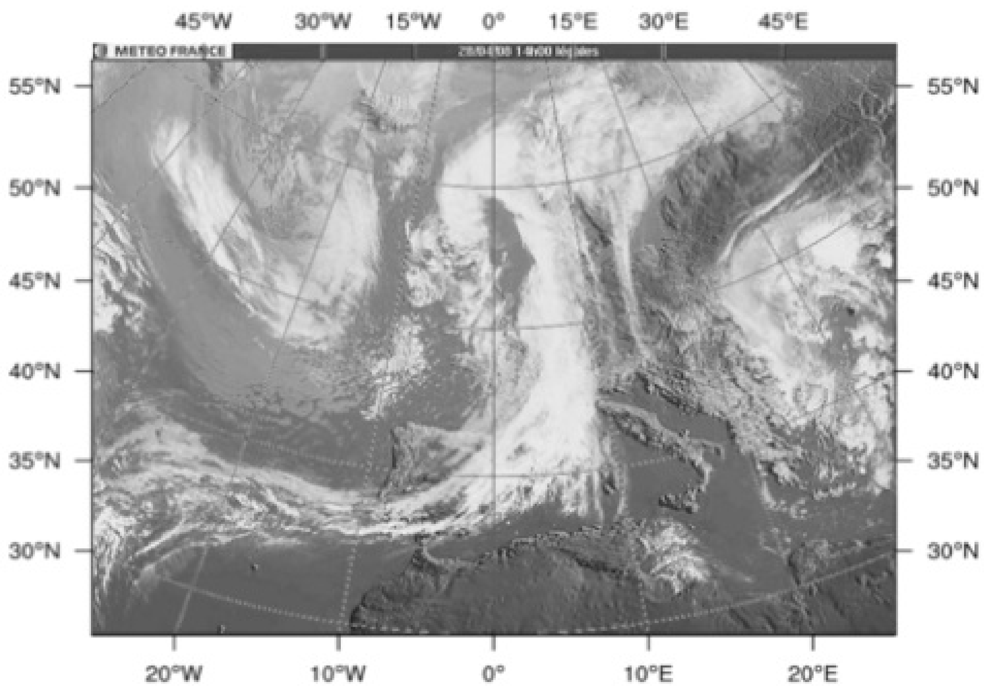

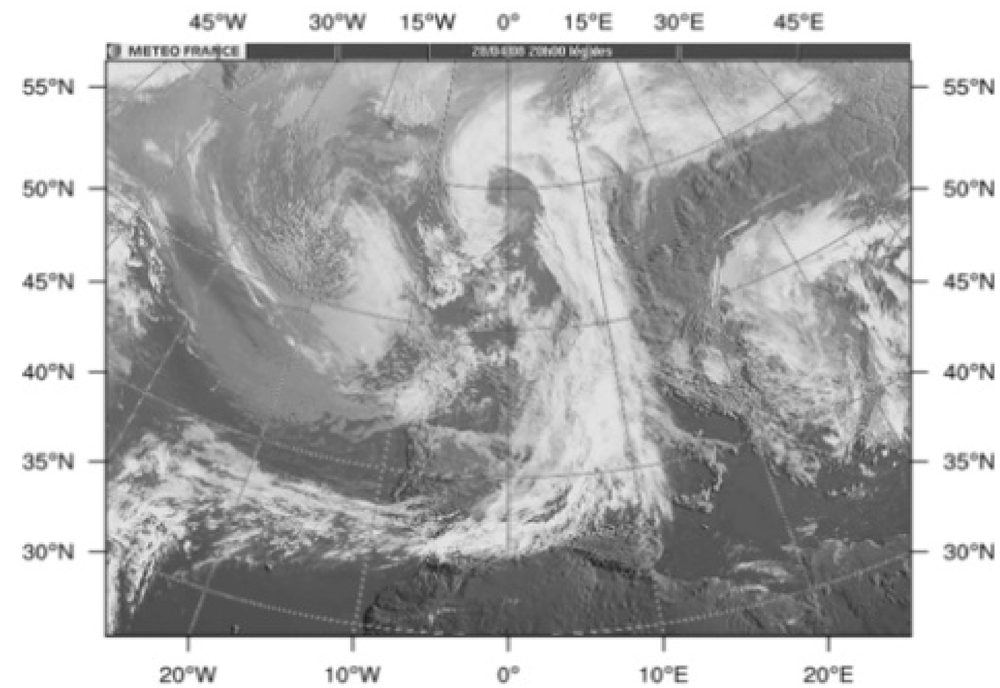

Meaningful distances between such images can be of the form:

$$E(\rho,\sigma) = \min_T \int \|x - T(x)\|^2 \rho(x)\,\mathrm{d}\,x + \mathrm{E}_{\mathrm{regu}}(T) \text{ where } T_{\#}\rho = \sigma.$$

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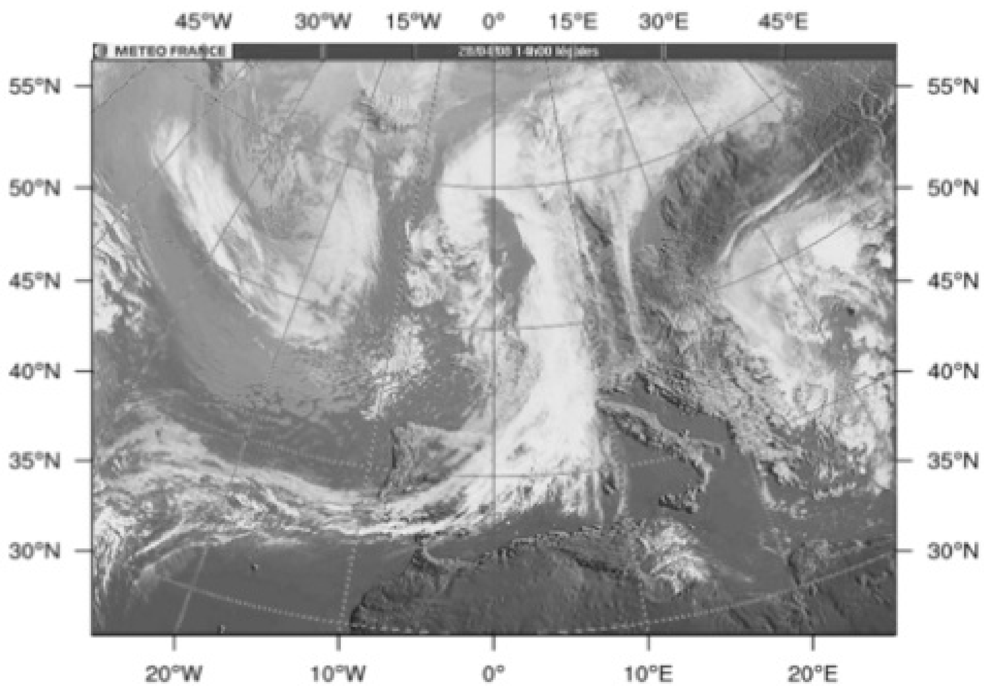

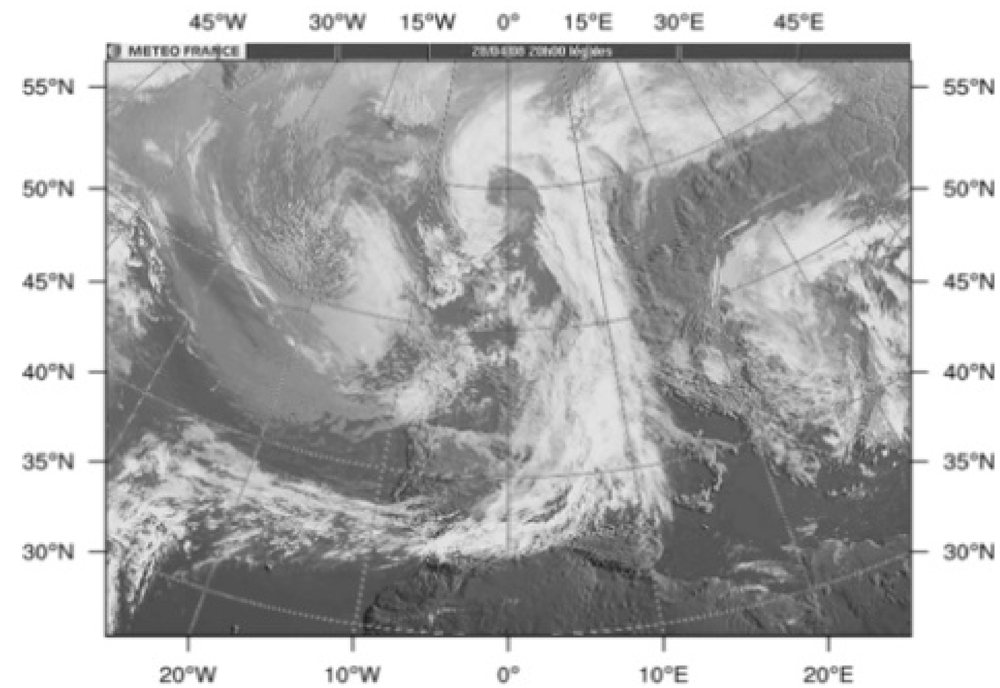

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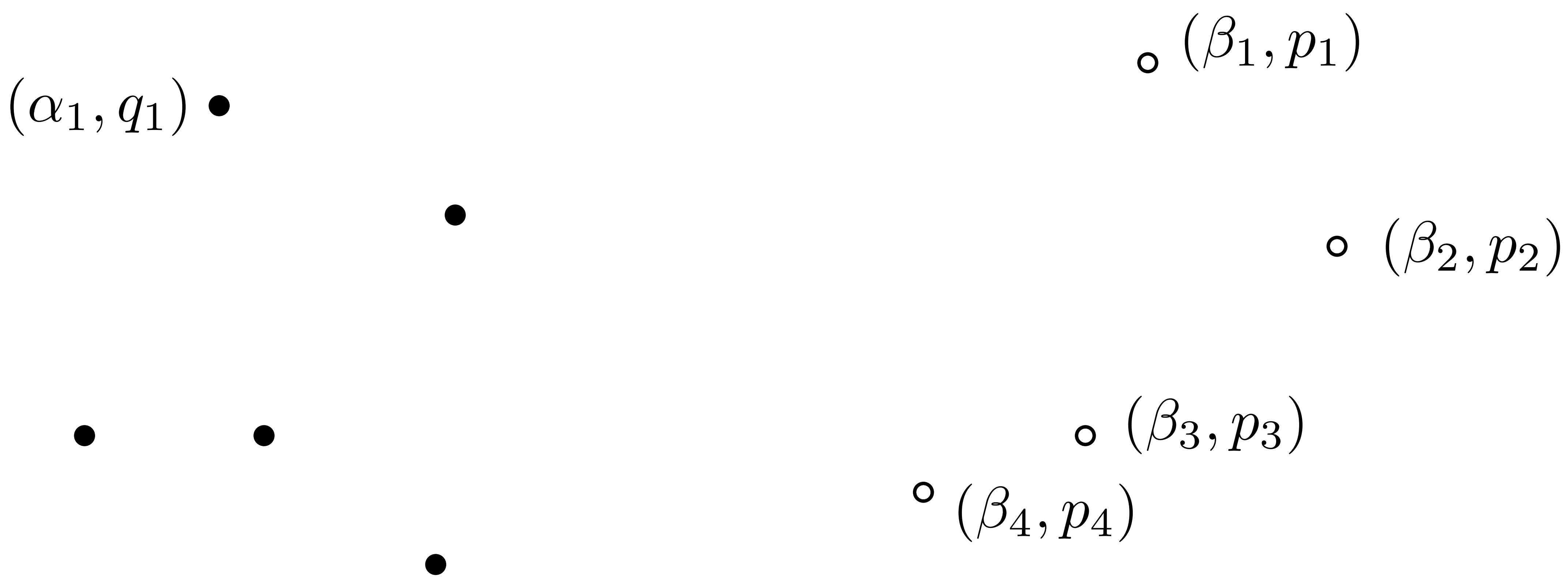
ANR project TOMMI (LJK / MAP5)

0. L^2 Optimal Transport

L^2 Optimal Transport

Source measure μ **Target measure ν**



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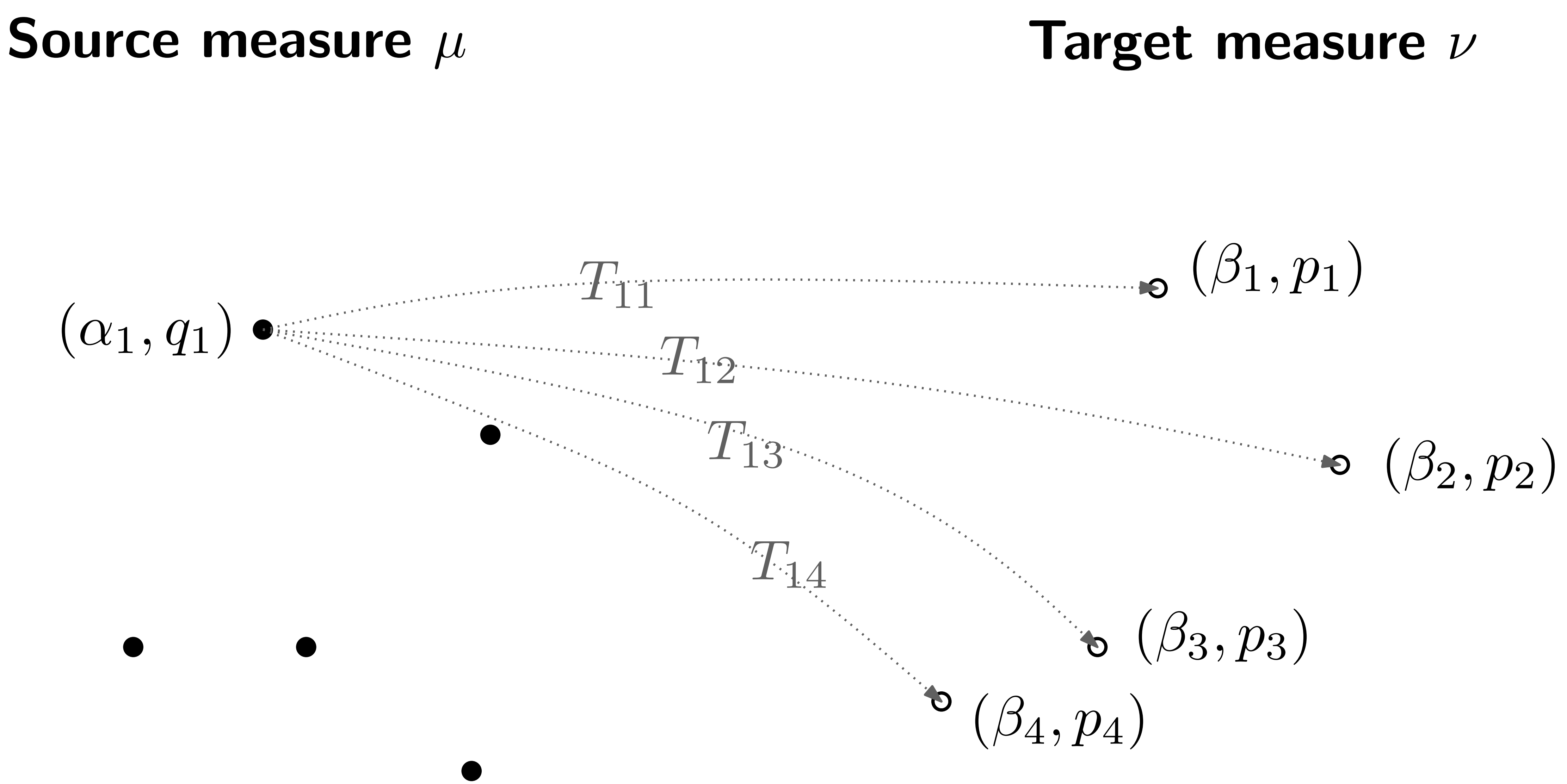

Transport plan: a matrix (T_{ij}) satisfiying $\sum_i T_{ij} = \alpha_i$ and $\sum_j T_{ij} = \beta_i$.

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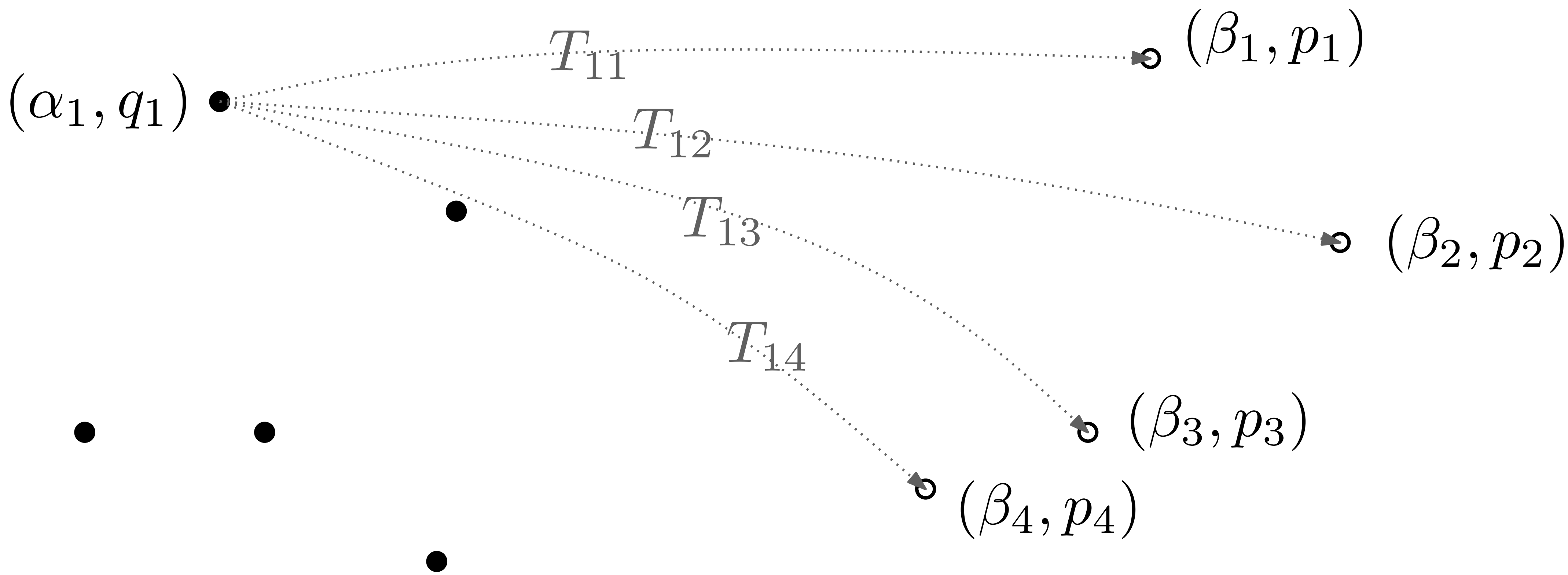

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Cost: $c(T) = \sum_{i,j} T_{ij} \|q_i - p_j\|^2$.

Wasserstein: $\mathrm{W}_2(\mu, \nu) := (\min_T c(T))^{1/2}$

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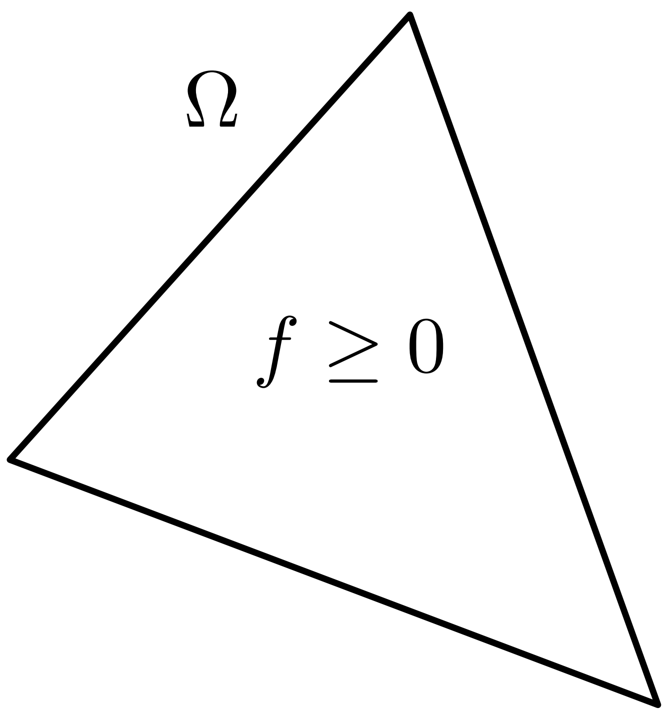
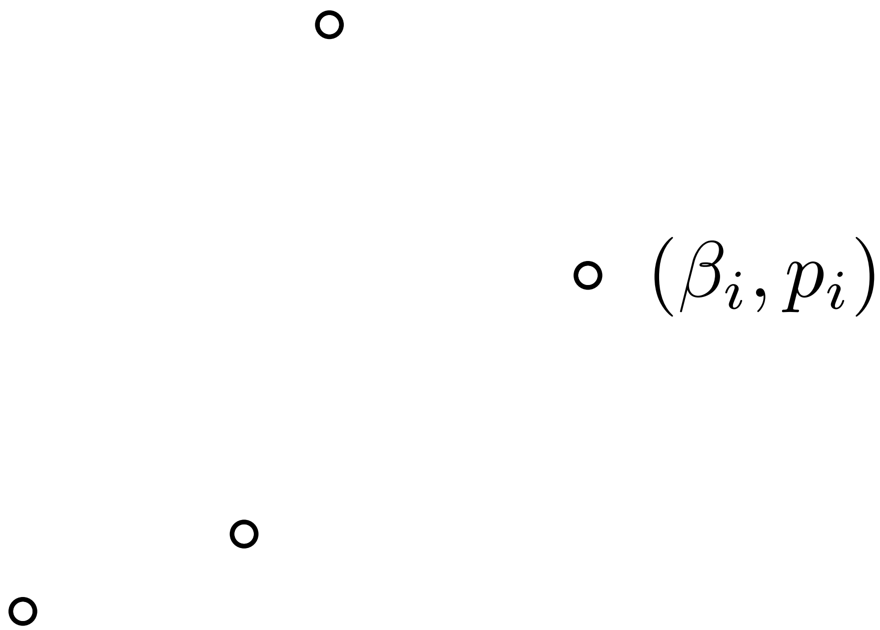


$\mu(A) = \int_{\Omega \cap A} f(x) \, \mathrm{d}\, x$

Target measure ν



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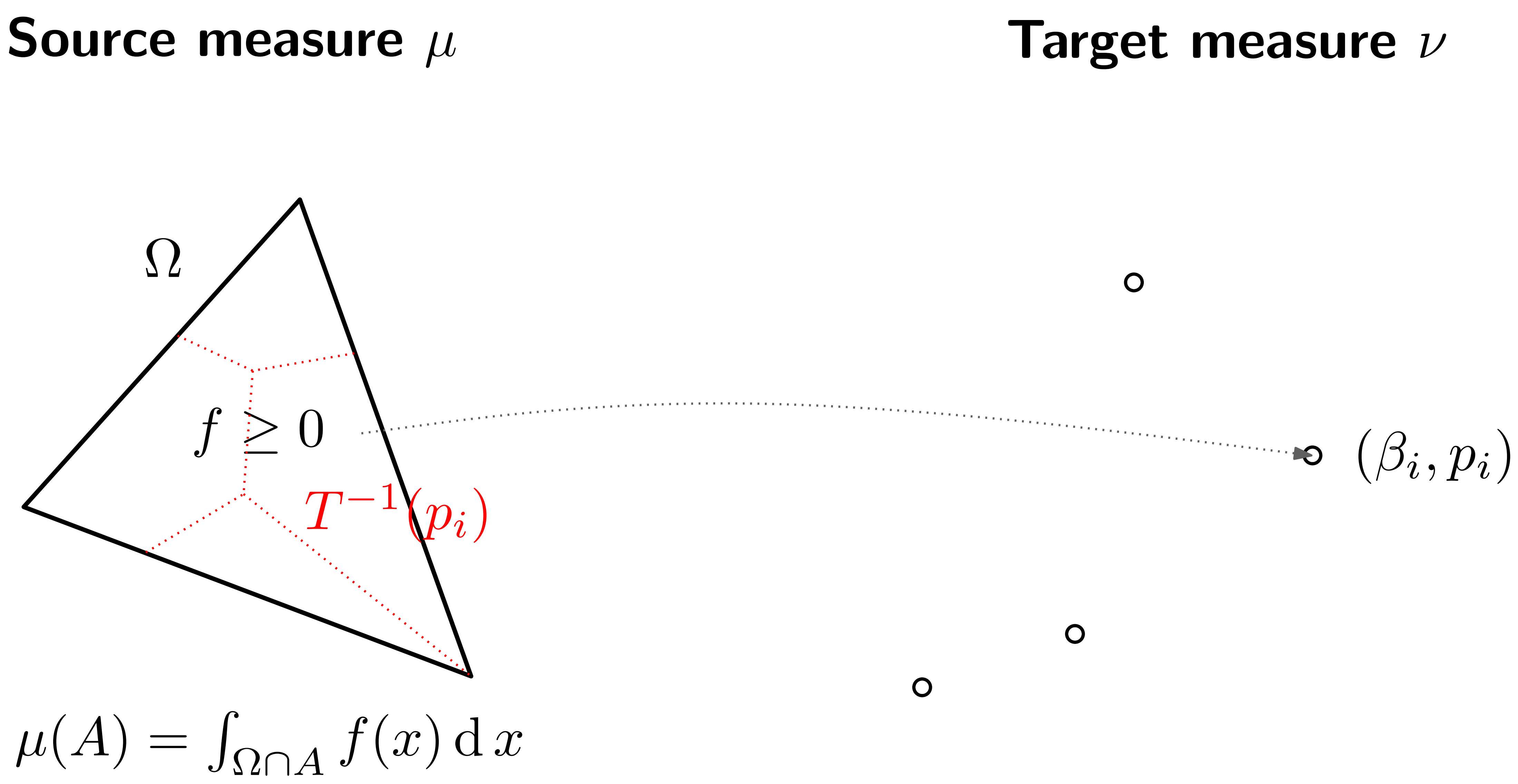

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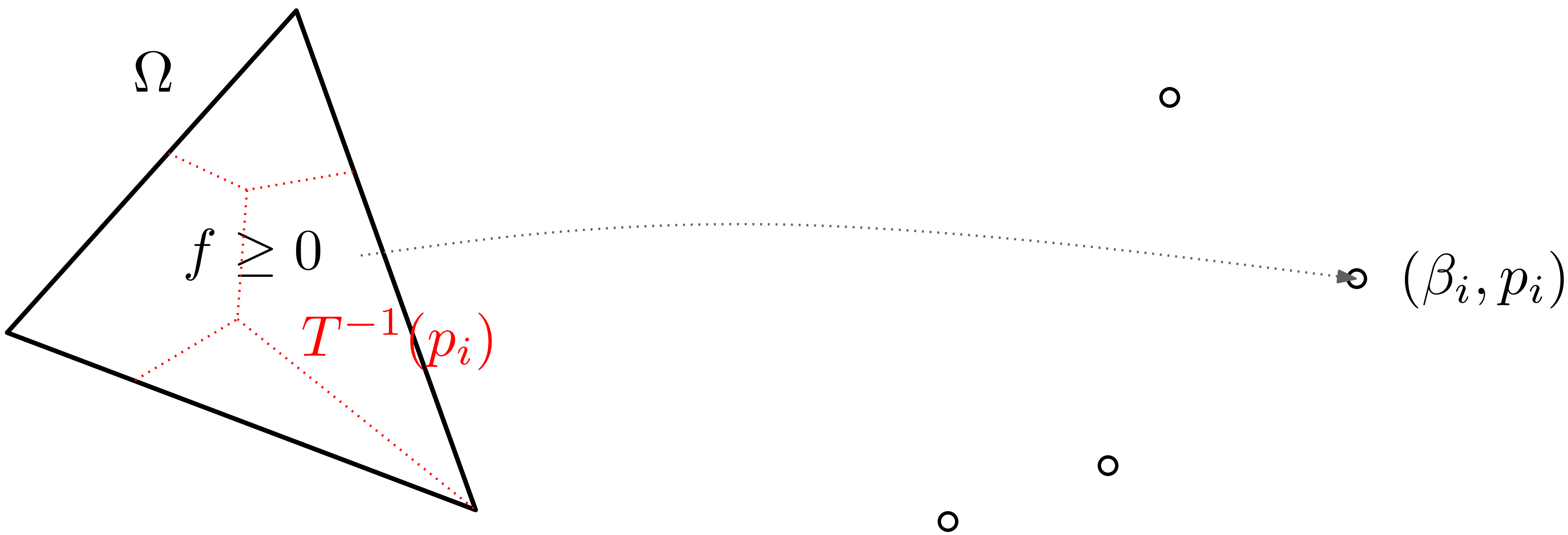

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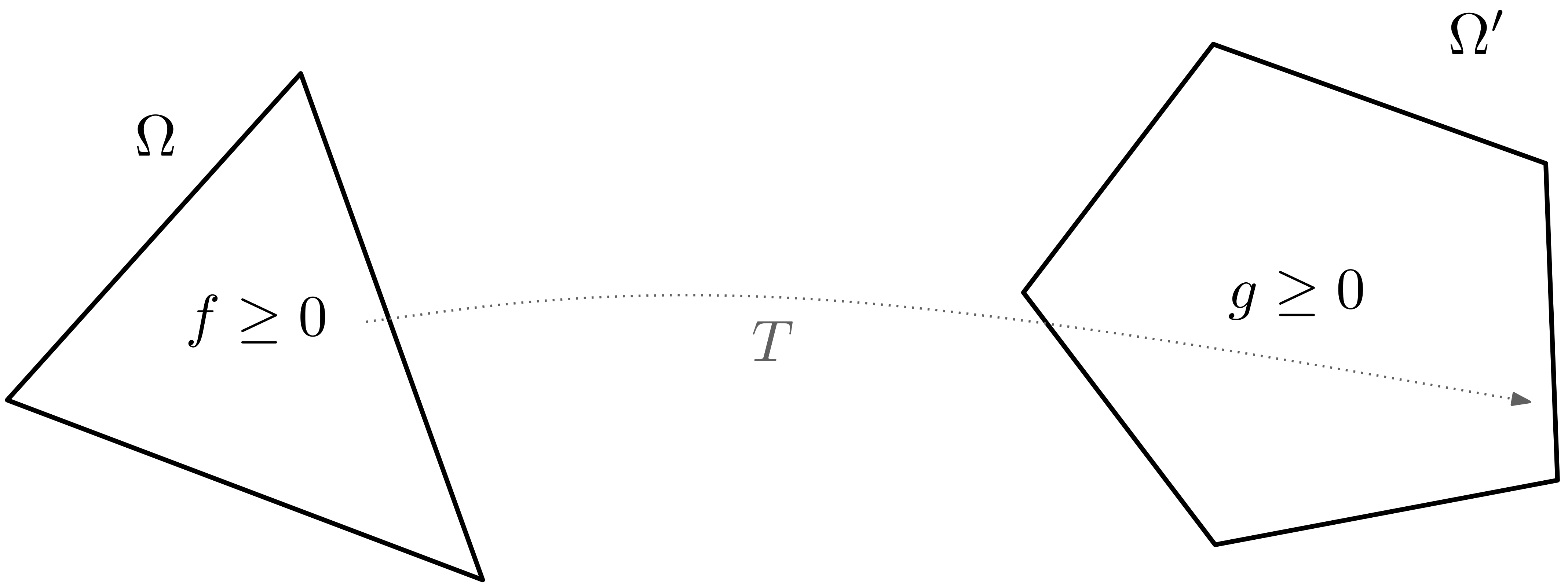

Transport plan: a map $T : \Omega \to \Omega'$ such that $\det(\mathrm{d}\, T(x)) = g(T(x))/f(x)$.

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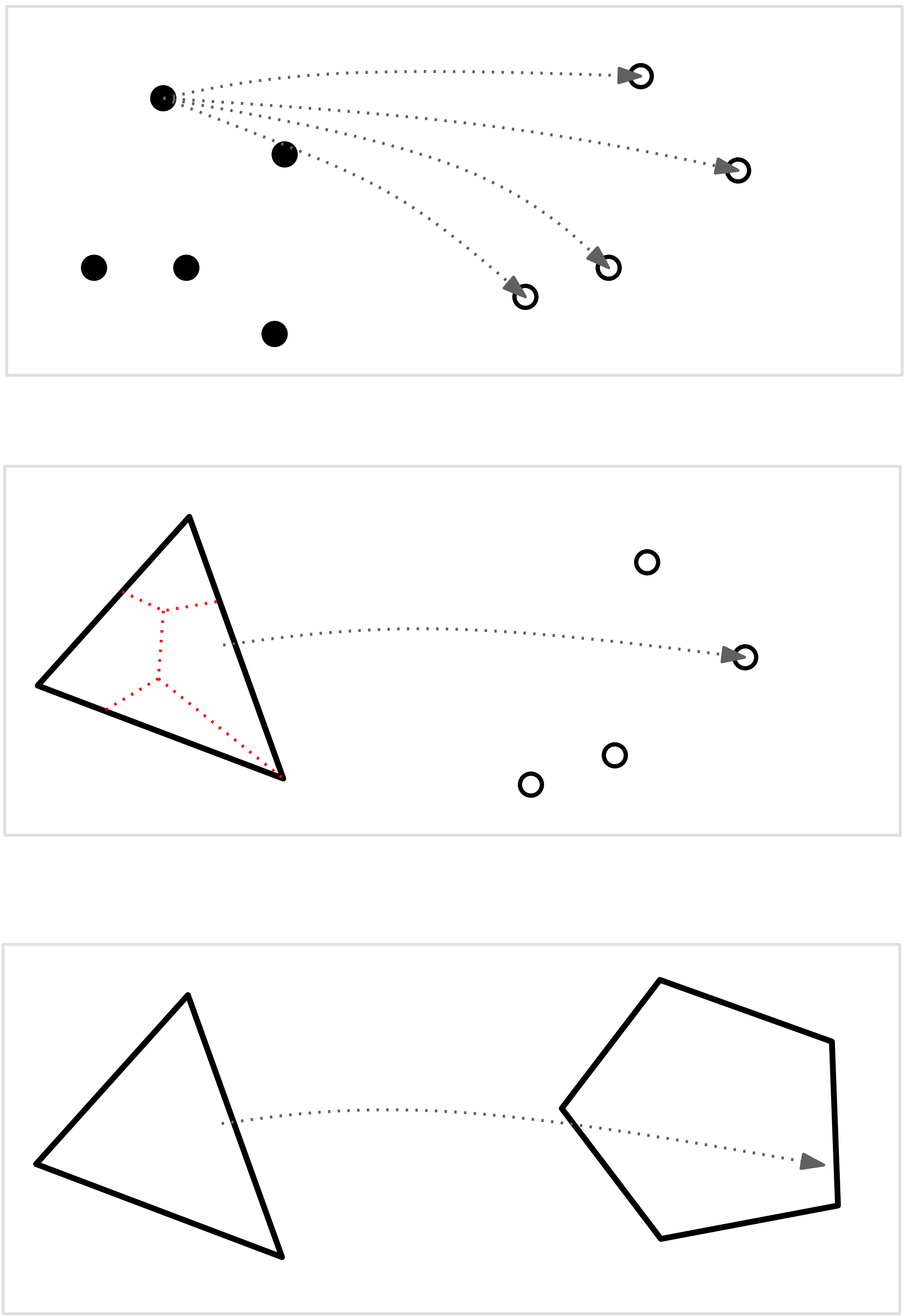

General α_i, β_j: linear programming

For $\alpha_i, \beta_j = 1$ and $p_i, q_j \in \mathbb{Z}^d$:

Hungarian algorithm, Bertsekas 'auction' algorithm

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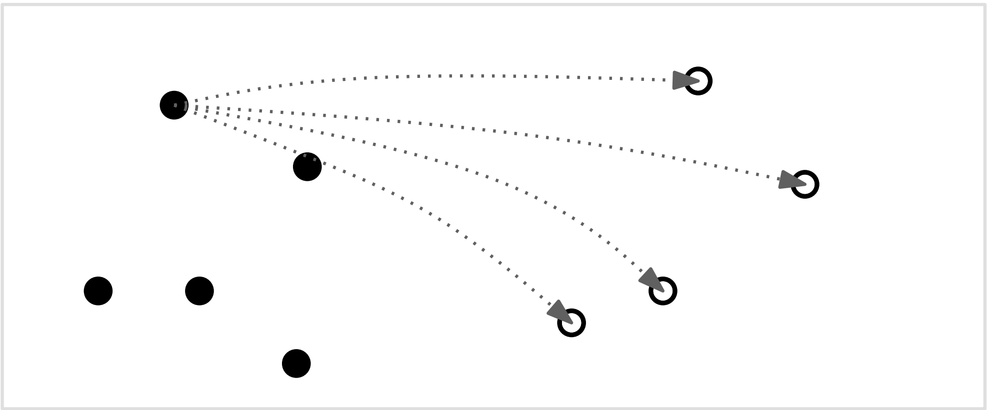

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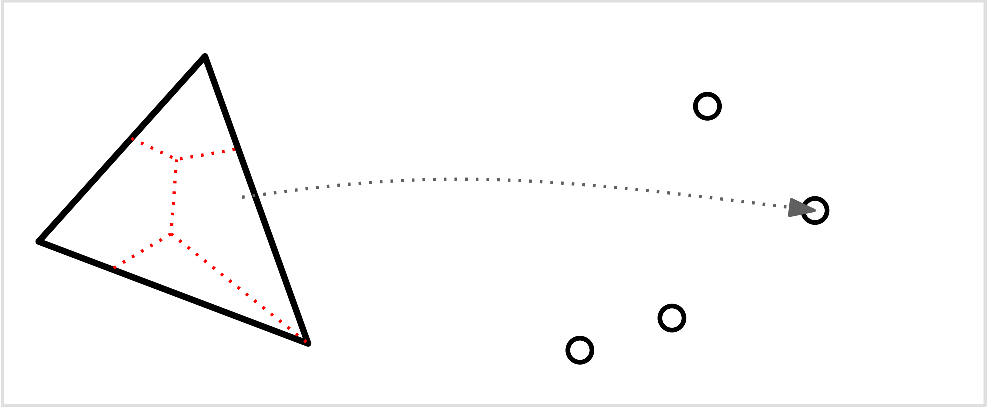
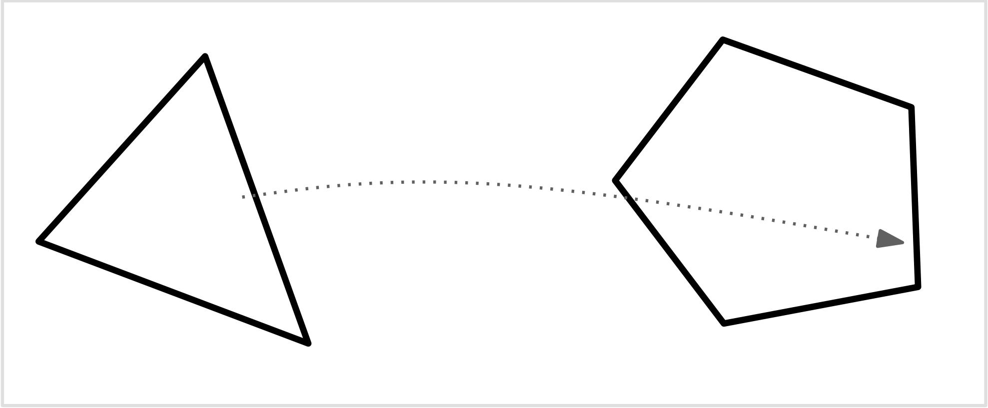




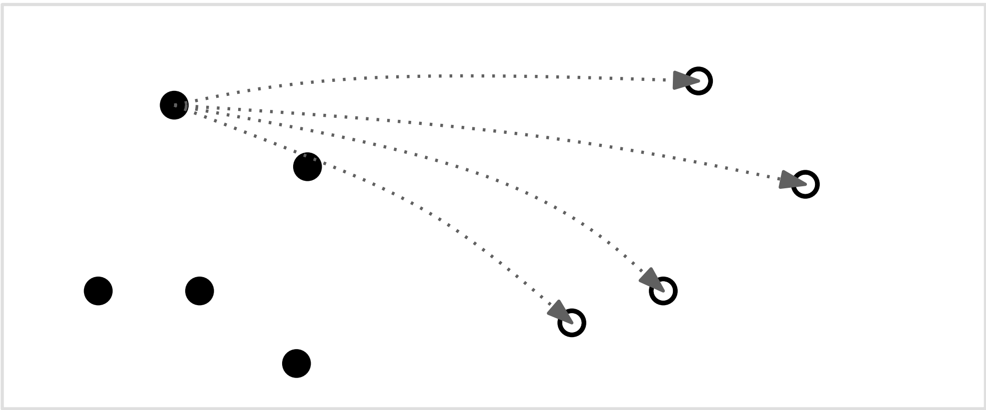
Smooth f, g with positive lower bound:

Benamou-Brenier '00

Loeper '05

Angenent-Haker-Tannenbaum '03

L^2 Optimal Transport



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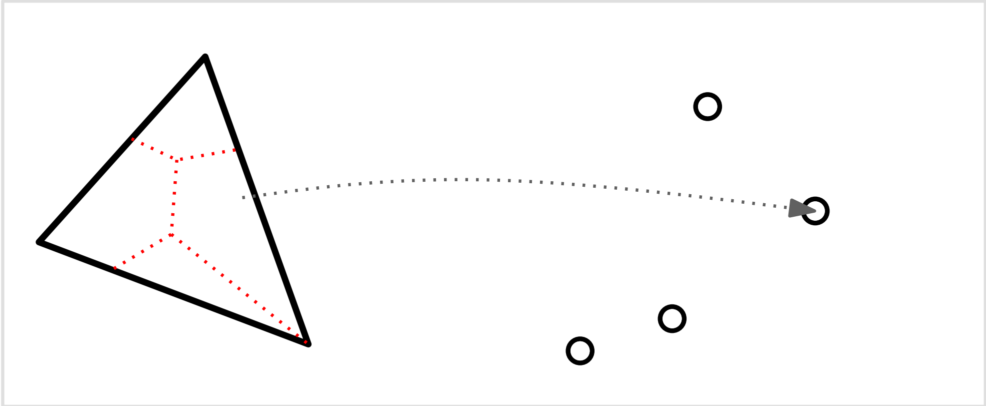
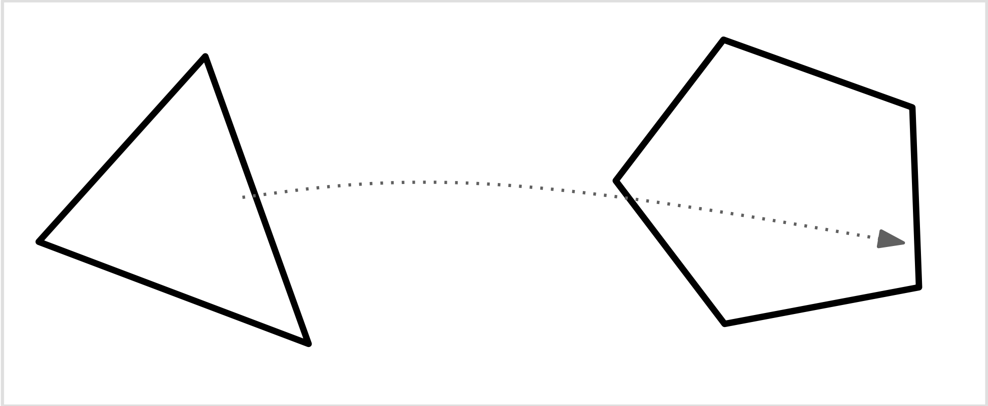


Source with density, discrete target:

Aurenhammer, Hoffmann, Aronov '98

McCann, Gangbo 98



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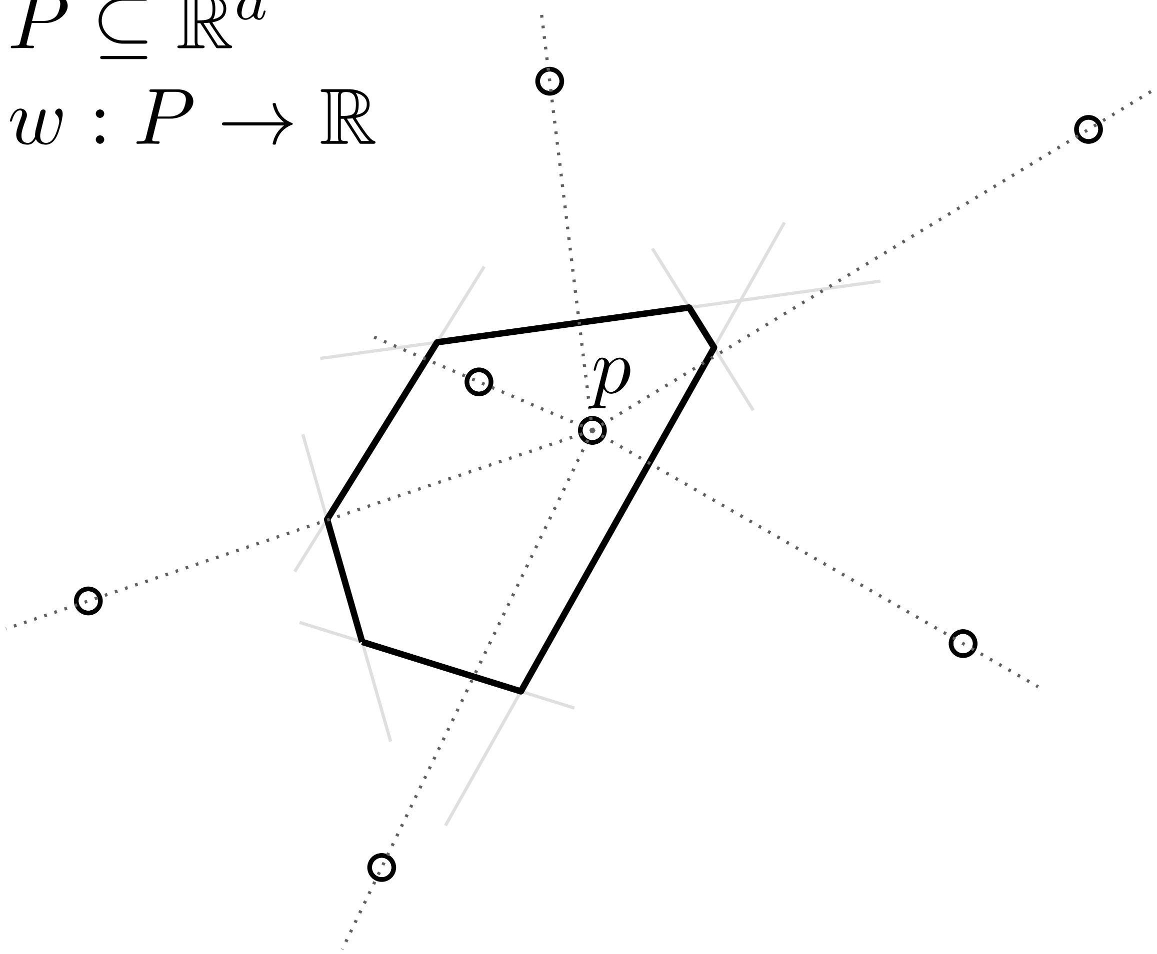
1. Optimal Transport via Convex Programming

Power Diagrams and Optimal Transport

$P \subseteq \mathbb{R}^d$

$w : P \to \mathbb{R}$



Transport map:

$$T_P^w(x) := \arg\min_{p \in P} \|x - p\|^2 + w(p)$$

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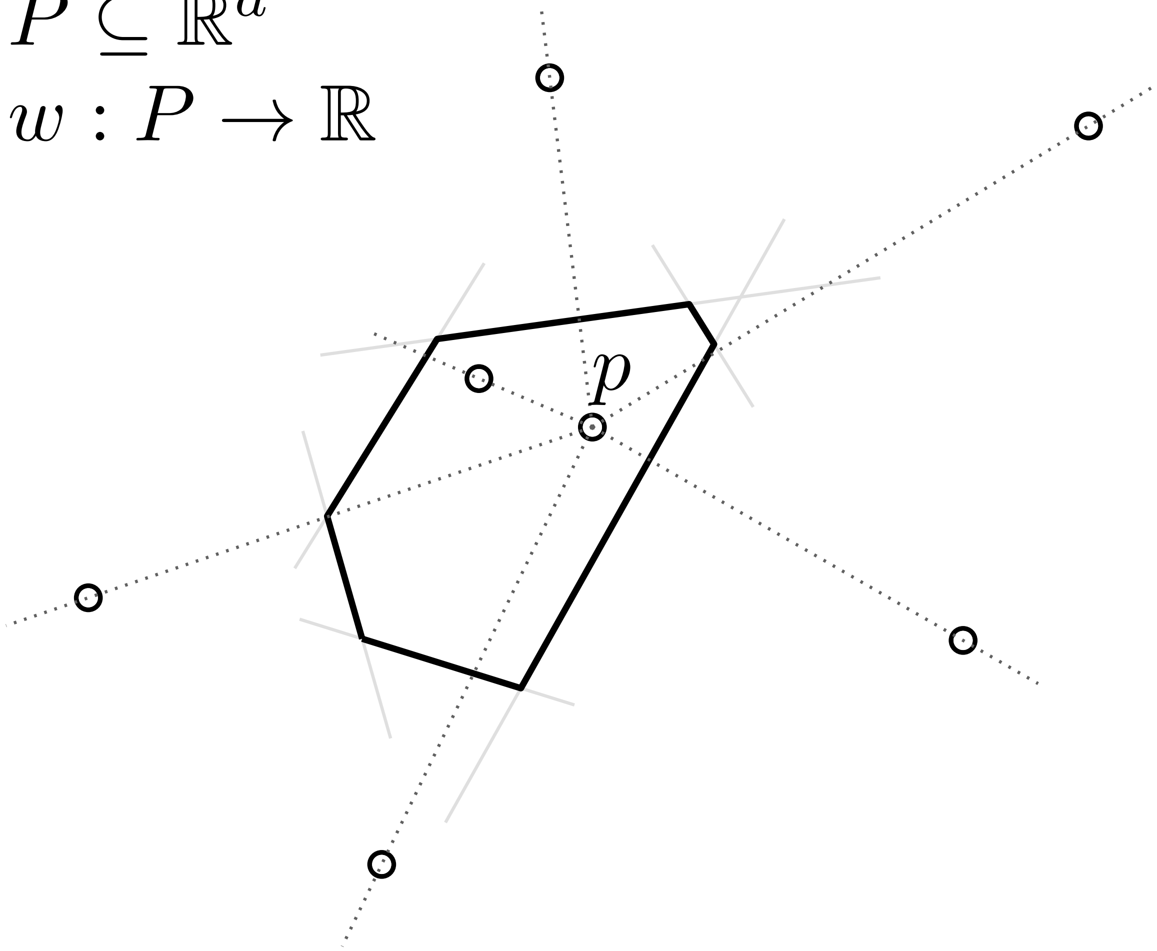

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$$\mathrm{Vor}_P^w(p) := \{x \in \mathbb{R}^d ; T_P^w(x) = p\}$$

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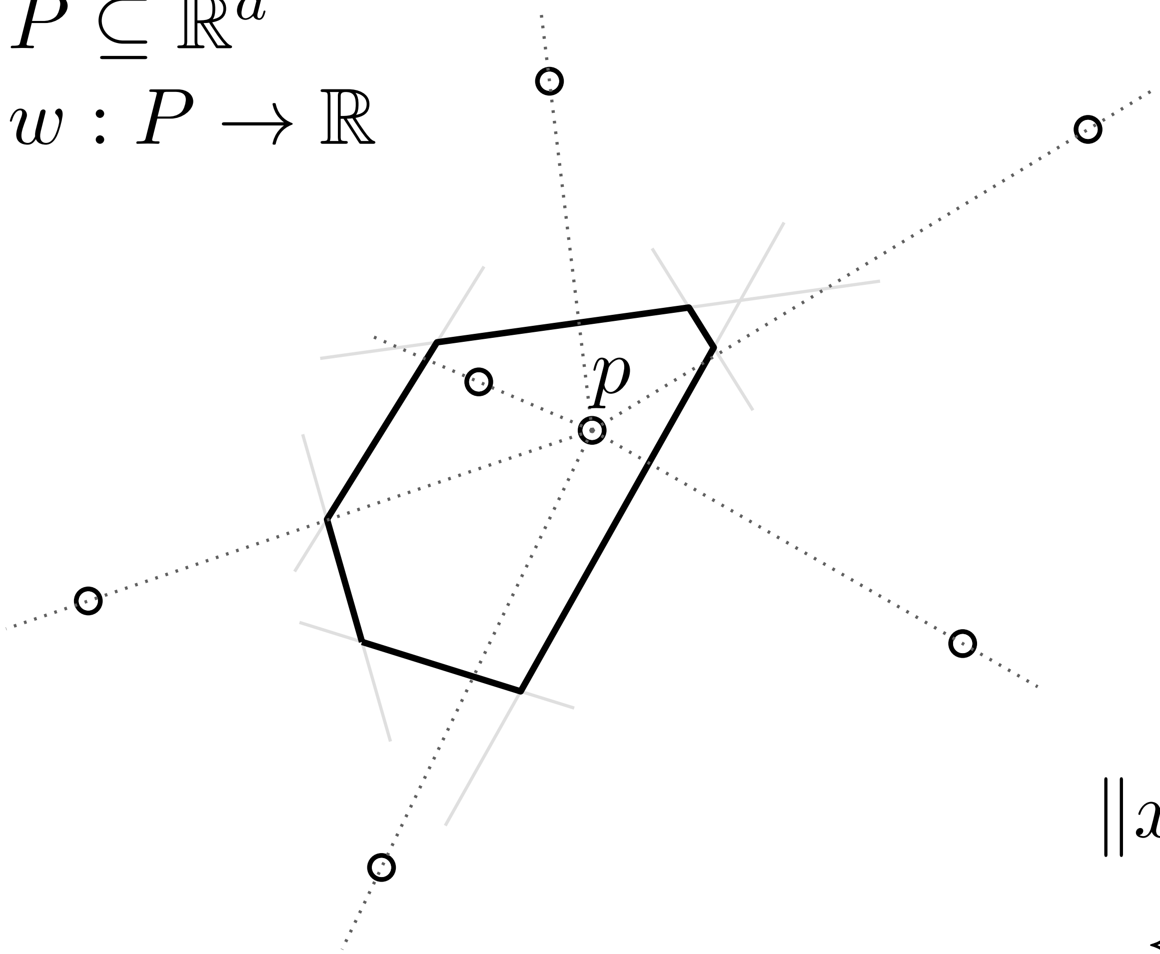

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$$\|x - p\|^2 + w(p) \leq \|x - q\|^2 + w(q)$$

$$\iff 2\langle x | q - p \rangle \leq w(q) - w(p)$$

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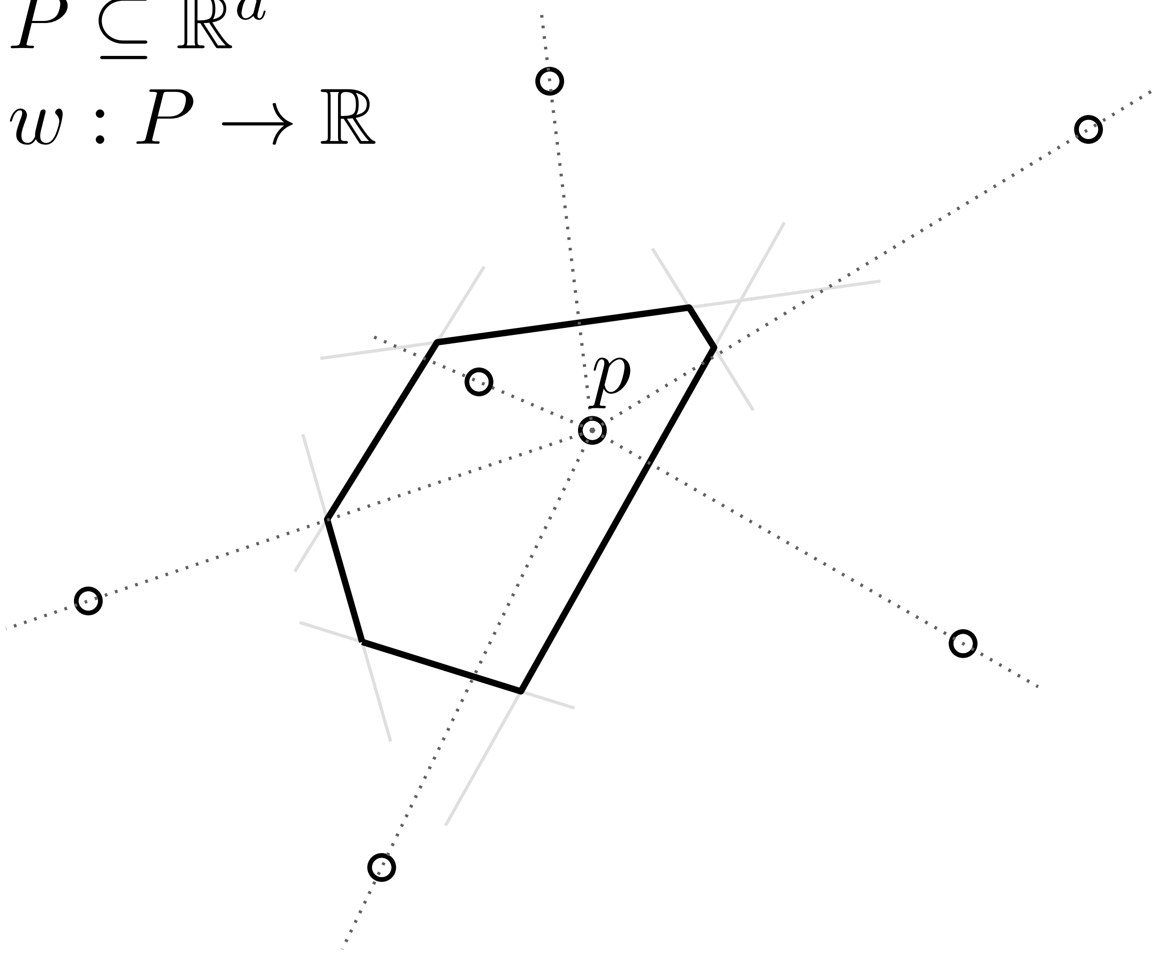

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Lemma: Given a measure μ with density and (P, w), the map T_P^w is an optimal transport between μ and

$$\nu := \sum_{p \in P} \mu(\mathrm{Vor}_P^w(p))\delta_p \qquad \text{(i.e. } \nu = T_{P\#}^w \mu\text{)}$$

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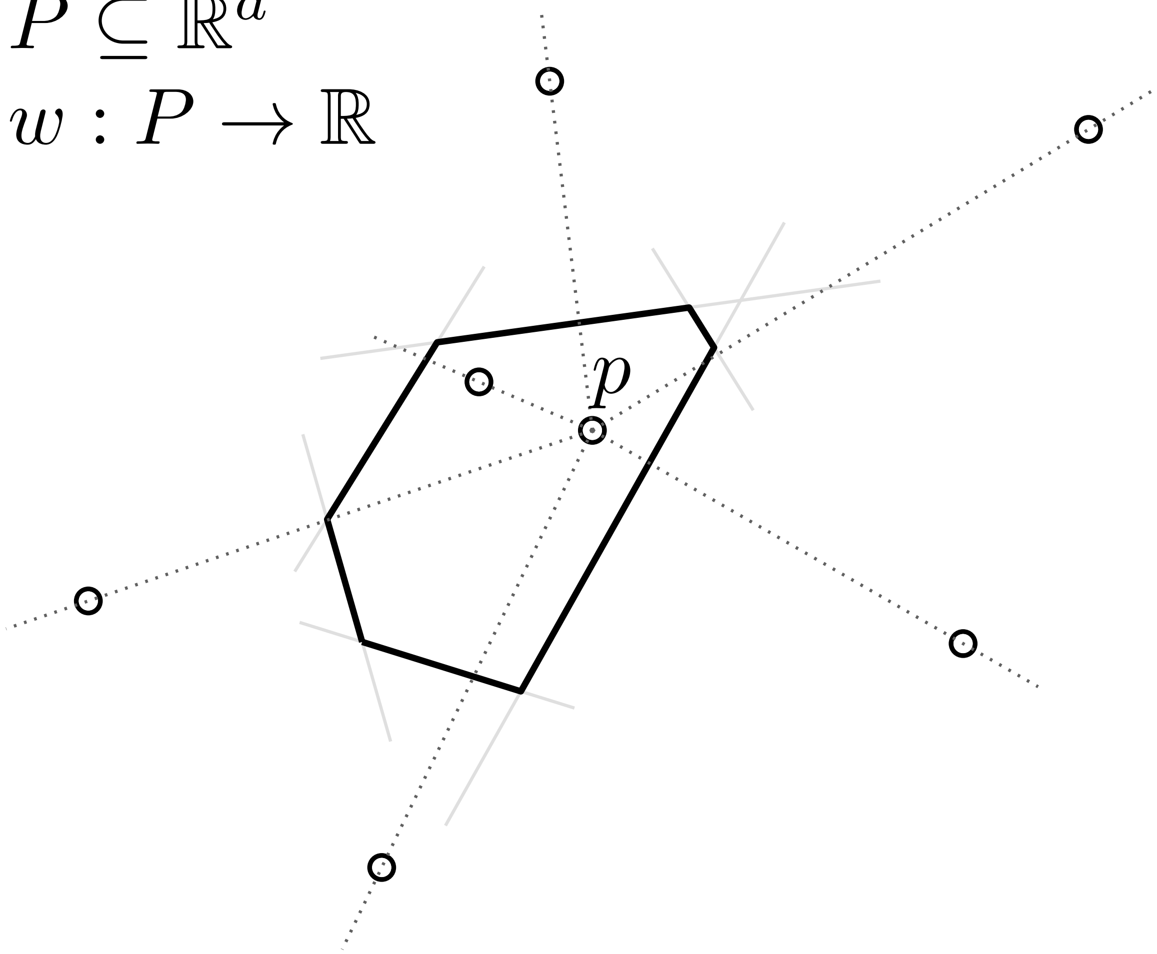

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Theorem: Given a measure μ with density and a discrete measure $\nu = \sum_{p \in P} \alpha_p \delta_p$, there exists $w : P \to \mathbb{R}$ s.t.

$$\forall p \in P, \ \alpha_p = \mu(\mathrm{Vor}_P^w(p)) \qquad \text{(i.e. } \nu = T_{P\#}^w \mu\text{)}$$

Aurenhammer, Hoffman, Aronov '94 Brenier '92

Optimal Transport via Convex Programming

Kantorovich Duality: Given two probability measure μ, ν

$$\text{Wass}_2(\mu, \nu) = \max_{\mathbb{R}^d} \int v(x) \,\mathrm{d}\, \mu(x) - \int w(p) \,\mathrm{d}\, \nu(p)$$

where v, w are such that $v(x) - w(p) \leq \|x - p\|^2$.

Optimal Transport via Convex Programming

Kantorovich Duality: Given two probability measure μ, ν

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Discrete case: μ with density and $\nu = \sum_{p\in P} \alpha_p \delta_p$,

$$\Phi(w) := -\sum_{p\in P} \int_{\mathrm{Vor}_P^w(p)} [\|x - p\|^2 + w(p)] \,\mathrm{d}\,\mu(x) + \sum_{p\in P} \alpha_p w(p)$$

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Gradient: $\Phi(w+\varepsilon h) - \Phi(w) = \sum_{p\in P} h(p)\,(\mu(\mathrm{Vor}_P^w(p)) - \alpha_p)\,\varepsilon + \mathrm{O}(\varepsilon^2)$

|| changes in Power cells

I.e. $\nabla\Phi(w) = (\alpha_p - \mu(\mathrm{Vor}_P^w(p)))_{p\in P}$

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$\nabla\Phi(w)$ is actually a subgradient, i.e. the function Φ is convex

Implementation of Convex Programming

1. Computation of Φ and $\nabla\Phi$:

$$\int_{\Omega\cap\mathrm{Vor}_P^w(p)} f(x)\,\mathrm{d}\,x$$

$$\int_{\Omega\cap\mathrm{Vor}_P^w(p)} \|x-p\|^2 f(x)\,\mathrm{d}\,x$$

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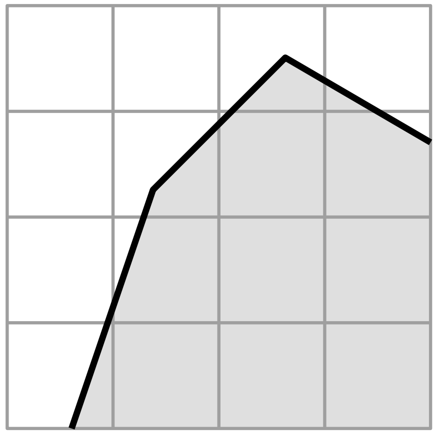
$f = 1$: Power diagram, Fast intersection of polygons

CGAL

O'Rourke, Chien, Olson, Naddor '82

$f =$ grayscale image: Piecewise constant on pixels

Modification of Bresenham algorithm to compute exact pixel coverage



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2. Iterative unconstrained convex programming:

- Choice of an initial weight vector, e.g. $w_0(p) := 0$ for all p.
- Computation of descent direction d_k

  steepest descent $-\nabla\Phi(w_k)$, Newton $-[\mathrm{D}^2\Phi(w_k)]^{-1}(\nabla\Phi(x_k))$, quasi-Newton...

  L-BFGS: low-storage version of the BFGS quasi-Newton scheme

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- Computation of descent direction d_k
- Computation of time step s_k

  optimal $s_k = \arg\min_s \Phi(w_k + s d_k)$, fixed $s_k = \mathrm{cst}$

  in practice: backtracking line-search (e.g. Wolfe condition)

Implementation of Convex Programming

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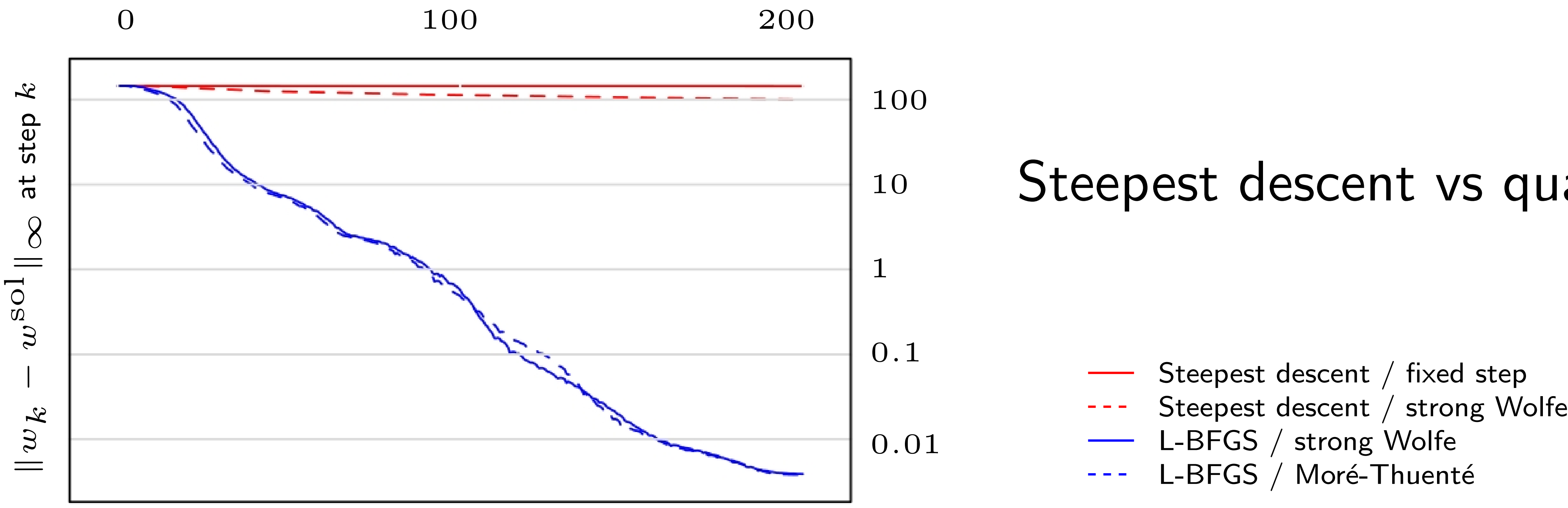
Comparison of Convex Optimization Methods



Steepest descent vs quasi-Newton

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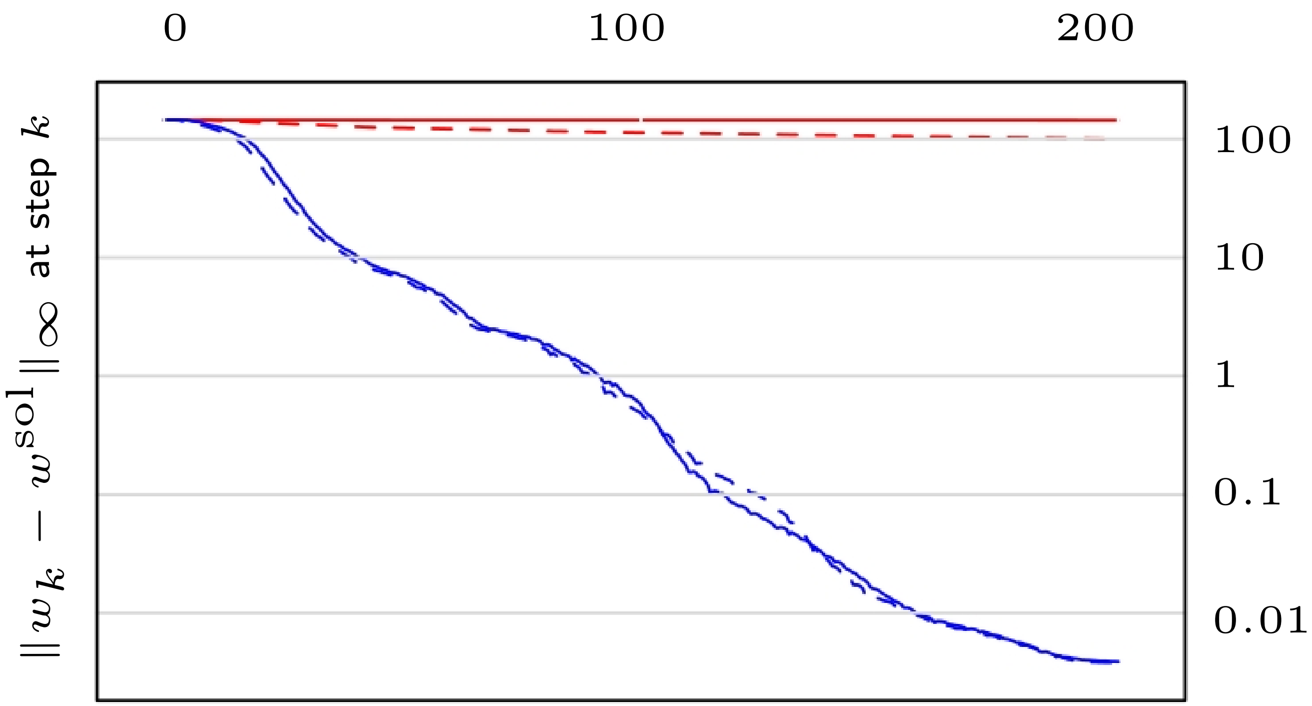
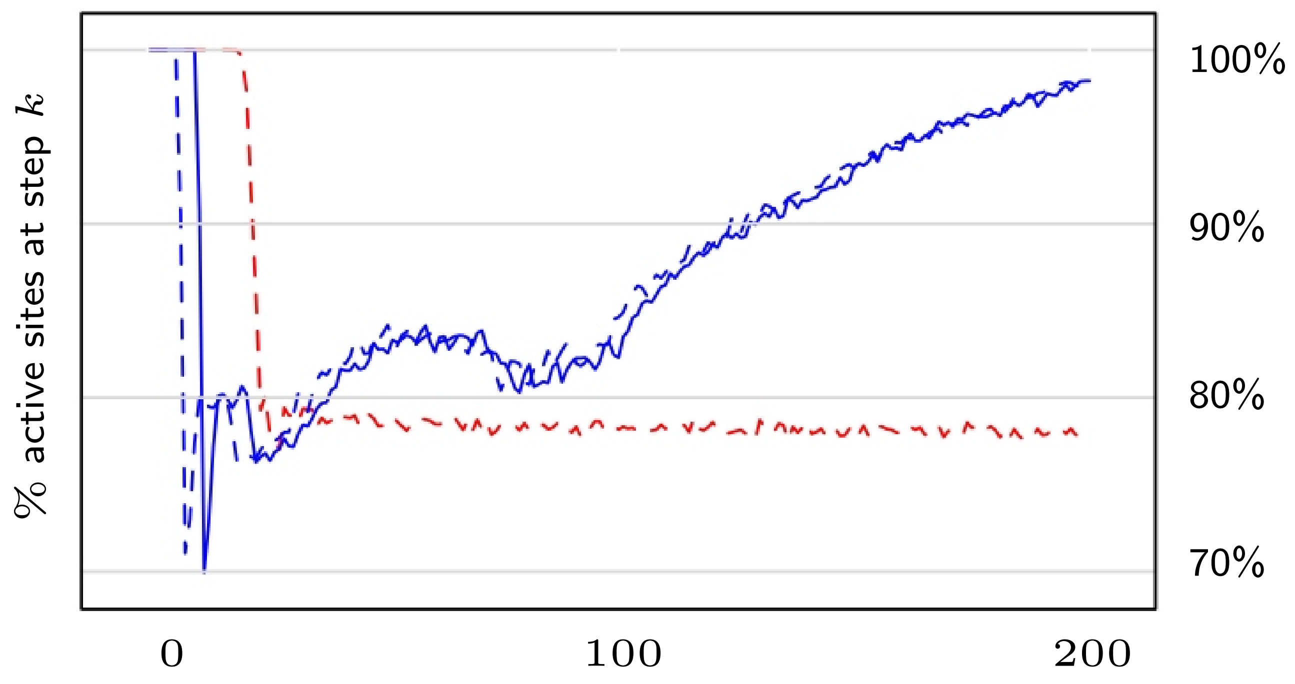


Steepest descent vs quasi-Newton

Number of sites with non-empty Power cell



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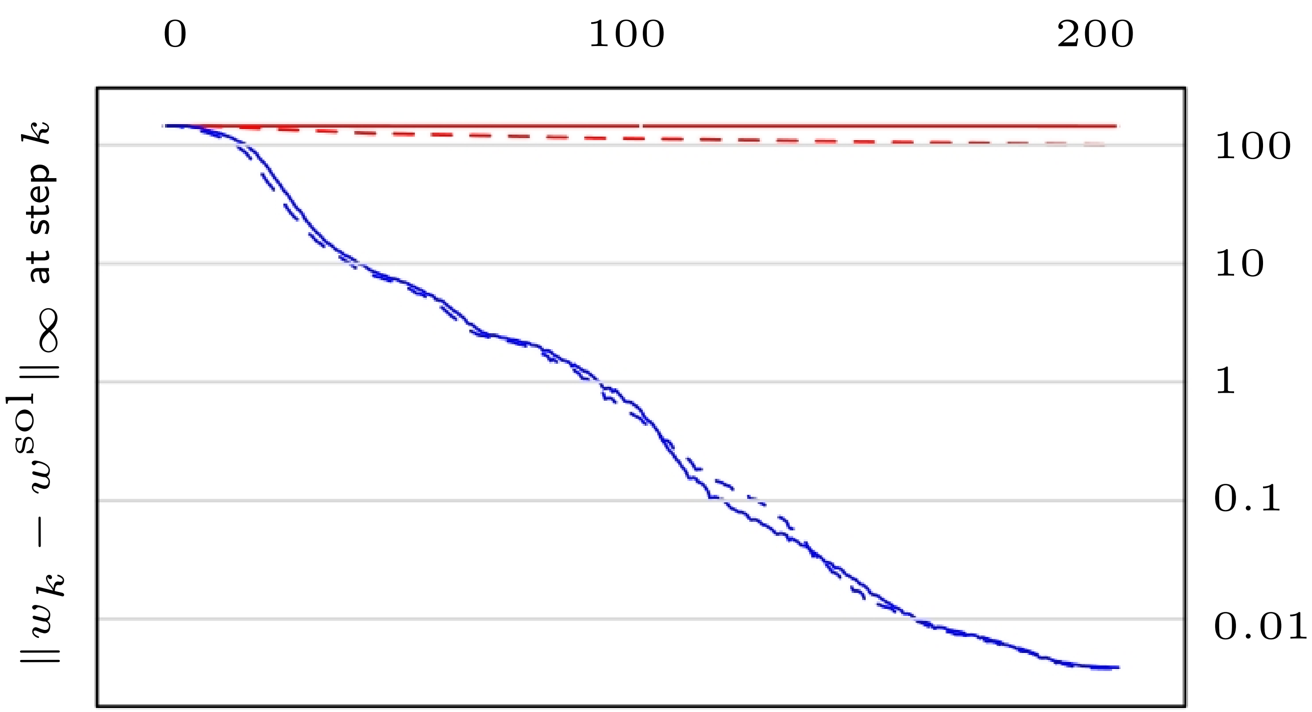

Steepest descent vs quasi-Newton

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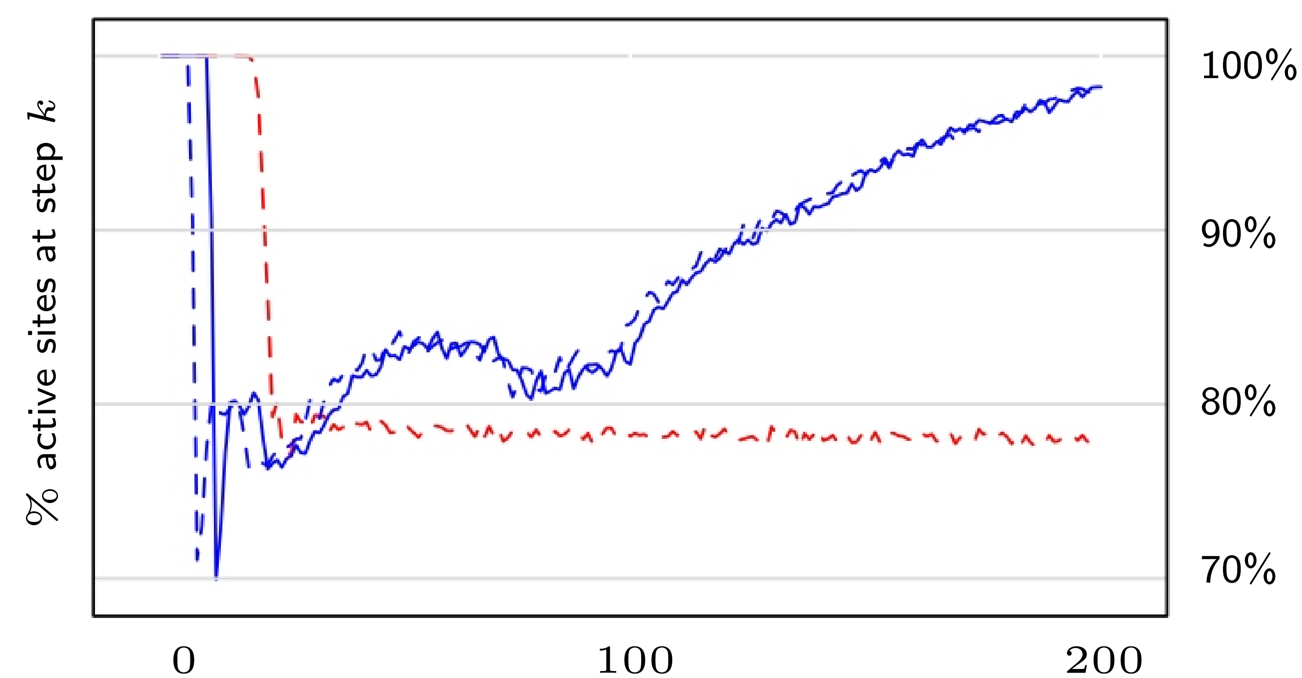
$\Longrightarrow$ Need to recompute completely the Power diagram at every step



2. Multiscale approach

An Approximation Theorem

Proposition: Suppose the following:

- μ probability with density $f \geq m > 0$ on a bounded connected domain Ω with piecewise smooth boundary.
- (ν_n) and ν_∞ are are supported on finite sets $P_n \subseteq \Omega$, and $\lim \mathrm{W}_2(\nu_n, \nu_\infty) = 0.$

Let w_n be weights that solve OT between μ and ν_n. Then,

$$\forall p_n \in P_n, \quad \lim p_n = p \in P_\infty \Longrightarrow w_\infty(p) = \lim w_n(p_n)$$

An Approximation Theorem

Proposition: Suppose the following:

- μ probability with density $f \geq m > 0$ on a bounded connected domain Ω with piecewise smooth boundary.
- (ν_n) and ν_∞ are are supported on finite sets $P_n \subseteq \Omega$, and $\lim \mathrm{W}_2(\nu_n, \nu_\infty) = 0.$

Let w_n be weights that solve OT between μ and ν_n. Then,

$$\forall p_n \in P_n, \quad \lim p_n = p \in P_\infty \Longrightarrow w_\infty(p) = \lim w_n(p_n)$$

- Weights are defined up to an additive constant.
- Open question: a quantitative version of this theorem.

An Approximation Theorem – Sketch of Proof

Proposition: Suppose [...] $\lim \mathrm{W}_2(\nu_n, \nu_\infty) = 0$. Then,

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Convex potential:

$$\phi_P^w(x) = \|x\|^2 - \min_{p \in P}(\|x - p\|^2 - w(p))$$
$$= \max_{p \in P} \langle x | p \rangle + \tfrac{1}{2}(w(p) - \|p\|^2)$$
$$\nabla \phi_P^w(x) = T_S^w$$

Zero-mean: we assume w.l.o.g. that $\int_\Omega \phi_S^w(x) \, \mathrm{d}\, \mu(x) = 0$

An Approximation Theorem – Sketch of Proof

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Zero-mean Convex potential: $\nabla \phi_n = T_n$ and $\int_\Omega \phi_n(x) f(x) \,\mathrm{d}\,\mu(x) = 0$.

- By stability of optimal transport plans, $\lim \|T_n - T_\infty\|_{\mathrm{L}^2(\mu)} = 0$.

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- By Poincaré inequality (assumptions on Ω and f),

$$\|\phi\|_{\mathrm{L}^2(\mu)} \leq \mathrm{cst} \times \|\nabla \phi\|_{\mathrm{L}^2(\mu)} \quad \text{provided that} \quad \int_\Omega \phi(x) \,\mathrm{d}\,\mu(x) = 0.$$

An Approximation Theorem – Sketch of Proof

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- Since ϕ_n and ϕ_∞ are Lipschitz, $\lim \|\phi_n - \phi_\infty\|_{\mathrm{L}^\infty(\Omega)} = 0$.

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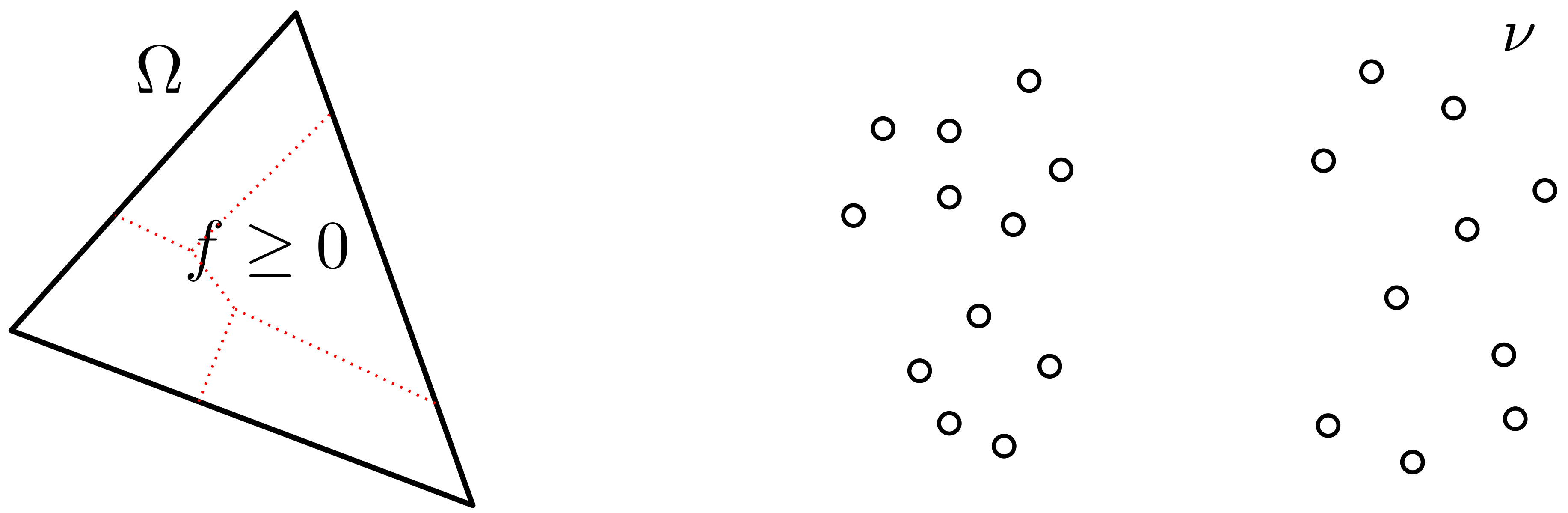
With a bit more work, this result implies the conclusion of the theorem.

Two-scale Approach for Optimization

Goal: Given a measure μ with density, and ν supported on P, $|P| = N$,

minimise $\Phi : w \mapsto \ldots$



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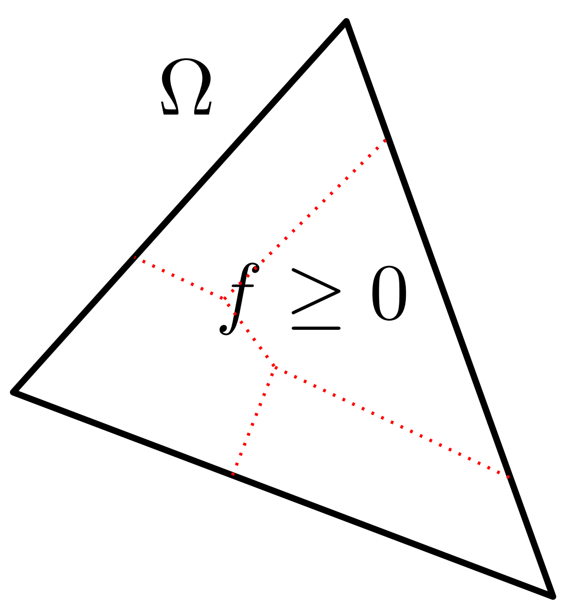
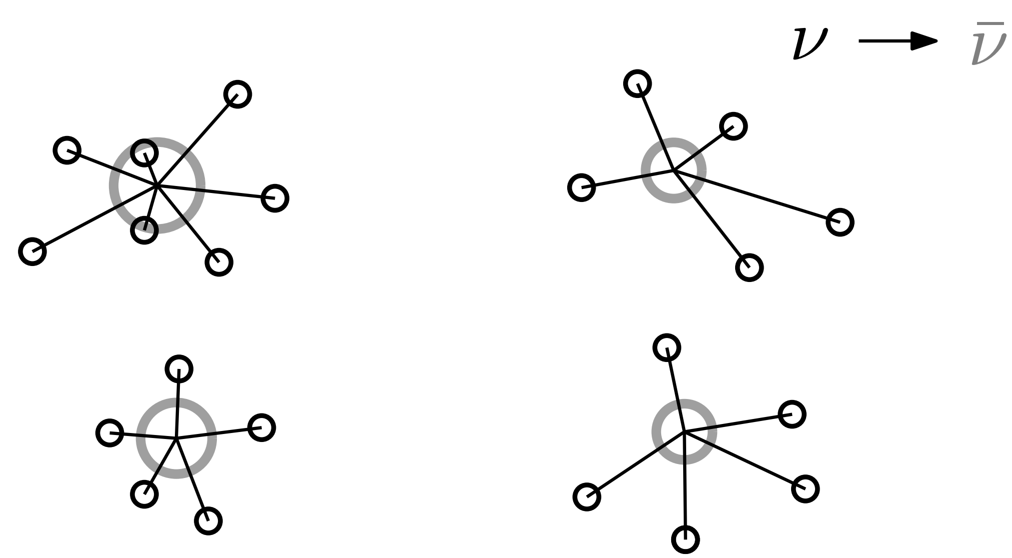
$$\bar{\nu} = \arg\min\{\mathrm{W}_2(\nu, \bar{\nu});\ |\mathrm{spt}(\bar{\nu})| \leq n\}$$





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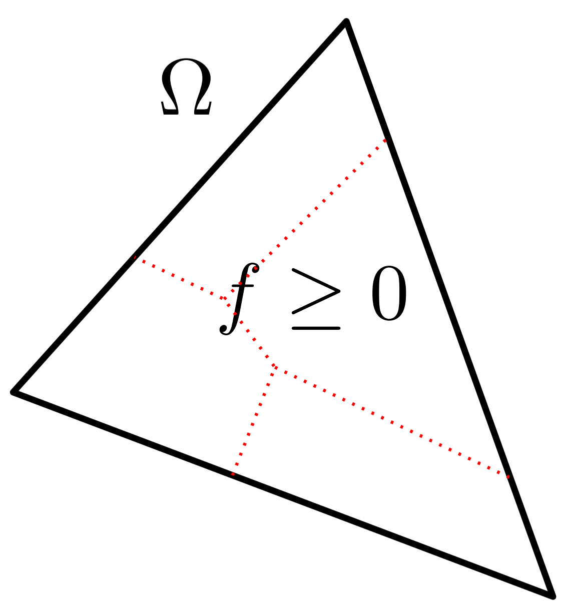
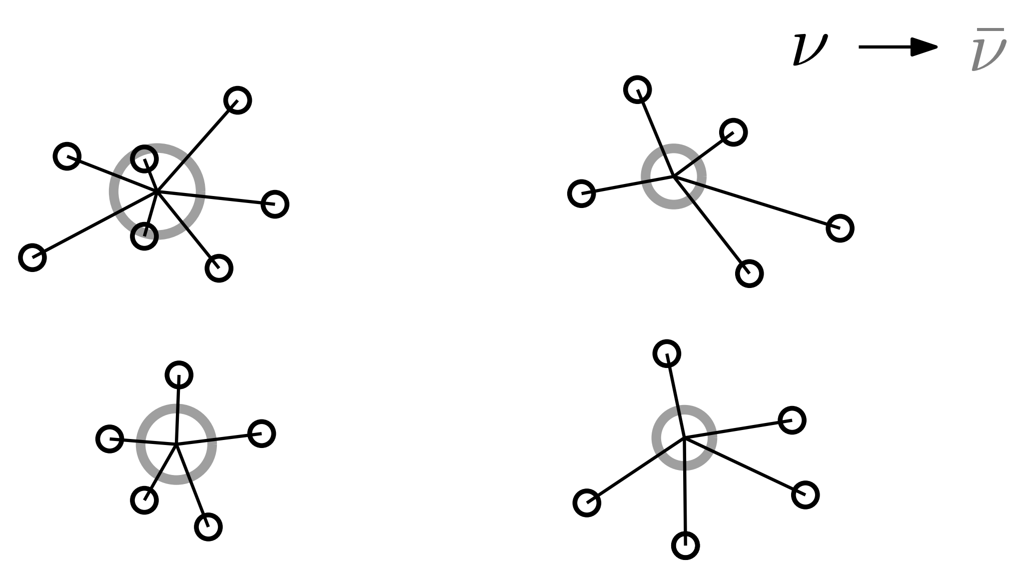
k-means / Lloyd's algorithm (local minimum)





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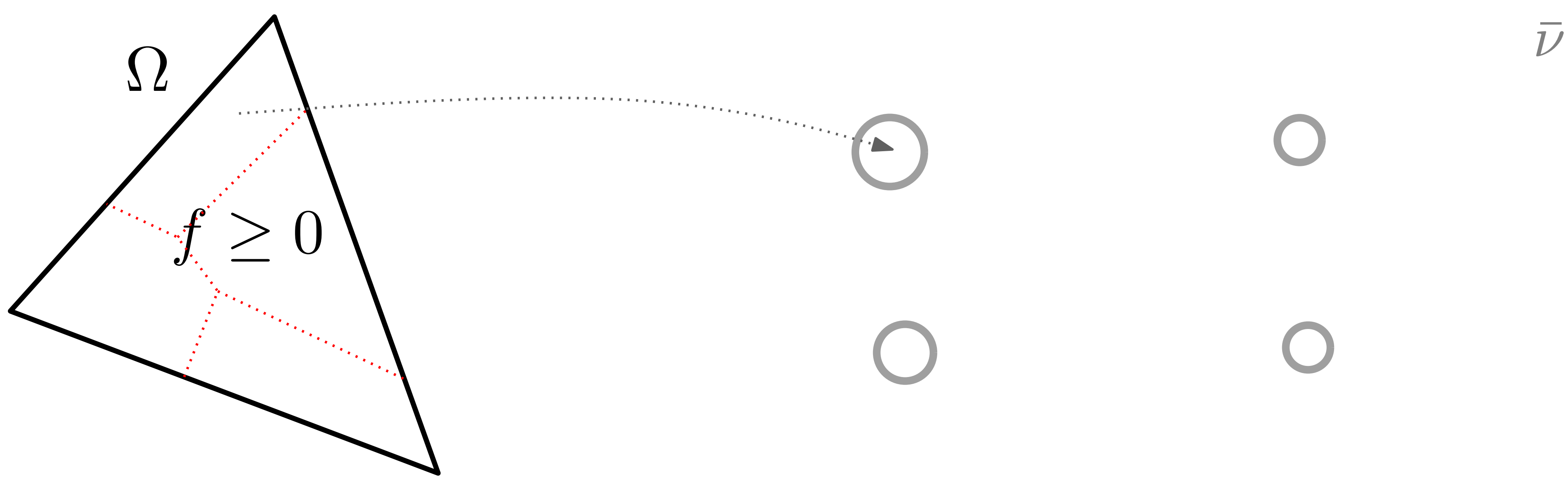

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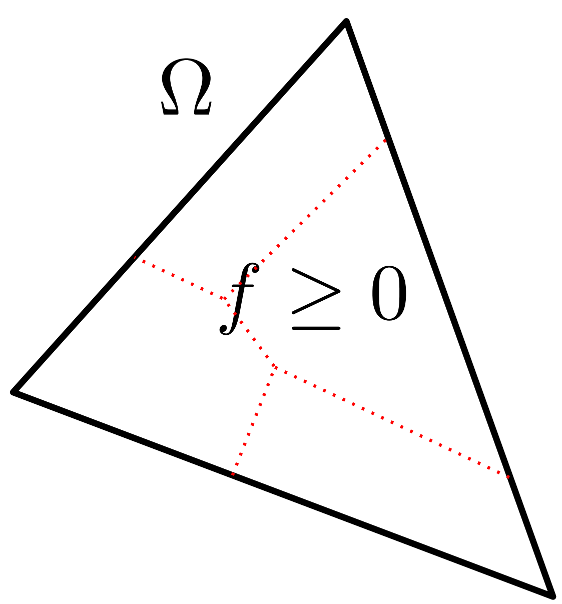

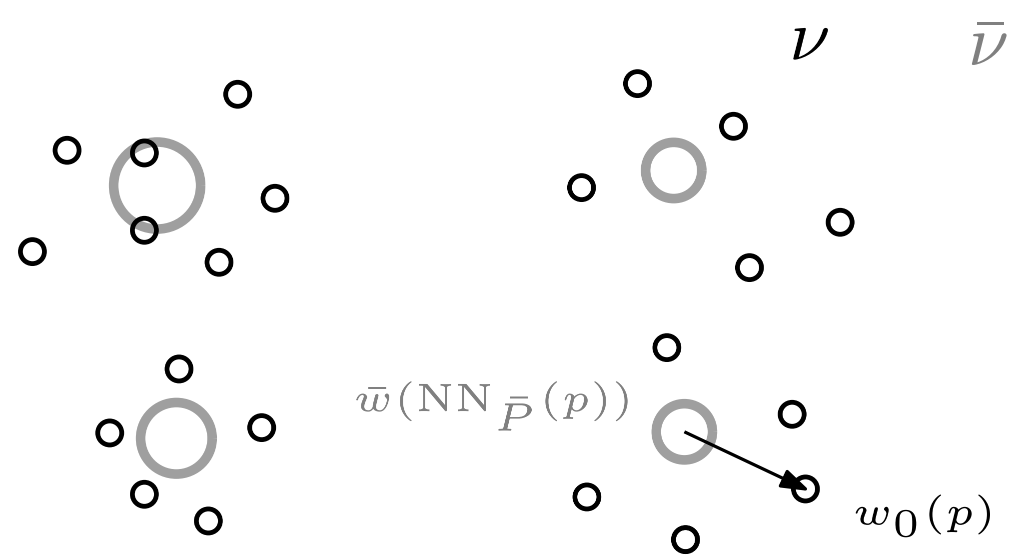

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Summary of the multiscale-scale algorithm

Input: a measure μ with density and a discrete measure ν on $\mathbb{R}^2$.

- Compute a sequence of discretizations of the target measure: $\nu_0 := \nu, \dots, \nu_L$, s.t. ν_ℓ is supported P_ℓ with $Nk^{-\ell}$ points.

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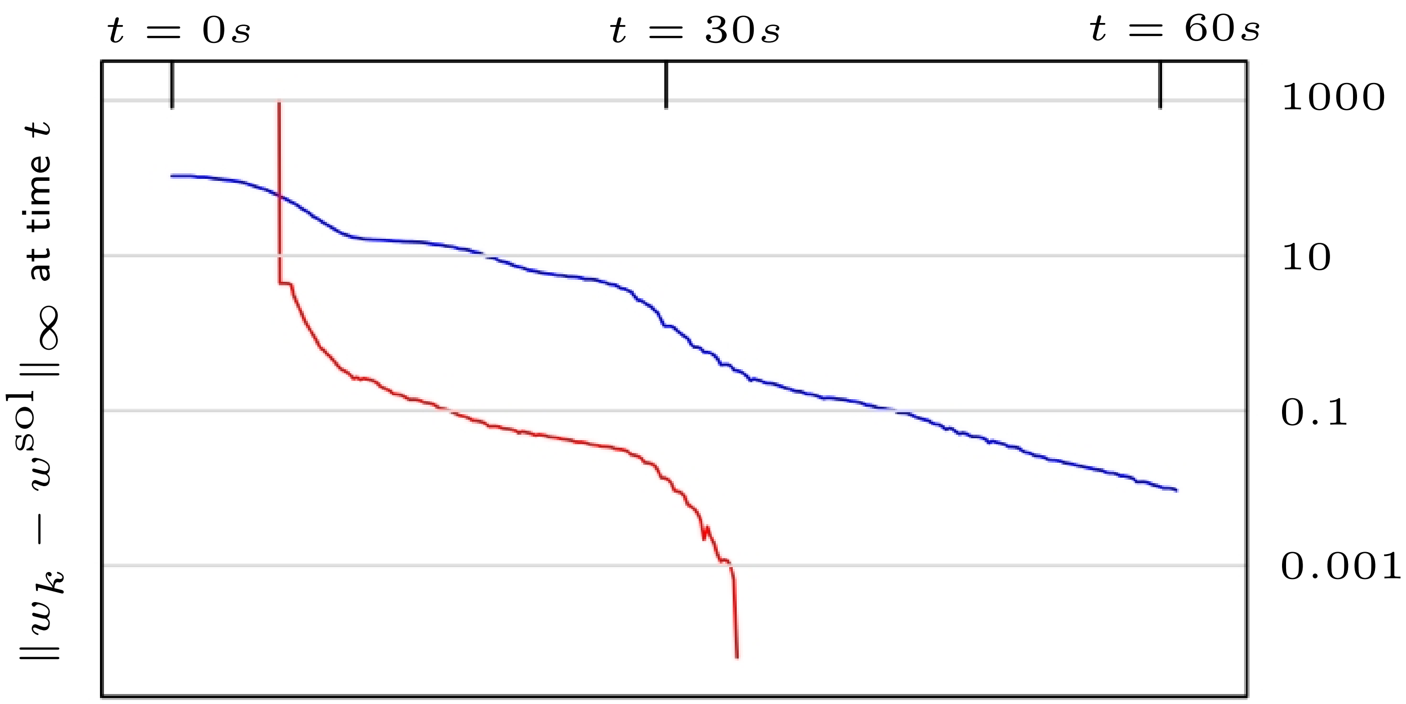
$\dots$

- Solve OT from μ to ν_ℓ starting from $w_\ell(p) := w_{\ell+1}(\mathrm{NN}_{P_\ell}(p))$.

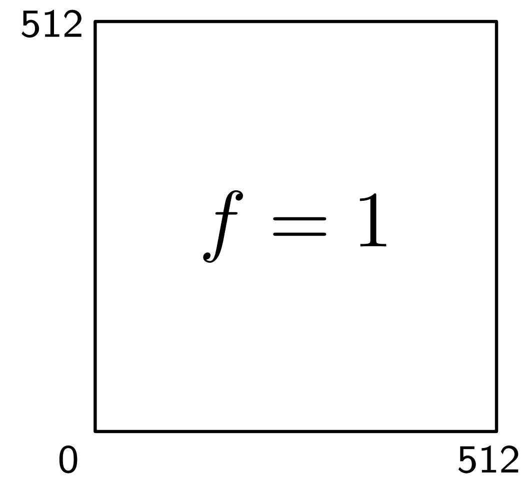
Remark: If the target measure is not discrete, one can obtain a first discretisation by an application of Lloyd's algorithm.

3. Experiments

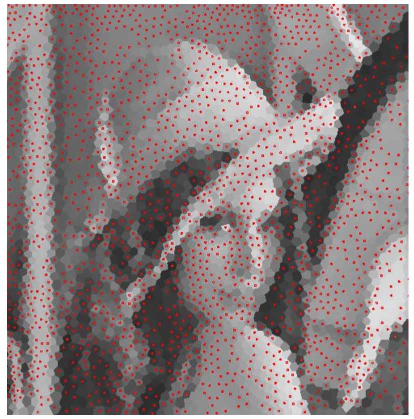
Multiscale vs Original — Convergence Speed



Source

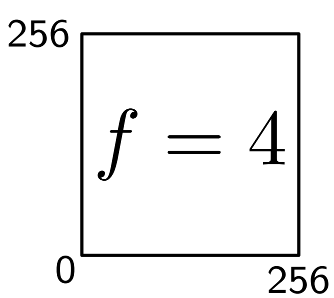


Target

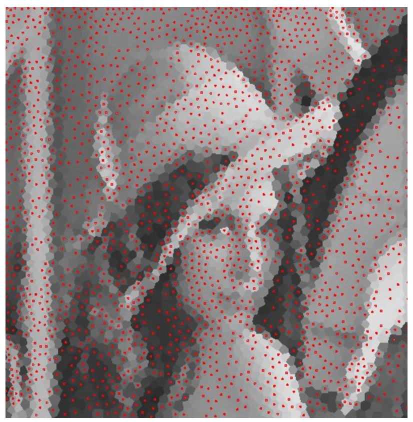


3k Lloyd sampling of Lena

Source



Target



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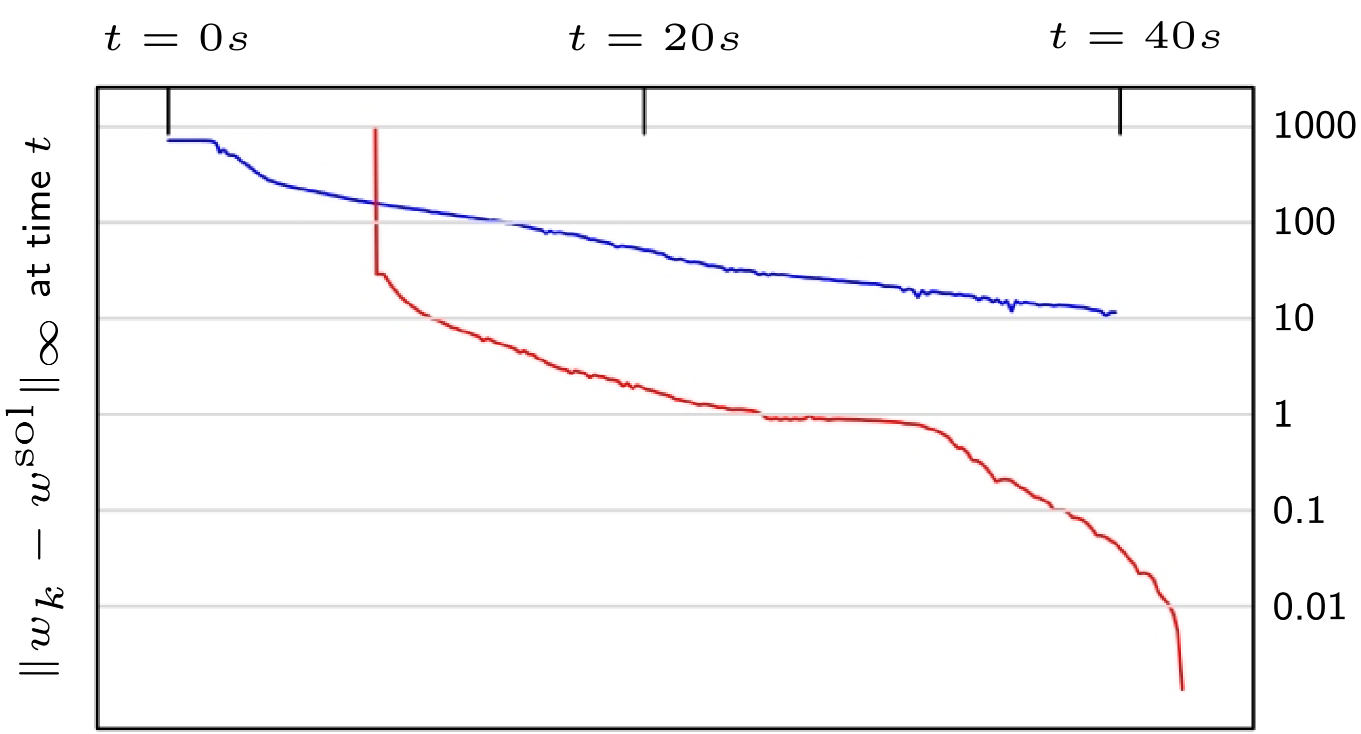
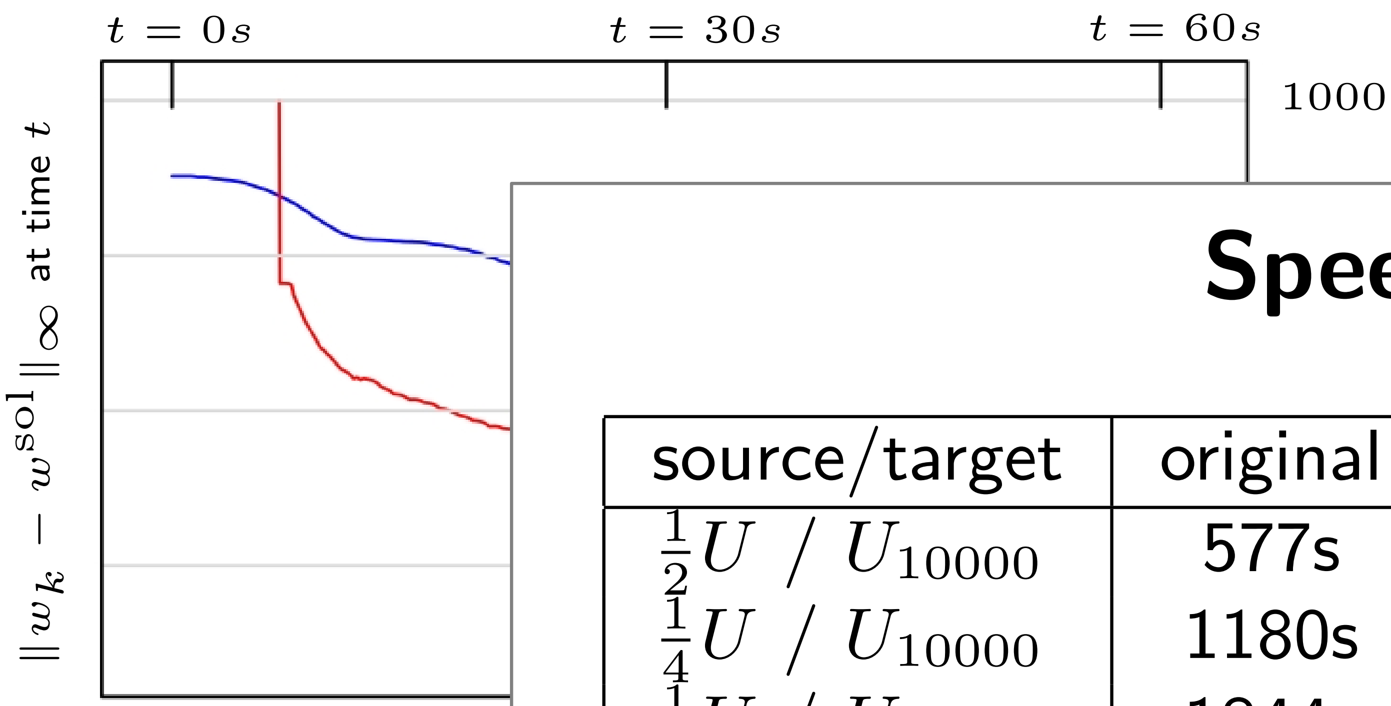
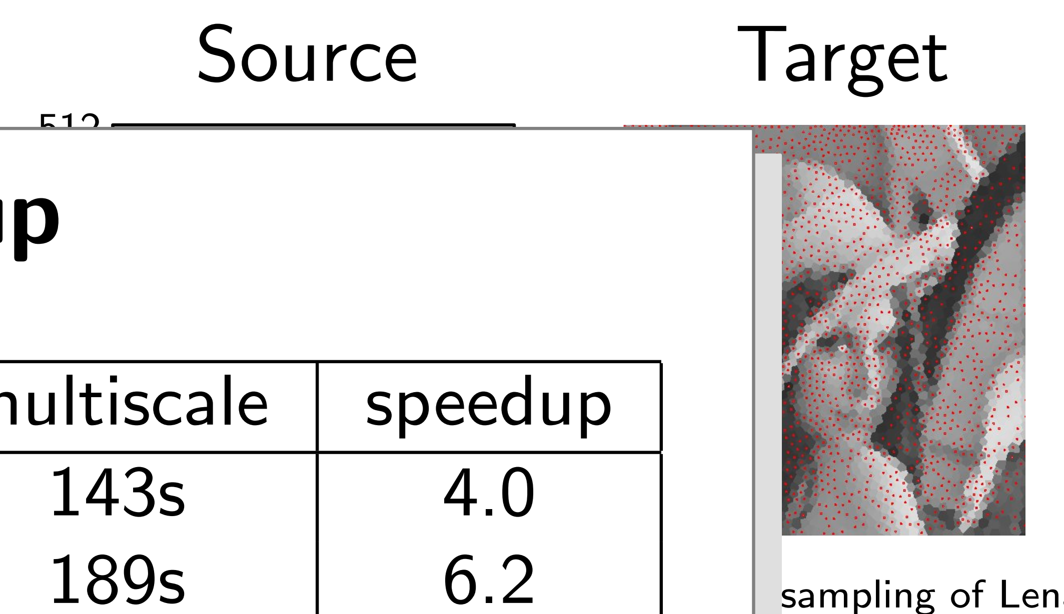


Multiscale vs Original — Convergence Speed





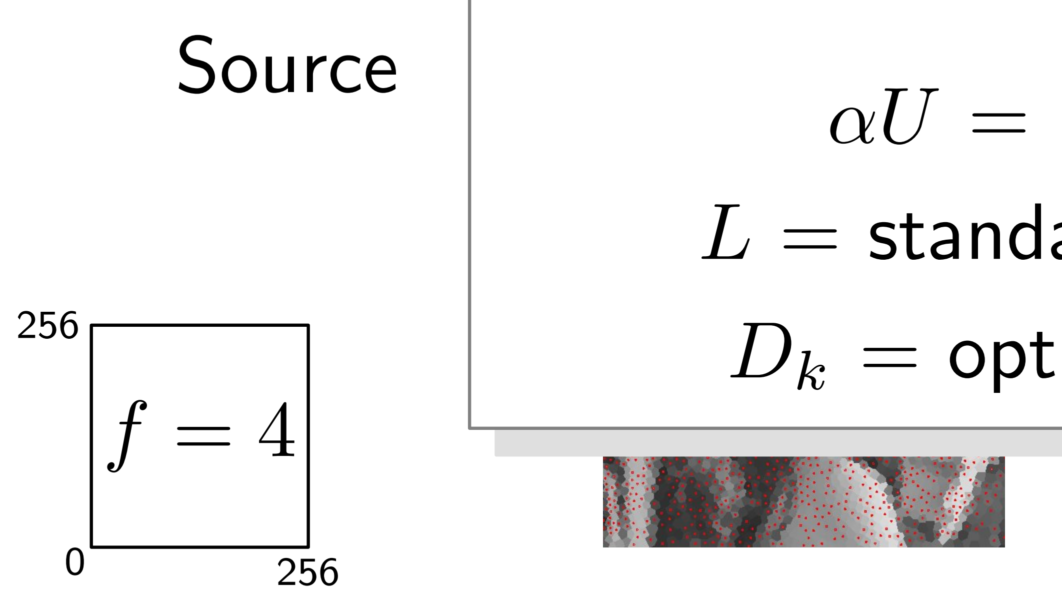
Speedup

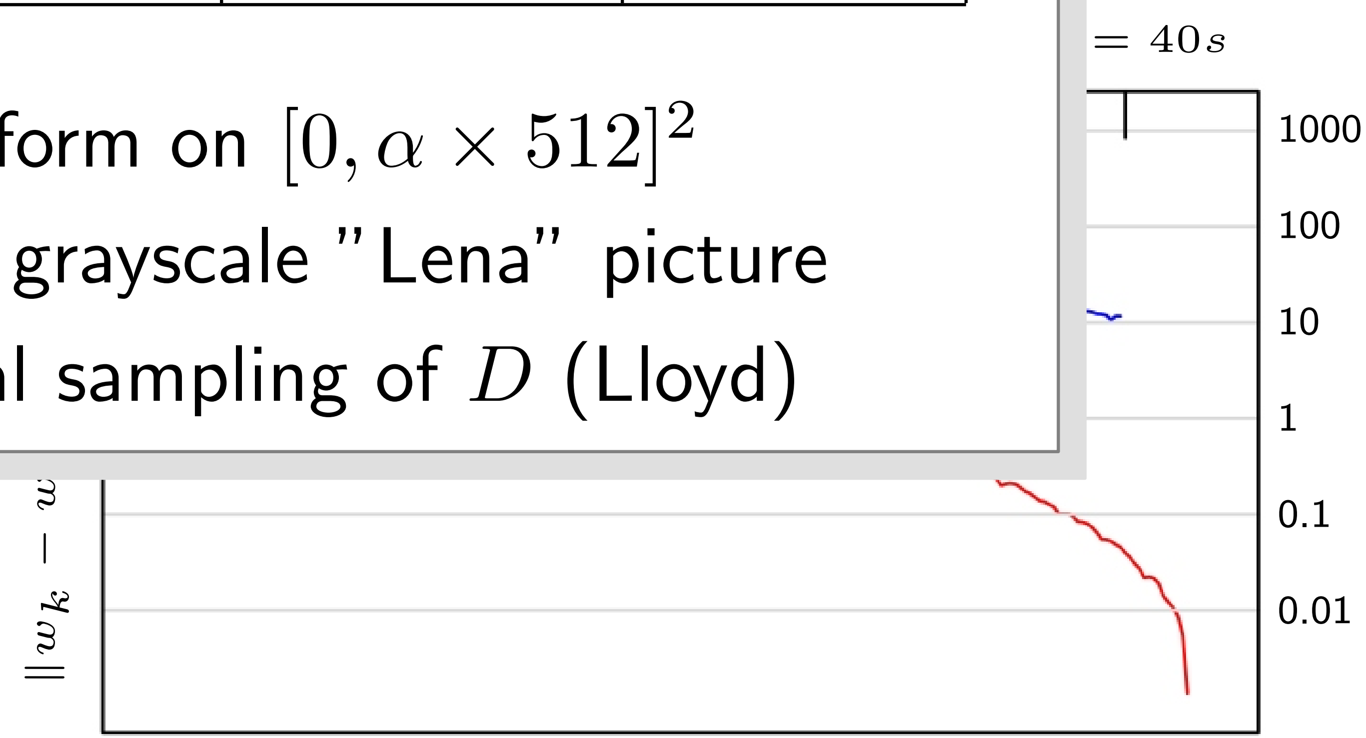
source/target	original	multiscale	speedup
$\frac{1}{2}U$ / U_{10000}	577s	143s	4.0
$\frac{1}{4}U$ / U_{10000}	1180s	189s	6.2
$\frac{1}{8}U$ / U_{10000}	1844s	241s	7.6
U / L_{10000}	216s	52s	4.1

$\alpha U =$ uniform on $[0, \alpha \times 512]^2$

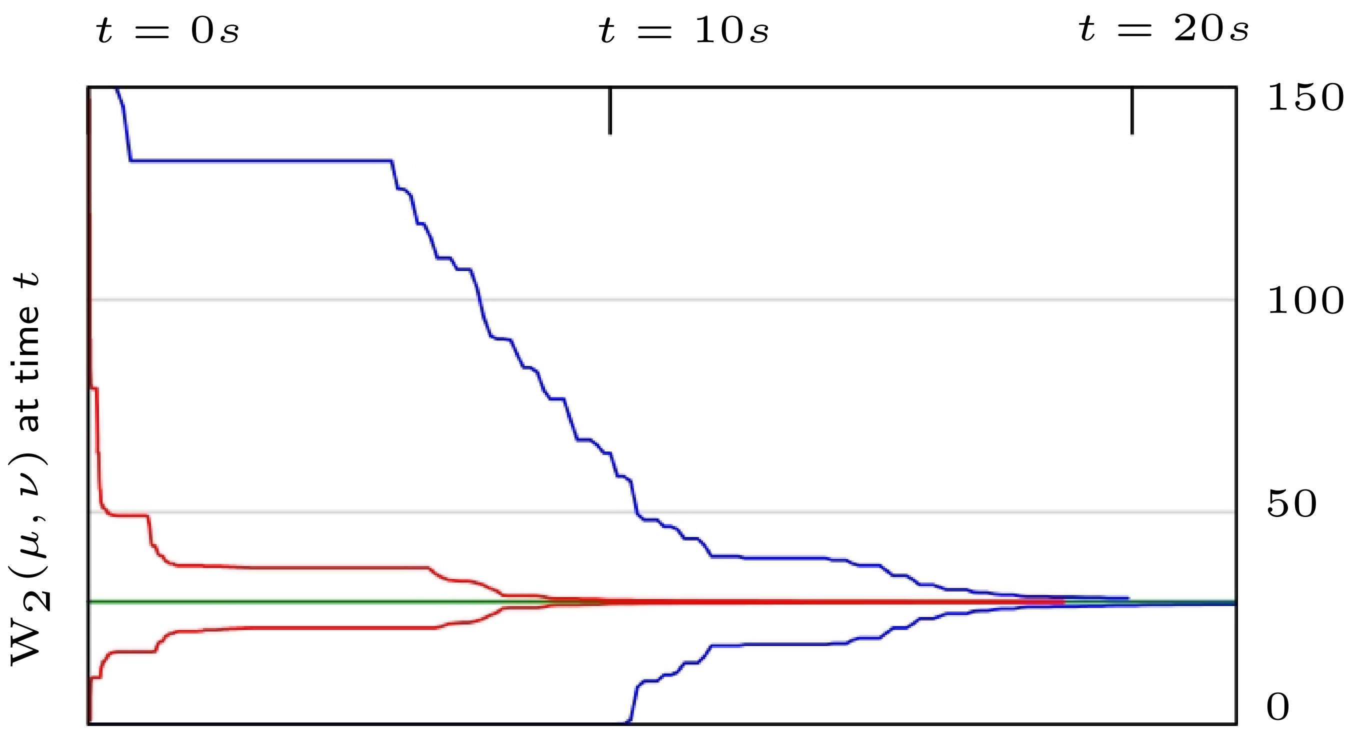
$L =$ standard grayscale "Lena" picture

$D_k =$ optimal sampling of D (Lloyd)

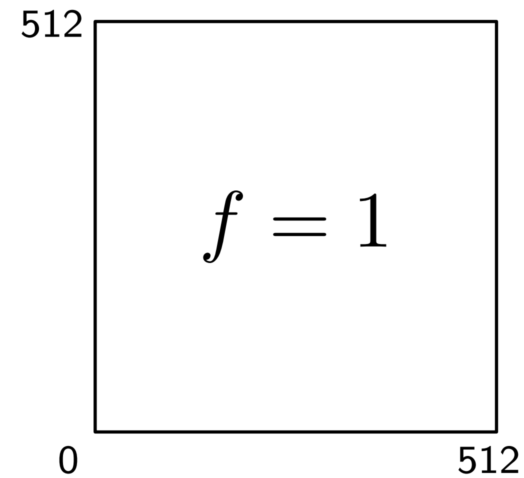




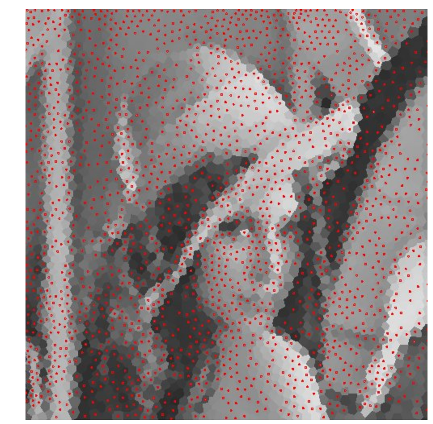
Multiscale vs Original — Wasserstein



Source

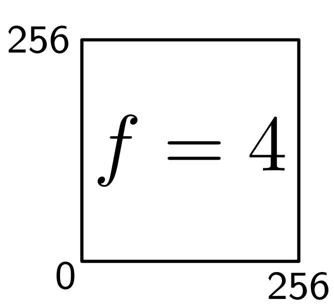


Target

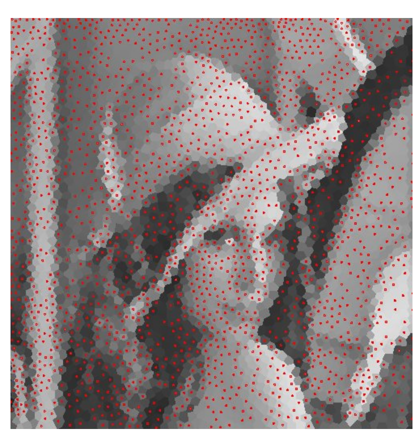


3k Lloyd sampling of Lena

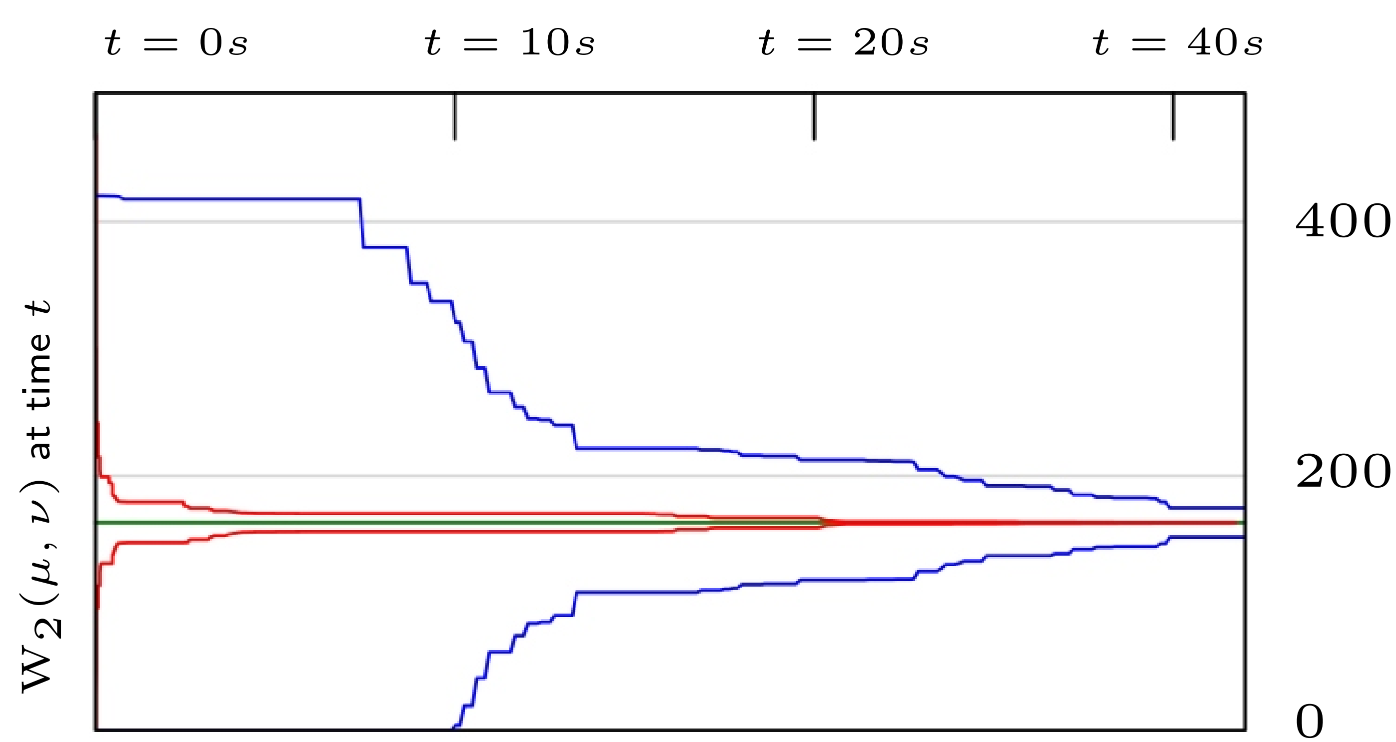
Source



Target



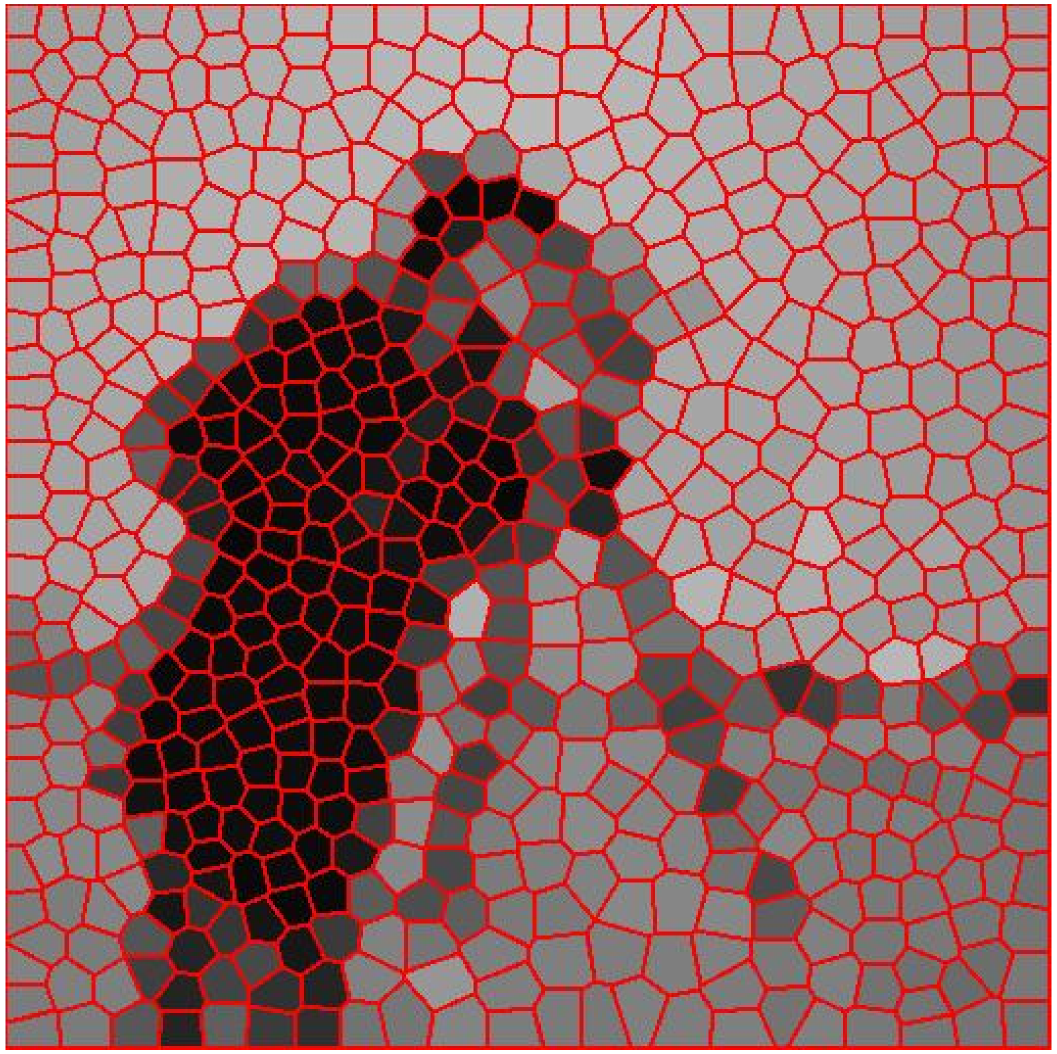
3k Lloyd sampling of Lena



Some Pictures of Optimal Transport Plans

Source: picture "Cameraman"
Target: Lloyd sampling of picture "Peppers" ($k = 625$)

"Displacement interpolation"
McCann '97



The mass of Dirac at p is spread onto $\mathrm{Vor}_P^w(p)$

$$w = w^{\mathrm{sol}}$$

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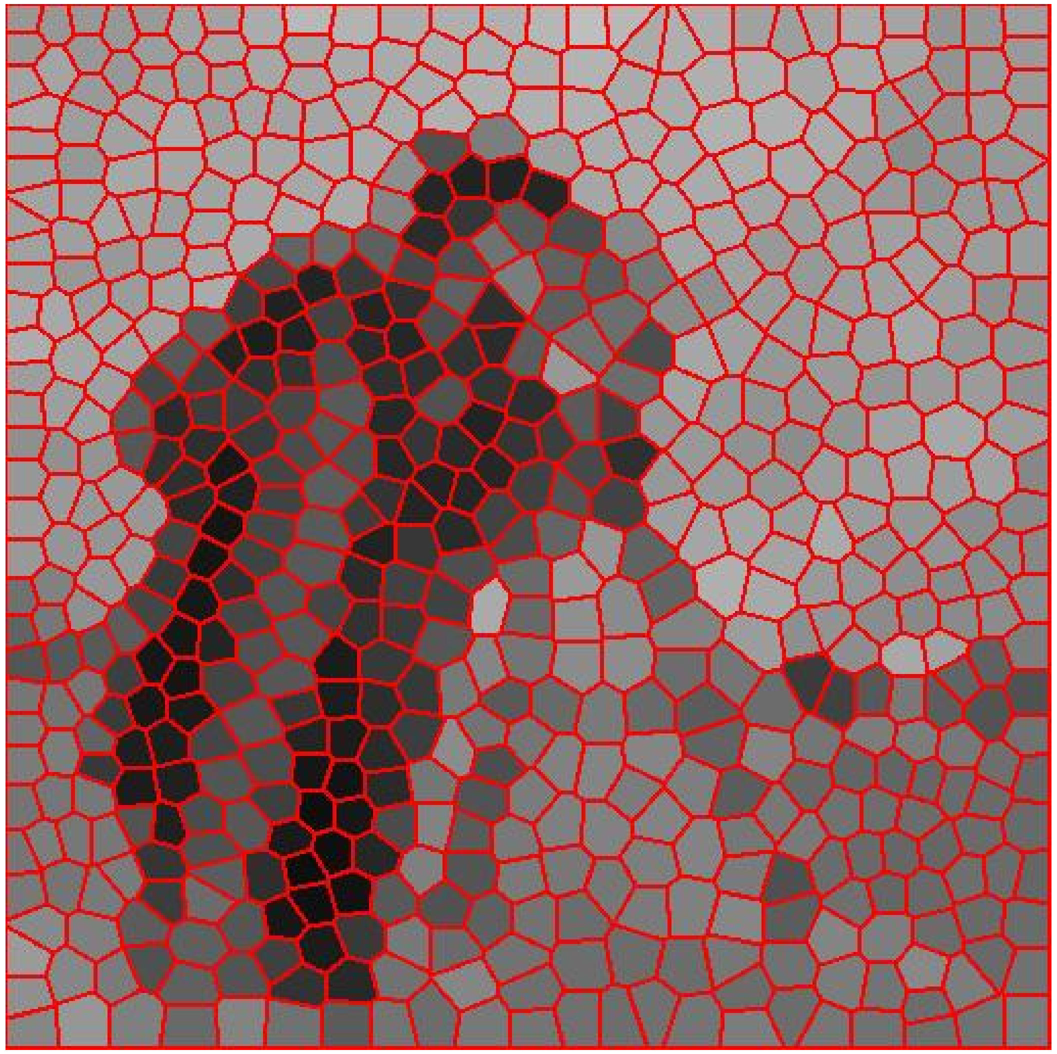

The mass of Dirac at p is spread onto $\mathrm{Vor}_P^w(p)$

$$w = w^{\mathrm{sol}} - \frac{1}{4} w^{\mathrm{sol}}$$

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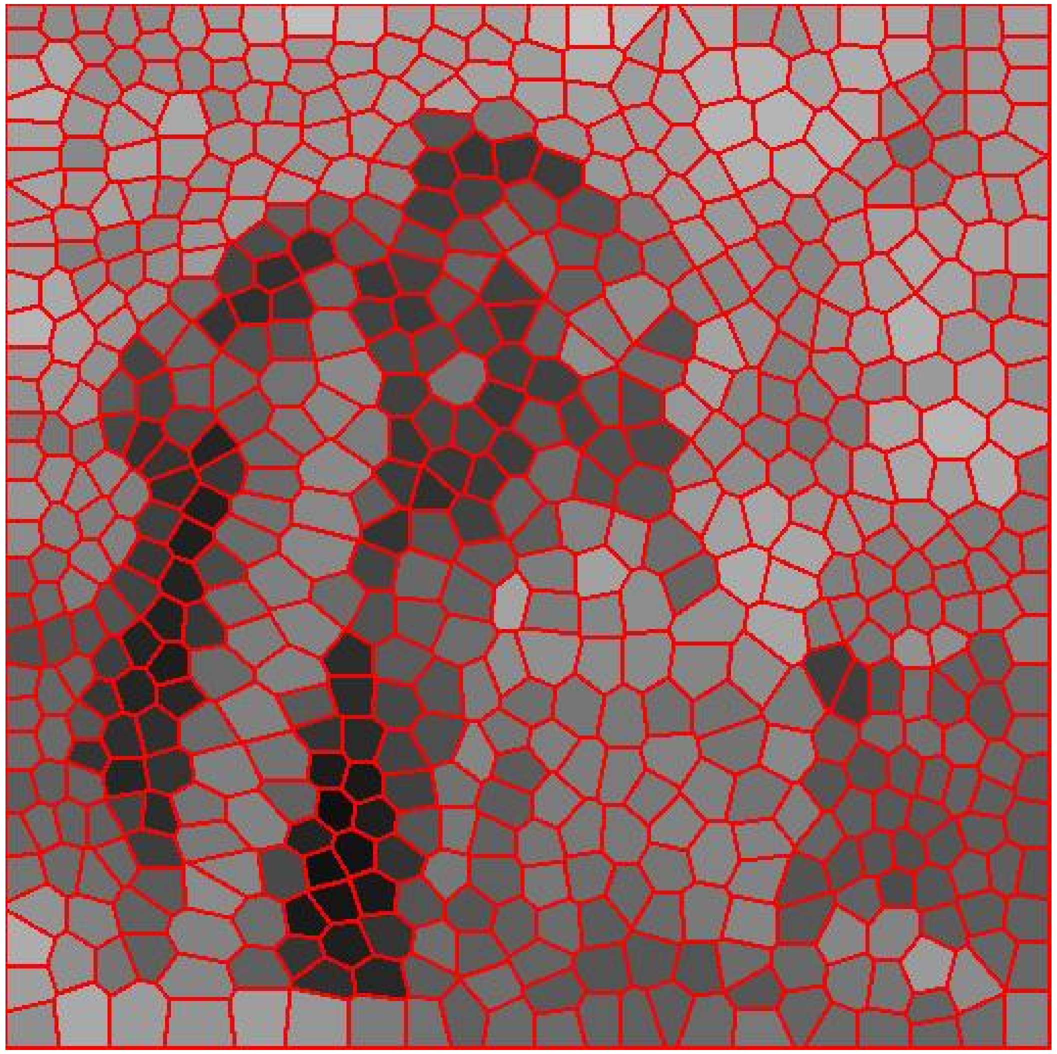
McCann '97



The mass of Dirac at p is spread onto $\mathrm{Vor}_P^w(p)$

$$w = w^{\mathrm{sol}} - \frac{1}{2} w^{\mathrm{sol}}$$

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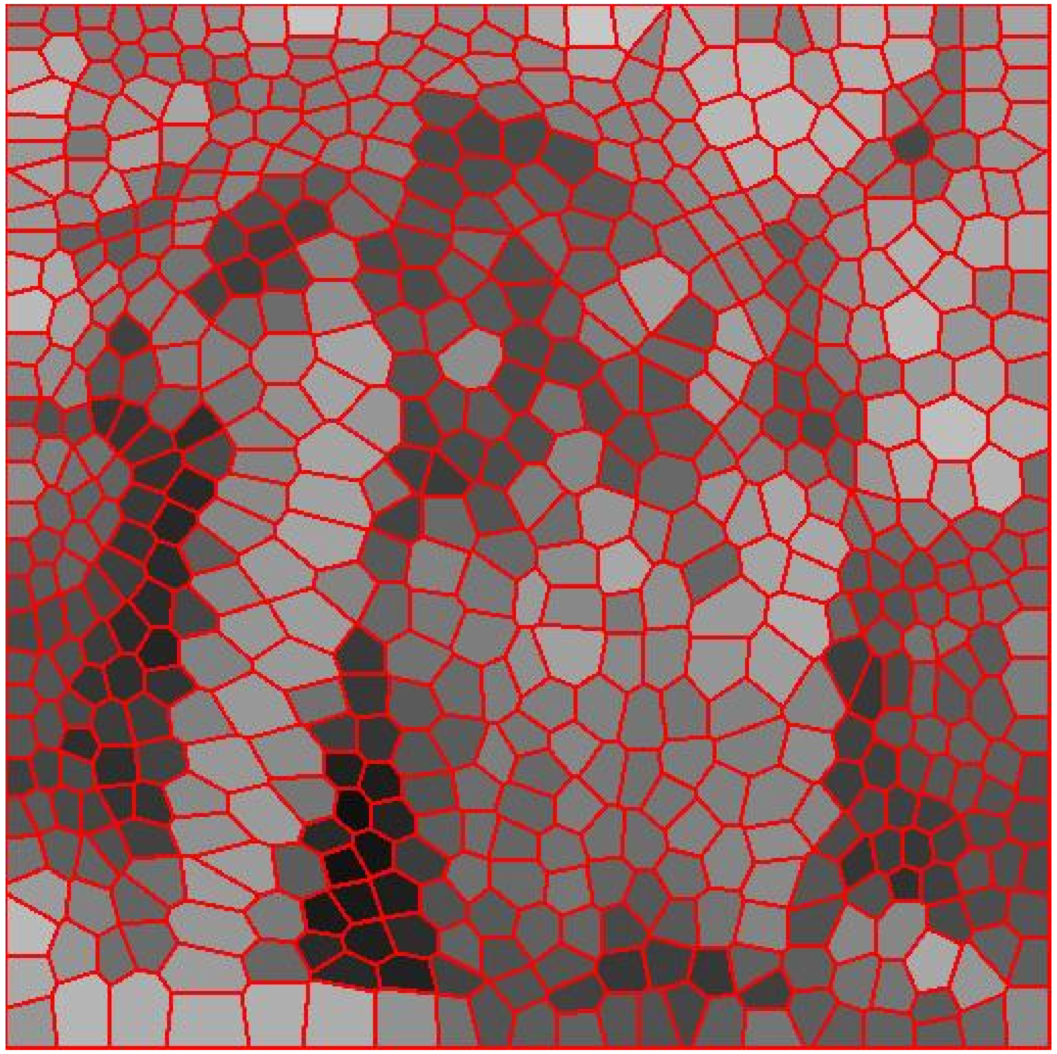
McCann '97



The mass of Dirac at p is spread onto $\mathrm{Vor}_P^w(p)$

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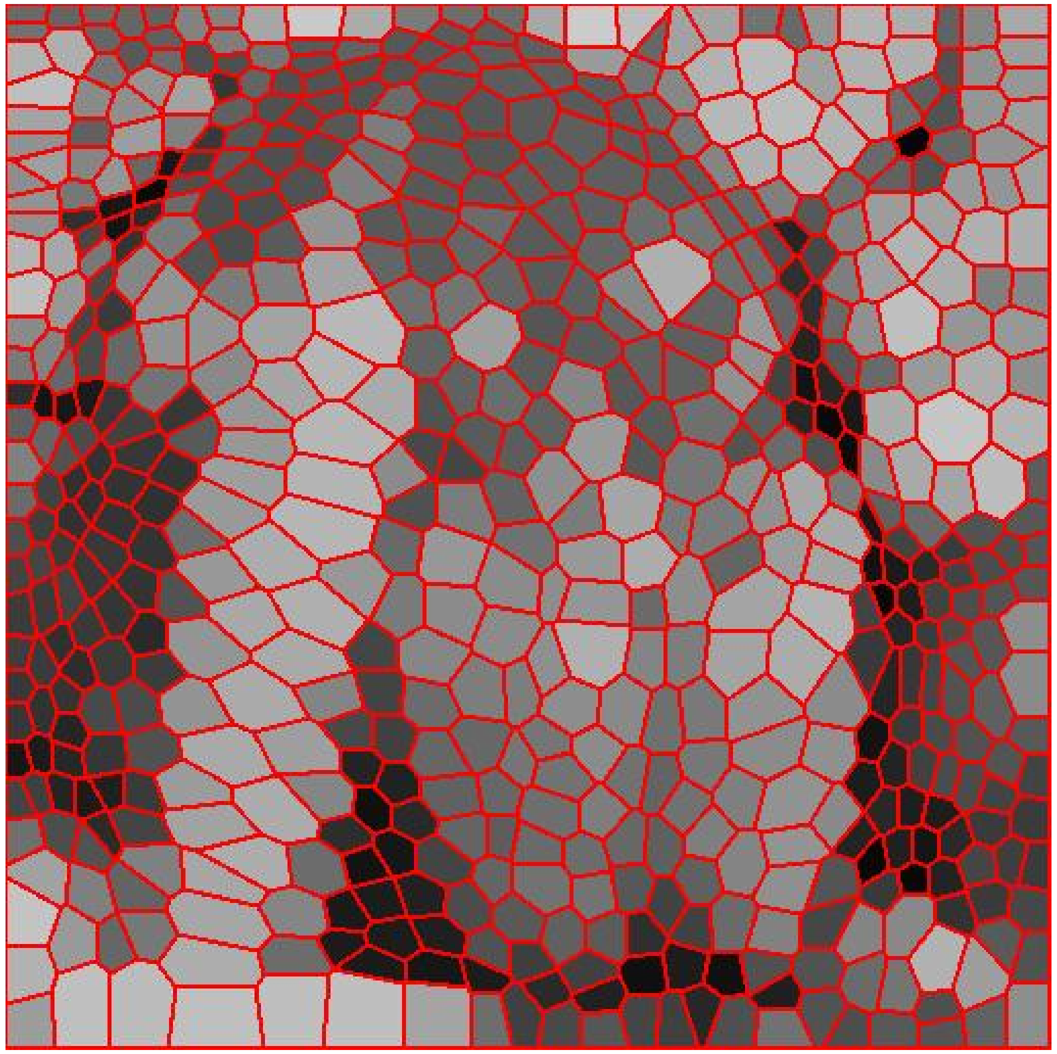
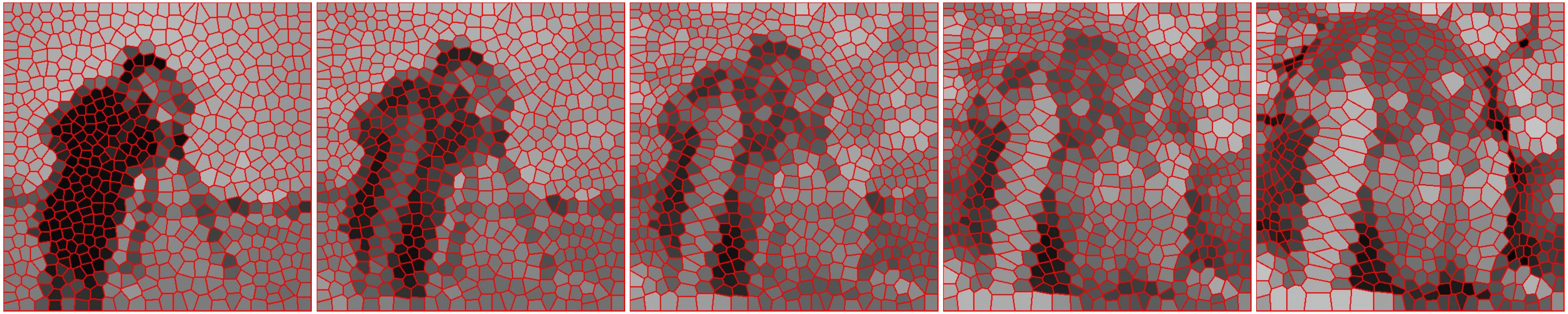


The mass of Dirac at p is spread onto $\mathrm{Vor}_P^w(p)$

$$w = 0$$

Some Pictures of Optimal Transport Plans

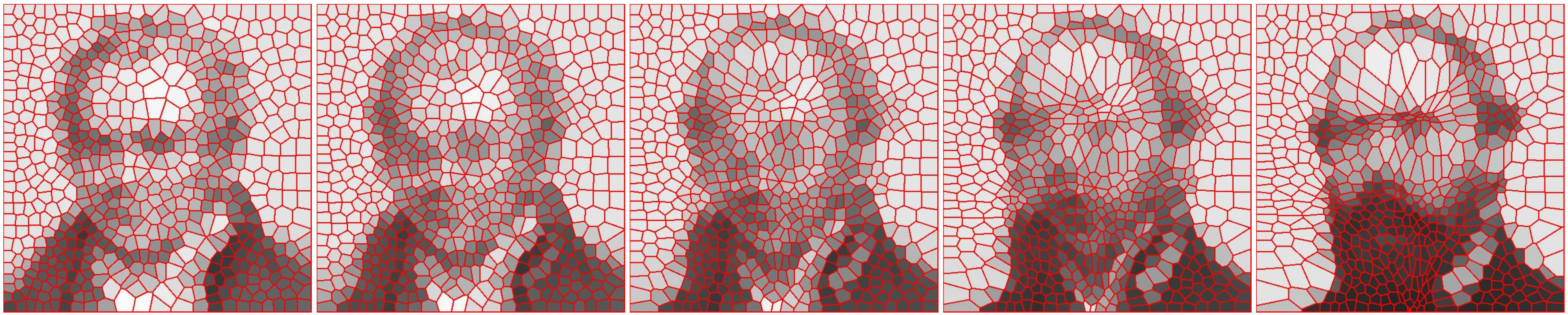
$$k = 625$$



$$k = 15000$$

Some Pictures of Optimal Transport Plans

$$k = 625$$



$$k = 15000$$

4. Assignment problem

ongoing work with Édouard Oudet

L^2 Assignment Problem

Problem: Given $P, Q \subseteq \mathbb{R}^d$ with $|P| = |Q| = N$,
find one-to-one $\sigma : Q \to P$ minimizing $\sum_{q \in Q} \|q - \sigma(q)\|^2$.

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$\iff$ minimising $\Psi(v, w) = -\sum_{q \in Q} v(q) + \sum_{p \in P} w(p)$

under the constraint $v(q) - w(p) \leq \|p - q\|^2$

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$\iff$ minimising $\Phi(w) = -\sum_{q \in Q} \min_{p \in P}(\|q - p\|^2 + w(p)) + \sum_{p \in P} w(p)$

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Dual formulation yields a non-smooth convex function

Φ is smooth at $w \iff \min_{p \in P}(\|q - p\|^2 + w(p))$ is unique for every $q \in Q$

L^2 Assignment Problem: A useful trick

Problem: min. $\Phi(w) = -\sum_{q\in Q} \min_{p\in P}(\|q-p\|^2 + w(p)) + \sum_{p\in P} w(p)$

- Suppose that $w \geq 0$ (Φ is invariant to shifts of w by a constant),

$$\min_{p\in P}(\|q-p\|^2 + w(p)) = \min_{p\in P} \|(q,0) - (p, \sqrt{w(p)})\|_{\mathbb{R}^{d+1}}$$

L^2 Assignment Problem: A useful trick

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- **kD-tree:** A data-structure for finding nearest neighbors.

Build time: $\mathrm{O}(N \log N)$

Query time: $\mathrm{O}(\log N)$ (for uniformly distributed points)

L^2 Assignment Problem: A useful trick

Problem: min. $\Phi(w) = -\sum_{q\in Q} \min_{p\in P}(\|q-p\|^2 + w(p)) + \sum_{p\in P} w(p)$

- Suppose that $w \geq 0$ (Φ is invariant to shifts of w by a constant),

$$\min_{p\in P}(\|q-p\|^2 + w(p)) = \min_{p\in P} \|(q,0) - (p, \sqrt{w(p)})\|_{\mathbb{R}^{d+1}}$$

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$\longrightarrow$ Efficient evaluation of $\Phi(w)$, $\nabla\Phi(w)$ and $\partial\Phi(w)$ (equality cases)

L^2 Assignment Problem: Auction algorithm

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Many improvements since Bertsekas' original Auction algorithm. We use the fastest to date: Bus-Tvrdik '11

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- Proposal: when this happens, turn to a *local linearization* of Ψ and use a LP solver.

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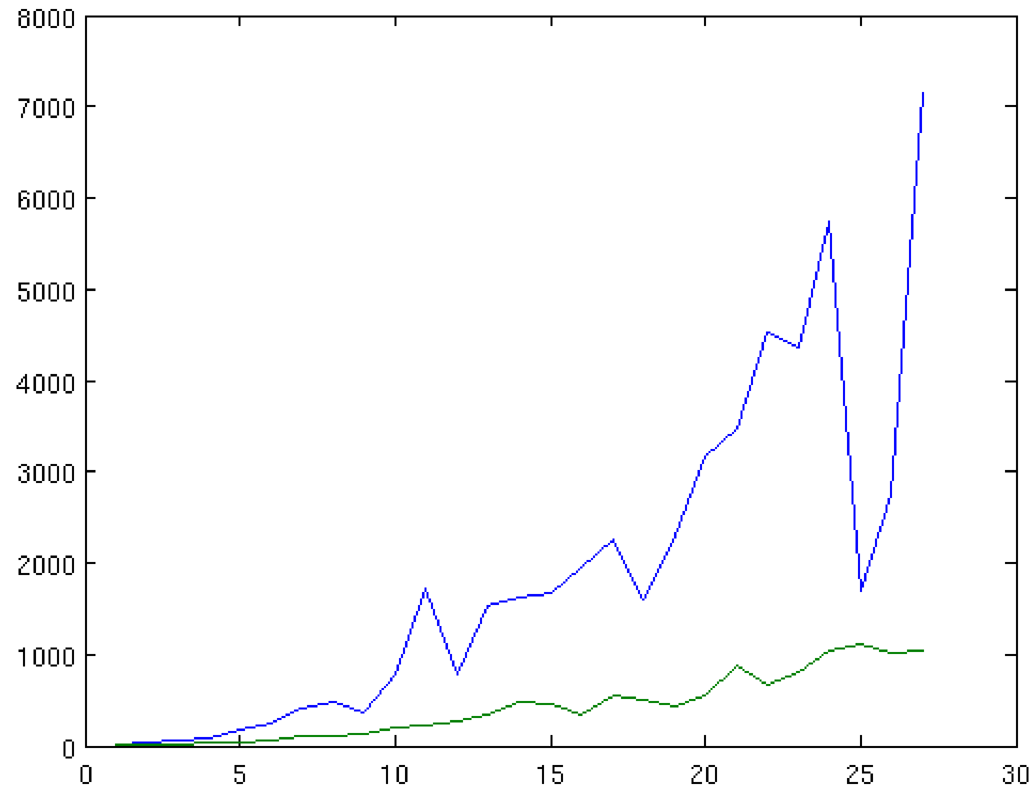
$$\forall p, q, \quad v(q) - w(p) \leq \|p-q\|^2$$

$$\Longleftrightarrow$$

$$\forall q, \forall 1 \leq i \leq k, \quad v(q) - w(p_i(q)) \leq \|p_i(q) - q\|^2 \quad \textbf{and} \quad \|\delta\| \leq \delta_0$$

L^2 Assignment Problem: Another approach?

Running time in seconds of **Auction** (blue) vs **LBFGS** and linearisation (green)



Data: P and Q are two random sample of N points in the cube $[0, 10^5]^3 \cap \mathbb{Z}^3$, for $N = 1k, \ldots, 30k$.

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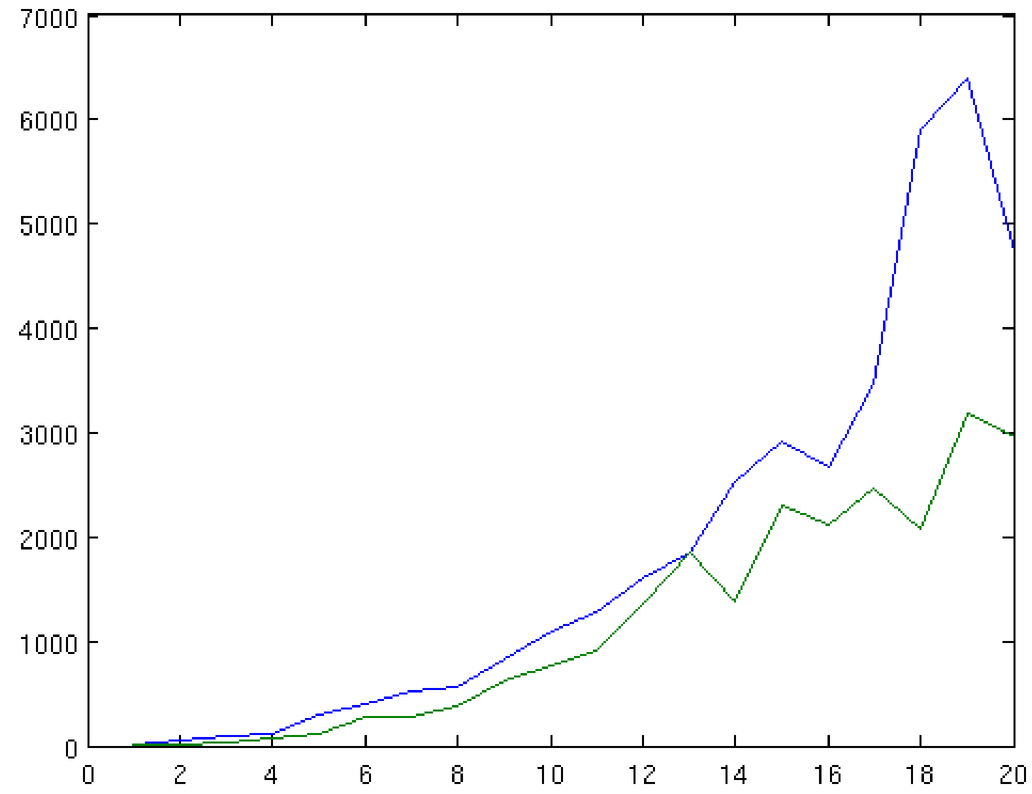

Data: P is a random sample of N points in the cube $[0, 10^5]^3 \cap \mathbb{Z}^3$, Q is obtained from a mixture of 15 isotropic Gaussian distributions for $N = 1k, \ldots, 20k$.

Conclusion

Open questions:

- Dependency of convergence speed on the "geometry" of μ and ν, i.e. quantitative stability theorem for OT plans ?
- Theory when both measures are discrete ? Complexity ?

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- Design of reflector antennas with prescribed far-field image.

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Thank you for your attention!