Inverse Scale Space Methods for Image Reconstruction

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Medical Image Reconstruction

Motivation for studying image reconstruction methods

Emerging medical imaging techniques

- PET
- SPECT
- MR
- Optical
- Raman
- CT
- ...

Spectacular insights

- Human body - e.g. a beating heart (from Jahn Müller)

Spectacular insights

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- Kids toys - e.g. a Kinder surprise egg

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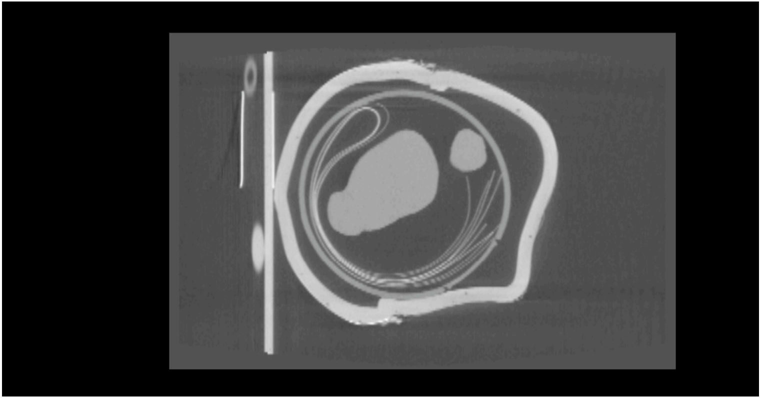

Figure: Scan and reconstruction by Jahn Müller

Spectacular insights

Spectacular insights

- Kids toys - e.g. a Kinder surprise egg

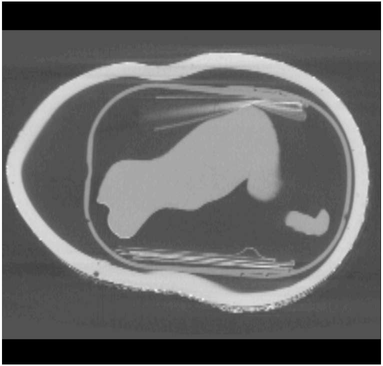

Figure: Scan and reconstruction by Jahn Müller

Surprise toy reconstruction (from Jahn Müller)

Inverse Problems

The mathematics behind the pretty pictures:

- Image reconstruction can often be formulated as recovering the desired image u from

$$f = Au + n_\delta \tag{1}$$

with

  - measured data f
  - noise n_δ
  - linear operator A (problem dependent, e.g. radon transform, blurring kernel, etc.)

$\rightarrow$ For all almost all applications **ill-posed**!

Inverse Problems

Image reconstruction as an inverse problem

Popular approach: Variational formulation.

$$\hat{u} = \arg\min_u \left(H_f(Au) + \alpha J(u) \right) \tag{2}$$

- Motivated from the maximum-likelihood estimate using Bayes model.
- Data fidelity term $H_f(Au)$ depending on the noise model,
  - e.g. $H_f(Au) = \frac{1}{2}\|Au - f\|_2^2$ for Gaussian noise.
- Regularization $J(u)$ determining what a 'good' solution $\hat{u}$ is,
  - e.g. low total variation $J(u) = \int |\nabla u| dx$,
  - e.g. sparsity via ℓ^1 regularization $J(u) = \|u\|_1$.

Inverse Problems

Problem of the formulation

$$\hat{u} = \arg\min_u \left(H_f(Au) + \alpha J(u) \right) \tag{3}$$

- Systematic loss of contrast!

Inverse Problems

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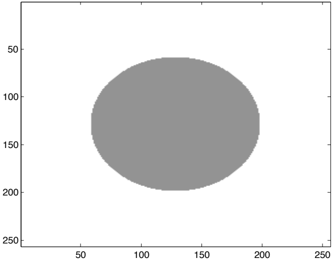
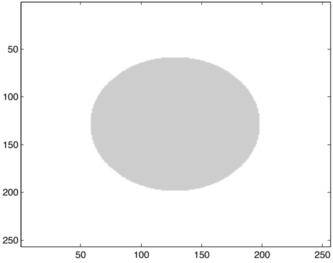
- Systematic loss of contrast!
- Example: TV denoising



(c) True image



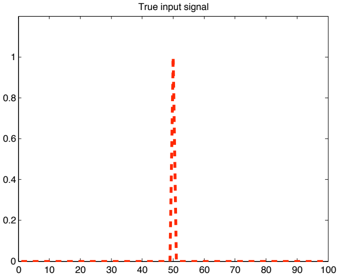
(d) Denoised image

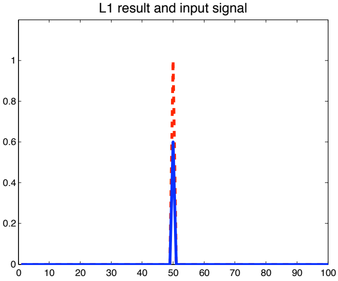
Inverse Problems

Problem of the formulation

$$\hat{u} = \arg\min_{u} \left(H_f(Au) + \alpha J(u) \right) \qquad (4)$$

- Systematic loss of contrast!
- Example: ℓ^1 minimization





Bregman iteration

Simple idea (from [4]) for fidelity $H_f(Au) = \frac{1}{2}\|Au - f\|^2$:

- Add back the error to amplify signal, iteratively denoise to reduce fine scale structures (=noise)

$$u^k = \arg\min_u \left(\frac{1}{2}\|Au - f^{k-1}\|^2 + \alpha J(u)\right)$$

$$f^k = f^{k-1} + (f - Au^k) \tag{5}$$

Stop before undesirable fine scale structures come back.

Bregman iteration

Adding back the noise to the signal is not as crazy as it sounds ...

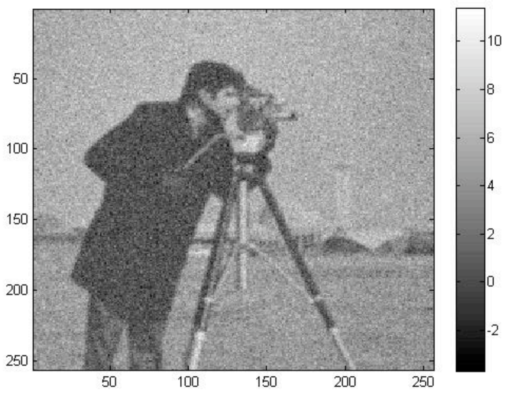


Figure: Noisy image

Bregman iteration

Adding back the noise to the signal is not as crazy as it sounds ...

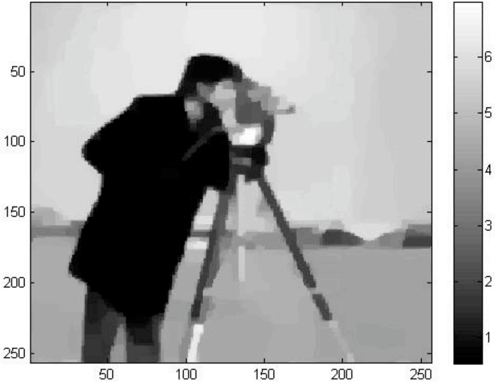


Figure: First denoised image - scale is reduced by almost 1/2

Bregman iteration

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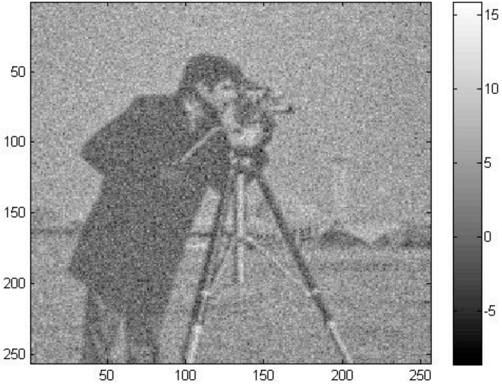

Figure: Add back the detected noise

Bregman iteration

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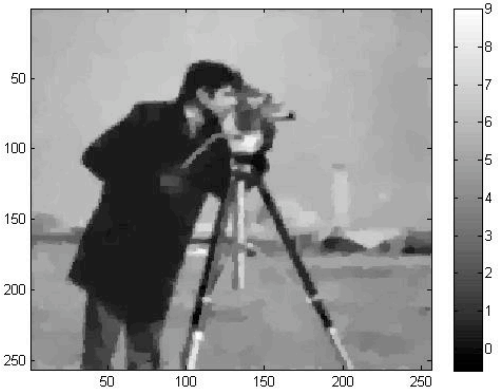
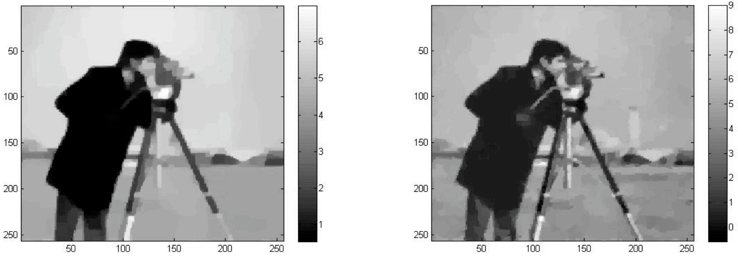


Figure: Scale of the original image was 0-9

Bregman iteration

Adding back the noise to the signal is not as crazy as it sounds ...



(a) TV (b) Bregman

Bregman iteration

The adding back the noise is equivalent to the so called **Bregman iteration** ([4])

$$u^k = \arg\min_u \left(\frac{1}{2}\|Au - f\|^2 + \alpha(J(u) - \langle p^{k-1}, u\rangle)\right), \tag{6}$$

with $p^{k-1} \in \partial J(u^{k-1})$.

- For this quadratic fidelity term it is equivalent to the Augmented Lagrangian method for determining the $J(\cdot)$ minimizing solution to $Au = f$.
- Numerical results improve when taking many iterations with large α.

Bregman iteration

Optimality condition to

$$u^k = \arg\min_u \left(\frac{1}{2}\|Au - f\|^2 + \alpha(J(u) - \langle p^{k-1}, u\rangle)\right), \tag{7}$$

can be written as

$$\frac{p^k - p^{k-1}}{\frac{1}{\alpha}} = A^T(f - Au^k) \tag{8}$$

with $p^k \in \partial J(u^k)$, $p^{k-1} \in \partial J(u^{k-1})$. For large α the LHS looks like the approximation of a derivative. Continuous formulation

$$\frac{d}{dt}p(t) = A^T(f - Au(t)) \qquad \text{such that } p(t) \in \partial J(u(t)) \tag{9}$$

is called the **inverse scale space flow** ([2]).

Inverse scale space flow

In recent work we showed

- The inverse scale space flow

$$\frac{d}{dt}p(t) = A^T(f - Au(t)) \qquad \text{such that } p(t) \in \partial J(u(t)) \quad (10)$$

  can be solved exactly (without discretization) for $J(\cdot)$ which are polyhedral ([1, 3]).
- Includes anisotropic TV, ℓ^1 regularization, linear equality or inequality constraints, ℓ^1-wavelet, framelet, shearlet, etc.

Inverse scale space flow

Example: The inverse scale space flow for ℓ^1 regularization

$$\frac{d}{dt}p(t) = A^T(f - Au(t)) \qquad \text{such that } p(t) \in \partial\|u(t)\|_1 \qquad (11)$$

- Important for the ℓ^1 flow: Characterization of the ℓ^1 subdifferential

$$p \in \partial\|u\|_1 \Leftrightarrow \begin{cases} |p_i| \leq 1 & \text{if } u_i = 0, \\ p_i = sign(u_i) & \text{else.} \end{cases} \qquad (12)$$

Inverse scale space flow

Example: The inverse scale space flow for ℓ^1 regularization

$$\frac{d}{dt}p(t) = A^T(f - Au(t)) \qquad \text{such that } p(t) \in \partial\|u(t)\|_1 \tag{13}$$

- We have $u(t) = 0$ for $t < \frac{1}{\|A^T f\|_\infty}$:
  For $u(t) = 0$ we have $\frac{d}{dt}p(t) = A^T f$, thus $p(t) = tA^T f$. Due to the characterization of the subdifferential $u(t) = 0$ is the only element for which $p(t) \in \partial\|u(t)\|_1$ as long as $\|p(t)\|_\infty < 1$ i.e. as long as $t < \frac{1}{\|A^T f\|_\infty}$.

Inverse scale space flow

Example: The inverse scale space flow for ℓ^1 regularization

- At $t^1 = \frac{1}{\|A^T f\|_\infty}$ we have (by continuity of $p(t)$) $p(t^1) = t^1 A^T f$. Now we determine the set $I = \{i \mid |p_i(t^1)| = 1\}$ and compute

$$u(t^1) = \arg\min_u \|AP_I u - f\|^2 \qquad (14)$$

under the constraints $u_i \geq 0$ if $p_i(t^1) = 1$ and $u_i \leq 0$ if $p_i(t^1) = -1$. This guarantees $p(t^1) \in \partial \|u(t^1)\|_1$.

Inverse scale space flow

Example: The inverse scale space flow for ℓ^1 regularization

- There exists some time t^2 such that

$$p(t) = p(t^1) + (t - t^1)A^T(f - Au(t^1)) \in \partial \|u(t^1)\|_1$$

for $t \leq t^2$. Why?

Inverse scale space flow

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  - For indices where $|p_i(t^1)| < 1$, we will have $|(p(t^1) + (t - t^1)A^T(f - Au(t^1)))_i| < 1$ for t small enough.

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  - If $u_i(t^1) > 0$ then the optimality condition for our optimization problem tells us $\left(A^T(f - Au(t^1))\right)_i = 0$ and thus

$$(p(t^1) + (t - t^1)A^T(f - Au(t^1)))_i = p_i(t^1) = 1$$

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$$(p(t^1) + (t - t^1)A^T(f - Au(t^1)))_i = p_i(t^1) = 1$$

  - If $u_i(t^1) = 0$ and $p_i(t^1) = 1$ then the optimality condition for our optimization problem tells us $A^T(f - Au(t^1)) \leq 0$ and thus

$$|(p(t^1) + (t - t^1)A^T(f - Au(t^1)))_i| \leq 1$$

for some t small enough.

Inverse scale space flow

Example: The inverse scale space flow for ℓ^1 regularization

- Iteratively repeating this argument shows that $u(t)$ is piecewise constant in time and $p(t)$ piecewise linear.
- For ℓ^1 the optimization problems only involve the (small) index set $\{i \mid |p_i(t^1)| = 1\}$.
- General ideas work for any polyhedral regularization.

Bregman iteration in application

Why Bregman iteration and the inverse scale space flow are important in application

- Bias of regularizations like total variation or ℓ^1 does not just lead to 'less pretty pictures', we have an actual loss of information!
- Examples for the improved contrast, image information and image quality recovery of Bregman iteration: PET image recovery.

Bregman iteration in application

Why Bregman iteration and the inverse scale space flow are important in application

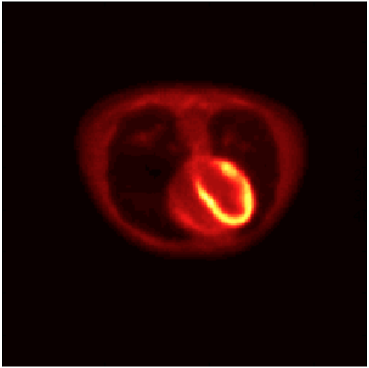

Figure: PET image, FDG 18 Thorax Scan, transversal, 20 min EM reconstruction[2]

[1] Images and techniques from Jahn Müller and Alex Sawatzky

Bregman iteration in application

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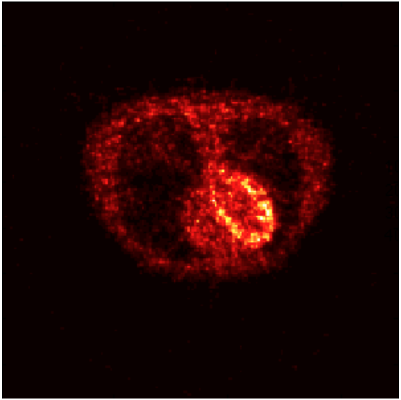

Figure: PET image, FDG 18 Thorax Scan, transversal, 5 second EM reconstruction[4]

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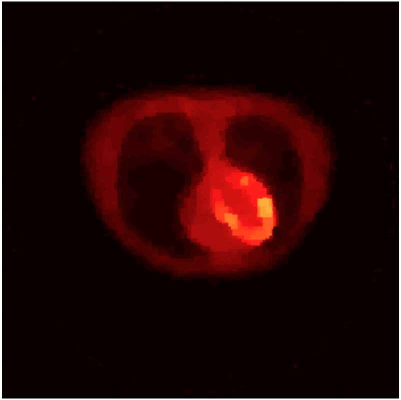

Figure: PET image, FDG 18 Thorax Scan, transversal, 5 second EM-TV reconstruction[6]

Bregman iteration in application

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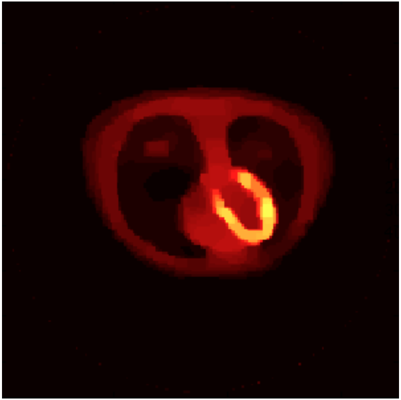

Figure: PET image, FDG 18 Thorax Scan, transversal, 5 second -Bregman-EM-TV reconstruction[8]

Bregman iteration in application

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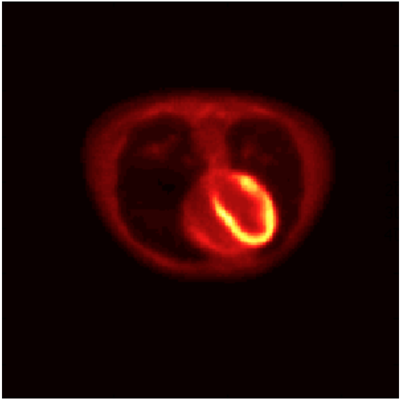
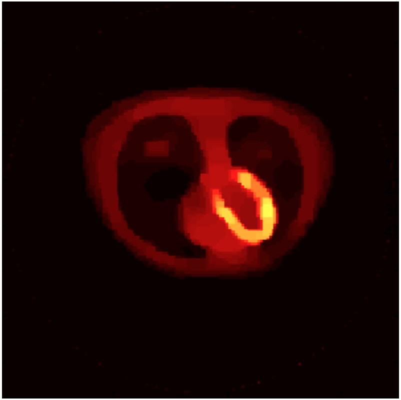


(a) EM



(b) Bregman EM-TV

Bregman iteration in application

Advantages

- Better image quality
- Similar images with lower count rates
- Reduction of the doses of the radioactive tracer
- Faster scans

Conclusions

Conclusions and future research

- Bregman iteration restores the information lost in typical regularization methods
- Can be formulated continuously in the inverse scale space flow
- Exact computation of the ℓ^1 inverse scale space flow by solving low dimensional non-negative least squares
- In the future: Apply continuous inverse scale space flow to image reconstruction techniques (either with TV or concepts like wavelets, framelets, etc.)

THANK YOU

THANK YOU!

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