INSTITUT D'ASTROPHYSIQUE DE PARIS
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Journée Détection et Ondes Gravitationnelles

GRAVITATIONAL WAVES AND GENERAL RELATIVITY

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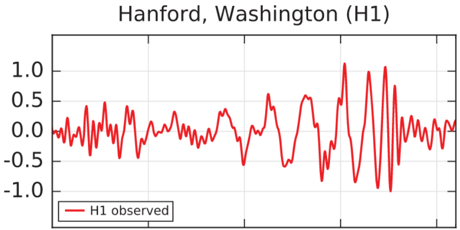
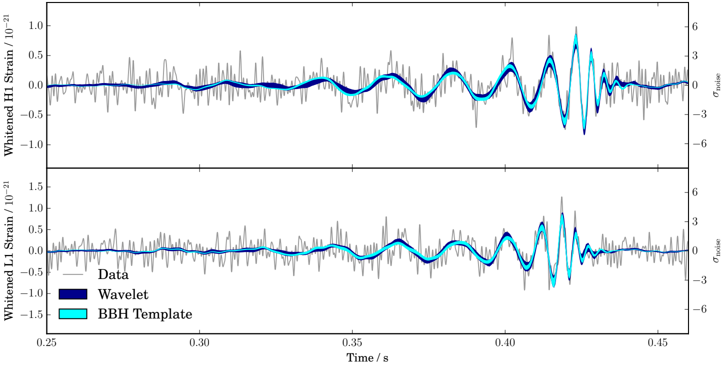
Institut d'Astrophysique de Paris

11 octobre 2017

Binary black-hole event GW150914 [LIGO/VIRGO collaboration 2016]





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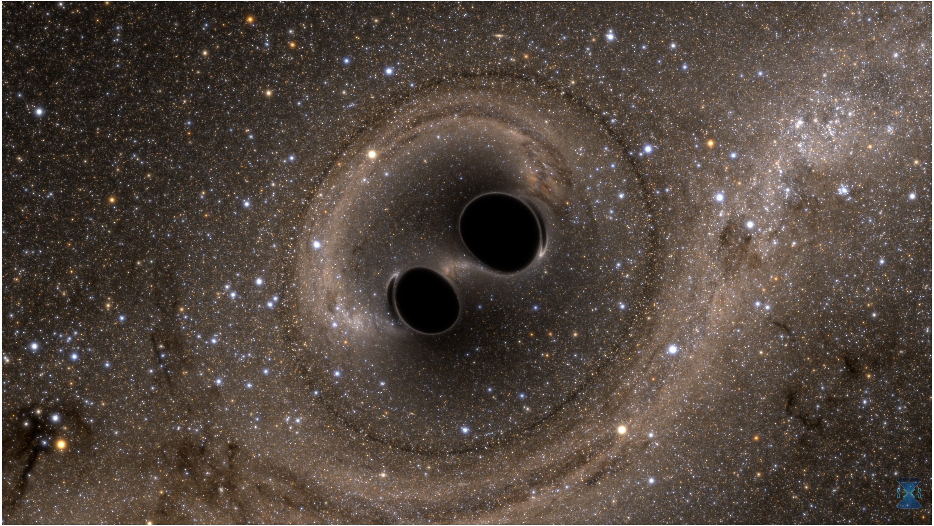

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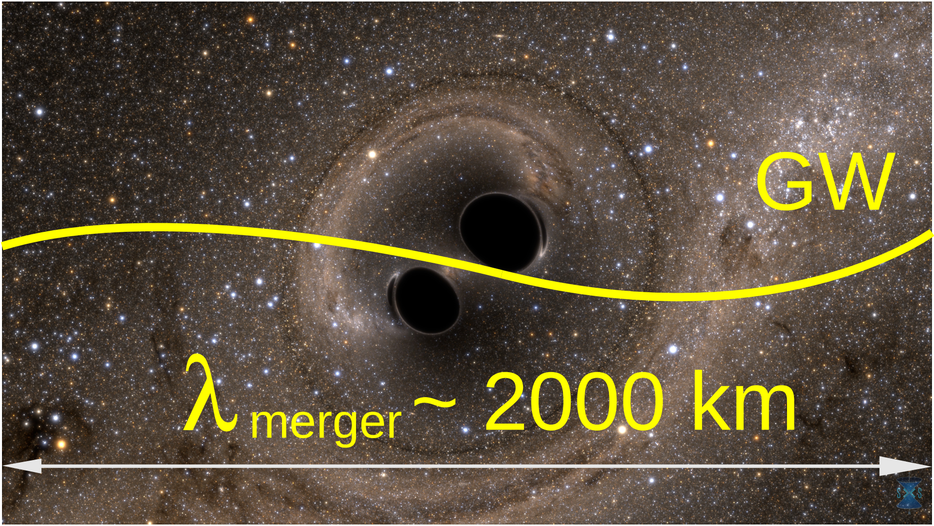
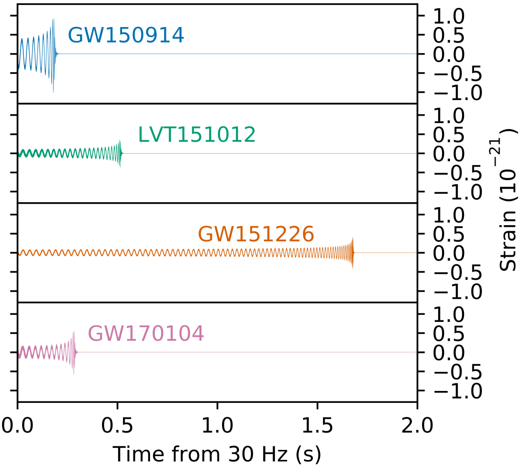


Gravitational wave events [LIGO/VIRGO collaboration 2016, 2017]



- For BH binaries the detectors are mostly sensitive to the merger phase and a few cycles are observed before coalescence
- For NS binaries the detectors will be sensitive to the inspiral phase prior the merger and tens of thousands of cycles will be observable

100 years of gravitational waves [Einstein 1916, 1918]

348 DOC. 32 INTEGRATION OF FIELD EQUATIONS

688 Sitzung der physikalisch-mathematischen Klasse vom 22. Juni 1916

Näherungsweise Integration der Feldgleichungen der Gravitation.

Von A. Einstein.

Bei der Behandlung der meisten speziellen (nicht prinzipiellen) Probleme auf dem Gebiete der Gravitationstheorie kann man sich damit begnügen, die $g_{\mu\nu}$ in erster Näherung zu berechnen. Dabei bedient man sich mit Vorteil der imaginären Zeitvariable $x_4 = it$ aus denselben Gründen wie in der speziellen Relativitätstheorie. Unter »erster Näherung« ist dabei verstanden, daß die durch die Gleichung

$$g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu} \quad (1)$$

⟸ small perturbation of Minkowski's metric

definierten Größen $\gamma_{\mu\nu}$, welche linearen orthogonalen Transformationen gegenüber Tensorcharakter besitzen, gegen 1 als kleine Größen behandelt werden können, deren Quadrate und Produkte gegen die ersten Potenzen vernachlässigt werden dürfen. Dabei ist $\delta_{\mu\nu} = 1$ bzw. $\delta_{\mu\nu} = 0$, je nachdem $\mu = \nu$ oder $\mu \neq \nu$.

Wir werden zeigen, daß diese $\gamma_{\mu\nu}$ in analoger Weise berechnet werden können wie die retardierten Potentiale der Elektrodynamik.

Quadrupole moment formalism [Einstein 1918; Landau & Lifchitz 1947]

$$4\pi R^2 \bar{\mathfrak{J}} = \frac{\varkappa}{40\pi}\left[\sum_{\mu\nu} \dddot{J}_{\mu\nu}^2 - \frac{1}{3}\left(\sum_{\mu} \dddot{J}_{\mu\mu}\right)^2\right].$$

1. Einstein quadrupole formula

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)^{\mathsf{GW}} = \frac{G}{5c^5}\left\{\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3} + \mathcal{O}\left(\frac{v}{c}\right)^2\right\}$$

2. Amplitude quadrupole formula

$$h_{ij}^{\mathsf{TT}} = \frac{2G}{c^4 D}\left\{\frac{\mathrm{d}^2 Q_{ij}}{\mathrm{d}t^2}\left(t - \frac{D}{c}\right) + \mathcal{O}\left(\frac{v}{c}\right)\right\}^{\mathsf{TT}} + \mathcal{O}\left(\frac{1}{D^2}\right)$$

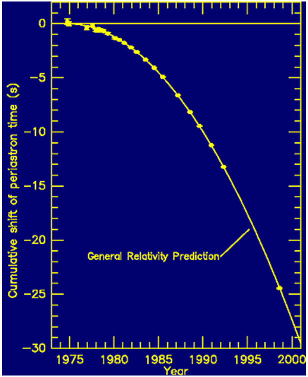
3. Radiation reaction formula [Chandrasekhar & Esposito 1970; Burke & Thorne 1970]

$$F_i^{\mathsf{reac}} = -\frac{2G}{5c^5}\rho\, x^j \frac{\mathrm{d}^5 Q_{ij}}{\mathrm{d}t^5} + \mathcal{O}\left(\frac{v}{c}\right)^7$$

which is a 2.5PN $\sim (v/c)^5$ effect in the source's equations of motion

The quadrupole formula works for the binary pulsar

[Taylor & Weisberg 1982]



$$\dot{P} = -\frac{192\pi}{5c^5}\nu\left(\frac{2\pi G\,M}{P}\right)^{5/3}\frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}} \approx -2.4\times 10^{-12}$$

[Peters & Mathews 1963, Esposito & Harrison 1975, Wagoner 1975, Damour & Deruelle 1983]

The quadrupole formula works also for GW150914 !

1. The GW frequency is given in terms of the chirp mass $\mathcal{M} = \mu^{3/5} M^{2/5}$ by

$$f = \frac{1}{\pi} \left[\frac{256}{5} \frac{G\mathcal{M}^{5/3}}{c^5} (t_\mathsf{f} - t) \right]^{-3/8}$$

2. Therefore the chirp mass is directly measured as

$$\boxed{\mathcal{M} = \left[\frac{5}{96} \frac{c^5}{G\pi^{8/3}} f^{-11/3} \dot{f} \right]^{3/5}}$$

   which gives $\mathcal{M} = 30 M_\odot$ thus $M \geqslant 70 M_\odot$

3. The GW amplitude is predicted to be

$$h_\mathsf{eff} \sim 4.1 \times 10^{-22} \left(\frac{\mathcal{M}}{M_\odot} \right)^{5/6} \left(\frac{100\,\mathsf{Mpc}}{D} \right) \left(\frac{100\,\mathsf{Hz}}{f_\mathsf{merger}} \right)^{-1/6} \sim 1.6 \times 10^{-21}$$

4. The distance $D = 400\,\mathsf{Mpc}$ is measured from the signal itself

Total energy radiated away by GW150914

1. The ADM energy of space-time is constant and reads (at any t)

$$E_{\mathsf{ADM}} = (m_1+m_2)c^2 - \frac{G m_1 m_2}{2r} + \frac{G}{5c^5}\int_{-\infty}^{t} \mathrm{d}t' \left(Q_{ij}^{(3)}\right)^2(t')$$

2. Initially $E_{\mathsf{ADM}} = (m_1+m_2)c^2$ while finally (at time t_{f})

$$E_{\mathsf{ADM}} = M_{\mathsf{f}} c^2 + \frac{G}{5c^5}\int_{-\infty}^{t_{\mathsf{f}}} \mathrm{d}t' \left(Q_{ij}^{(3)}\right)^2(t')$$

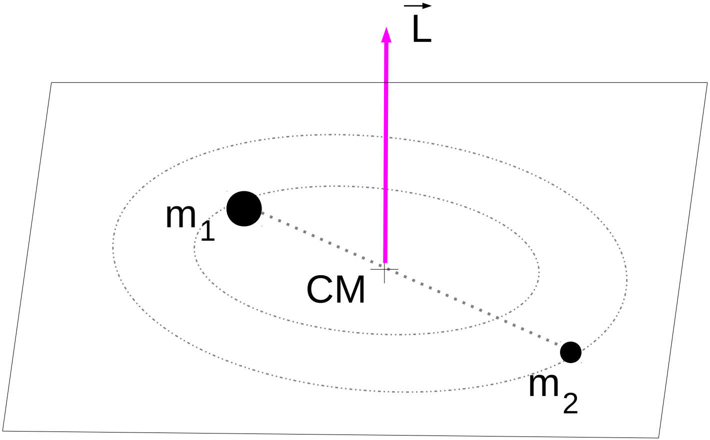
3. The total energy radiated in GW is

$$\boxed{\Delta E^{\mathsf{GW}} = (m_1+m_2-M_{\mathsf{f}})c^2 = \frac{G}{5c^5}\int_{-\infty}^{t_{\mathsf{f}}} \mathrm{d}t' \left(Q_{ij}^{(3)}\right)^2(t') = \frac{G m_1 m_2}{2 r_{\mathsf{f}}}}$$

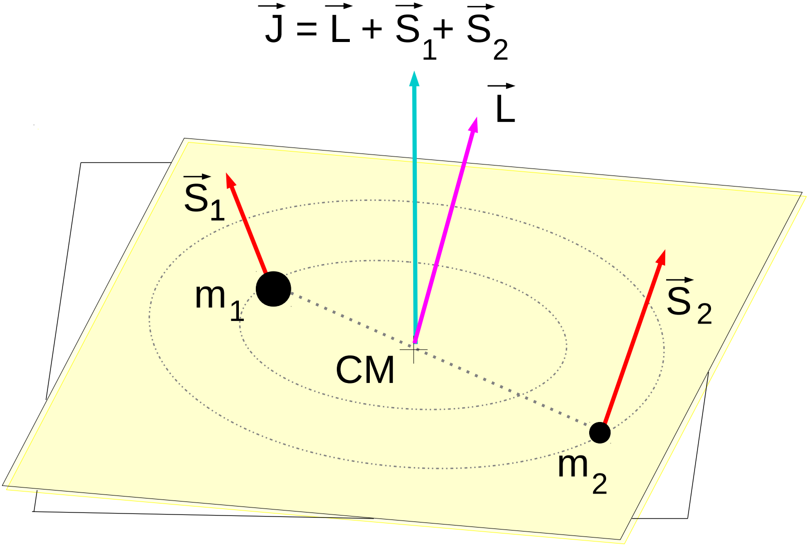
4. The total power released is

$$\boxed{P^{\mathsf{GW}} \sim \frac{3 M_\odot c^2}{0.2\,\mathsf{s}} \sim 10^{49}\,\mathsf{W} \sim 10^{-3}\,\frac{c^5}{G}}$$

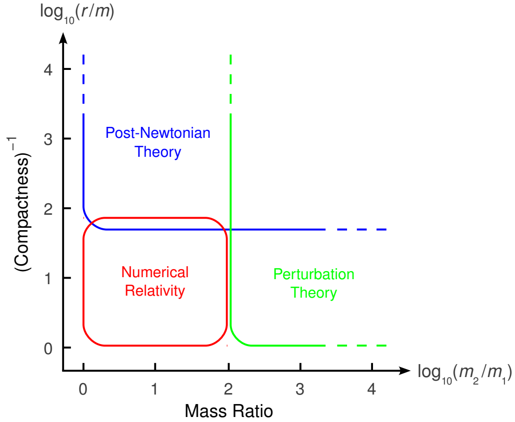
Modelling the compact binary dynamics



Modelling the compact binary dynamics



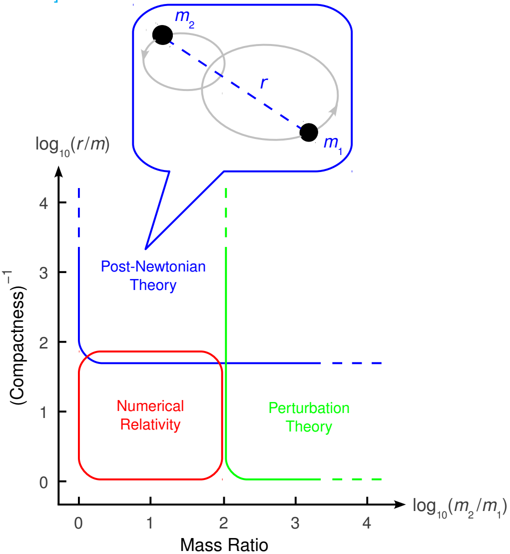
Methods to compute GW templates



[courtesy Alexandre Le Tiec]

Methods to compute GW templates

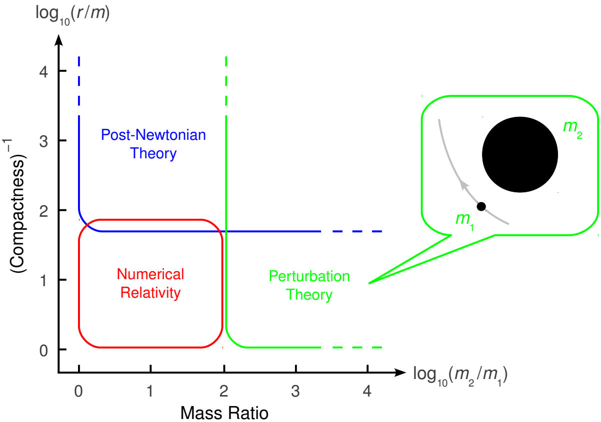
[see Blanchet 2014 for a review]



[courtesy Alexandre Le Tiec]

Methods to compute GW templates

[Detweiler 2008; Barack 2009]



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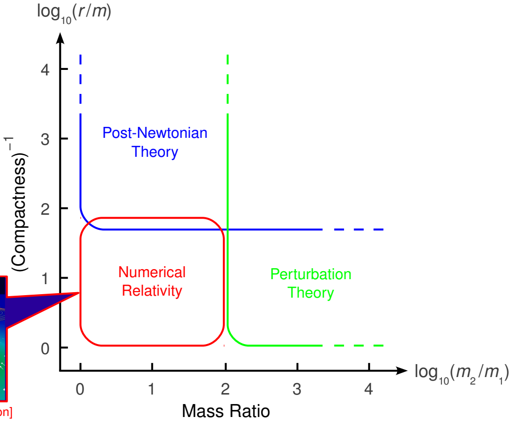
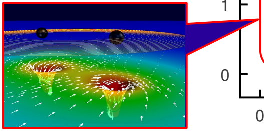
Methods to compute GW templates



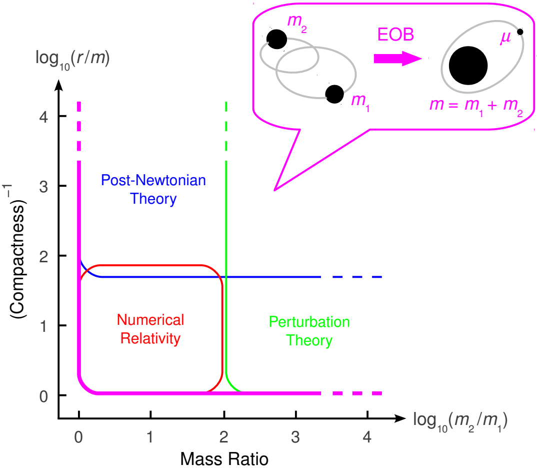


[Caltech/Cornell/CITA collaboration]

[courtesy Alexandre Le Tiec]

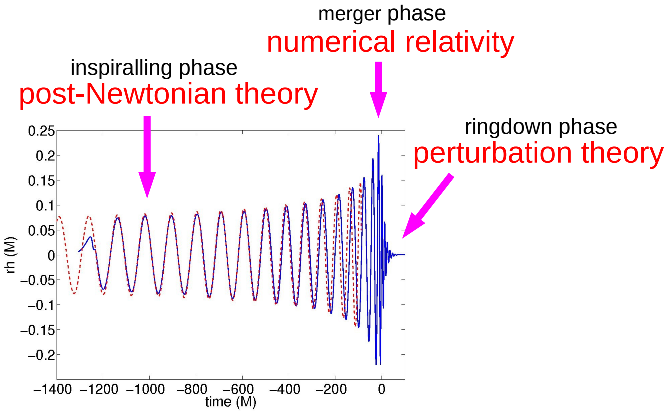
Methods to compute GW templates

[Buonanno & Damour 1998]



[courtesy Alexandre Le Tiec]

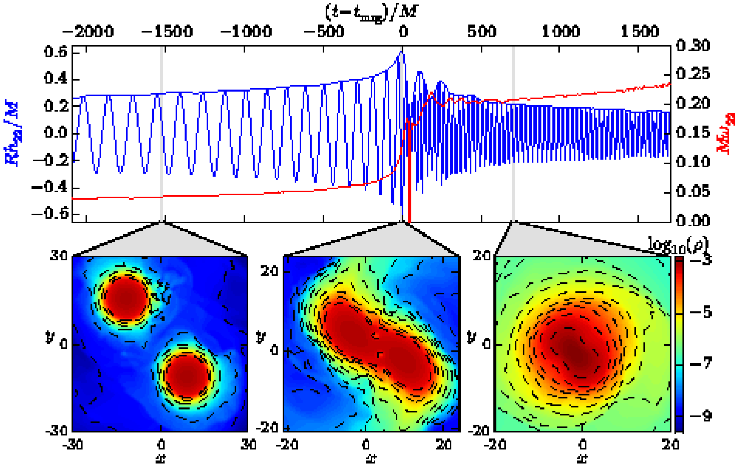
The gravitational chirp of compact binaries



- The post-Newtonian expansion for the early inspiral is matched to a full numerical relativity (NR) calculation of the merger phase
- An important test of GR (no hair theorem) can be performed with the ringdown phase in the case of the merger of two black holes

Post-merger waveform of neutron star binaries

[Shibata *et al.*, Rezzolla *et al.* 1990-2010s]



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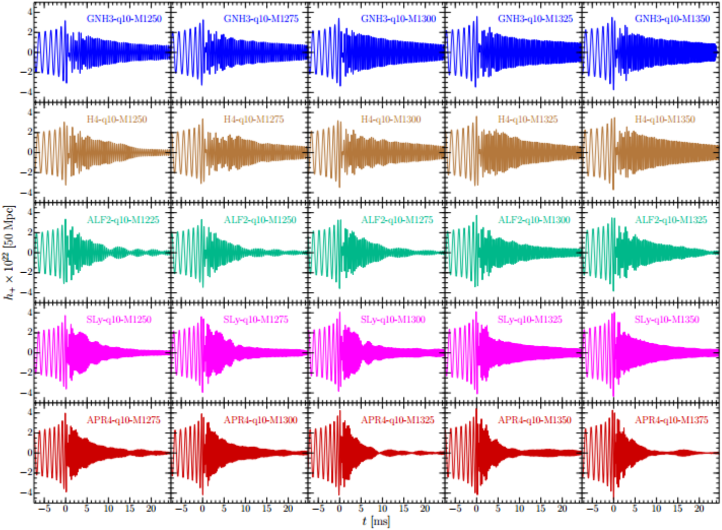
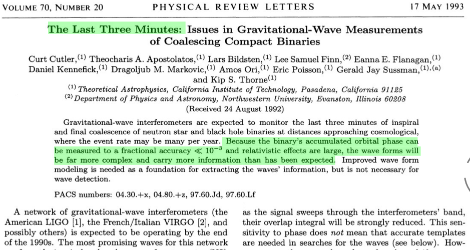


Why inspiralling binaries require high PN modelling

[Caltech "3mn paper" 1992; Blanchet & Schäfer 1993]



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The Last Three Minutes: Issues in Gravitational-Wave Measurements of Coalescing Compact Binaries

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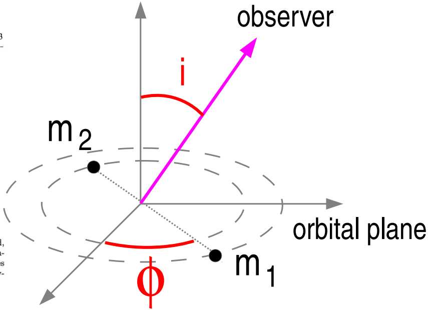
Gravitational-wave interferometers are expected to monitor the last three minutes of inspiral and final coalescence of neutron star and black hole binaries at distances approaching cosmological, where the event rate may be many per year. Because the binary's accumulated orbital phase can be measured to a fractional accuracy $\ll 10^{-3}$ and relativistic effects are large, the wave forms will be far more complex and carry more information than has been expected. Improved wave form modeling is needed as a foundation for extracting the waves' information, but is not necessary for wave detection.

PACS numbers: 04.30.+x, 04.80.+z, 97.60.Jd, 97.60.Lf

A network of gravitational-wave interferometers (the American LIGO [1], the French/Italian VIRGO [2], and possibly others) is expected to be operating by the end of the 1990s. The most promising waves for this network

as the signal sweeps through the interferometers' band, their overlap integral will be strongly reduced. This sensitivity to phase does *not* mean that accurate templates are needed in searches for the waves (see below). How-



$$\phi(t) = \phi_0 - \underbrace{\frac{M}{\mu}\left(\frac{GM\omega}{c^3}\right)^{-5/3}}_{\text{quadrupole formalism}} \left\{ 1 + \underbrace{\frac{1\text{PN}}{c^2} + \frac{1.5\text{PN}}{c^3} + \cdots + \frac{3\text{PN}}{c^6} + \cdots}_{\text{needs to be computed with 3PN precision at least}} \right\}$$

Here 3PN means 5.5PN as a radiation reaction effect !

Methods to compute PN equations of motion

1. ADM Hamiltonian canonical formalism [Ohta *et al.* 1973; Schäfer 1985]
2. EOM in harmonic coordinates [Damour & Deruelle 1985; Blanchet & Faye 1998, 2000]
3. Extended fluid balls [Grishchuk & Kopeikin 1986]
4. Surface-integral approach [Itoh, Futamase & Asada 2000]
5. Effective-field theory (EFT) [Goldberger & Rothstein 2006; Foffa & Sturani 2011]

- EOM derived in a general frame for arbitrary orbits
- Dimensional regularization is applied for UV divergences[1]
- Radiation-reaction dissipative effects added separately by matching
- Spin effects can be computed within a pole-dipole approximation
- Tidal effects incorporated at leading 5PN and sub-leading 6PN orders

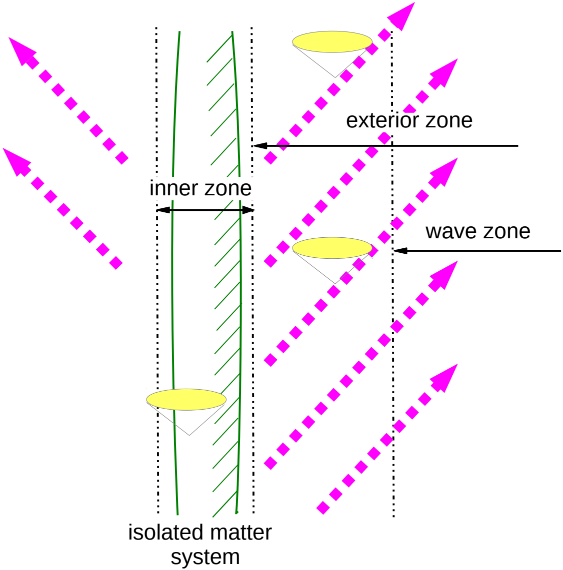
[1] Except in the surface-integral approach

Methods to compute PN radiation field

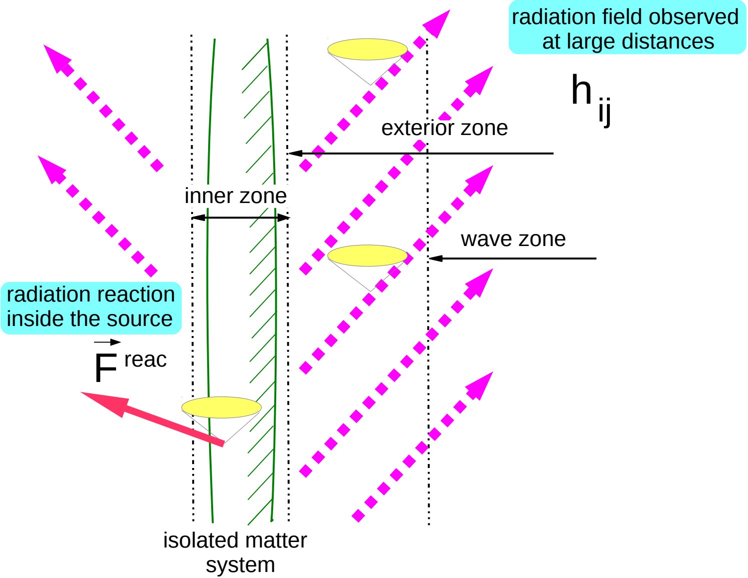
1. Multipolar-post-Minkowskian (MPM) & PN [Blanchet-Damour-Iyer 1986, . . . , 1998]
2. Direct iteration of the relaxed field equations (DIRE) [Will-Wiseman-Pati 1996, . . .]
3. Effective-field theory (EFT) [Hari Dass & Soni 1982; Goldberger & Ross 2010]

- Involves a machinery of tails and related non-linear effects
- Uses dimensional regularization to treat point-particle singularities
- Phase evolution relies on balance equations valid in adiabatic approximation
- Spin effects are incorporated within a pole-dipole approximation
- Provides polarization waveforms for DA & spin-weighted spherical harmonics decomposition for NR

Isolated matter system in general relativity

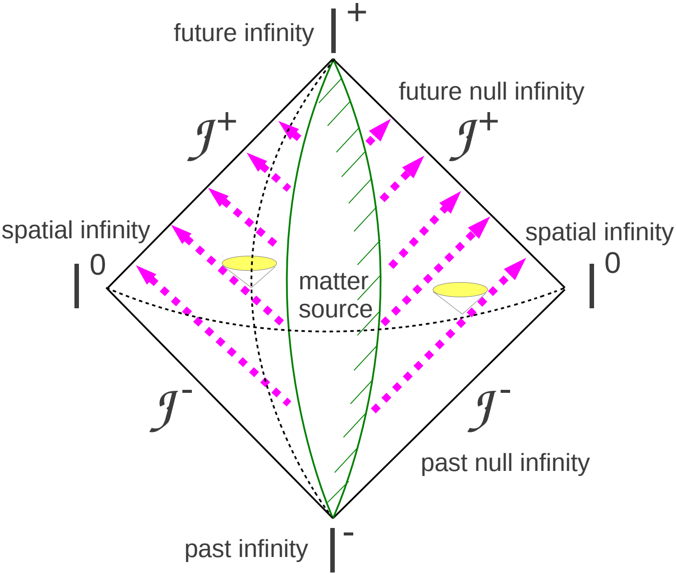


Isolated matter system in general relativity



Asymptotic structure of radiating space-time

[Bondi-Sachs-Penrose formalism 1960s]



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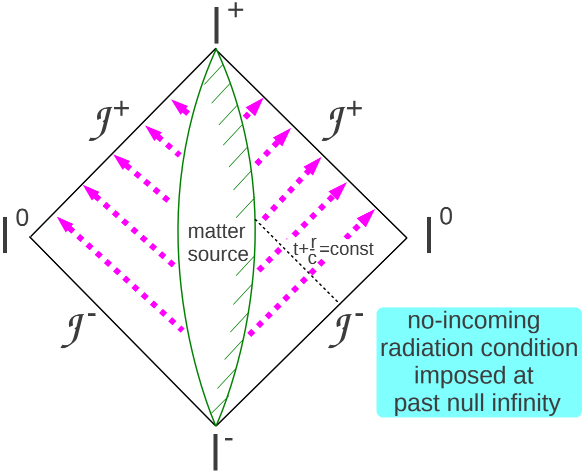


$$\lim_{\substack{r\to+\infty\\ t+\frac{r}{c}=\text{const}}} \left(\frac{\partial}{\partial r}+\frac{\partial}{c\partial t}\right)\left(r h^{\alpha\beta}\right)=0$$

Gauge-fixed Einstein field equations

- Start with the Einstein-Hilbert action with the matter term and add the harmonic coordinates gauge-fixing term

$$S = \frac{c^3}{16\pi G}\int \mathrm{d}^4x \Big(\sqrt{-g}\,R \underbrace{-\frac{1}{2}\mathfrak{g}_{\alpha\beta}\partial_\mu\mathfrak{g}^{\alpha\mu}\partial_\nu\mathfrak{g}^{\beta\nu}}_{\text{gauge-fixing term}}\Big) + S_{\text{mat}}$$

where $\mathfrak{g}^{\alpha\beta} = \sqrt{-g}g^{\alpha\beta}$ is the gothic metric

$$\mathfrak{g}^{\mu\nu}\partial^2_{\mu\nu}\mathfrak{g}^{\alpha\beta} = \frac{16\pi G}{c^4}|g|T^{\alpha\beta} + \overbrace{\Sigma^{\alpha\beta}[\mathfrak{g},\partial\mathfrak{g}]}^{\text{non-linear source term}}$$

$$\underbrace{\partial_\mu\mathfrak{g}^{\alpha\mu} = 0}_{\text{harmonic-gauge condition}}$$

- Such system of equations constitutes a well-posed problem ("problème bien posé") in the sense of Hadamard [Choquet-Bruhat 1952]

Perturbation around Minkowski space-time

- Assume the space-time slightly differs from Minkowski space-time $\eta_{\alpha\beta}$

$$\mathfrak{g}^{\alpha\beta} = \eta^{\alpha\beta} + h^{\alpha\beta} \qquad \text{with} \quad |h| \ll 1$$

$$\Box h^{\alpha\beta} = \frac{16\pi G}{c^4}|g|T^{\alpha\beta} + \overbrace{\Lambda^{\alpha\beta}[h, \partial h, \partial^2 h]}^{\text{non-linear source term}} \equiv \frac{16\pi G}{c^4} \underbrace{\tau^{\alpha\beta}}_{\text{stress-energy pseudo-tensor}}$$

$$\underbrace{\partial_\mu h^{\alpha\mu} = 0}_{\text{harmonic-gauge condition}}$$

- Such system can be resolved assuming Minkowskian boundary conditions of no incoming radiation imposed at at $\mathcal{I}^-$

Post-Minkowskian expansion [e.g. Bertotti & Plebanski 1960]

- Appropriate for weakly self-gravitating isolated matter sources

$$\gamma_{\text{PM}} \equiv \frac{GM}{c^2 a} \ll 1 \qquad \begin{cases} M \text{ mass of source} \\ a \text{ size of source} \end{cases}$$

$$\mathfrak{g}^{\alpha\beta} = \eta^{\alpha\beta} + \underbrace{\sum_{n=1}^{+\infty} G^n \, h^{\alpha\beta}_{(n)}}_{G \text{ labels the PM expansion}}$$

$$\Box h^{\alpha\beta}_{(n)} = \frac{16\pi G}{c^4} |g| T^{\alpha\beta}_{(n)} + \overbrace{\Lambda^{\alpha\beta}_{(n)}[h_{(1)}, \cdots, h_{(n-1)}]}^{\text{know from previous iterations}}$$

$$\partial_\mu h^{\alpha\mu}_{(n)} = 0$$

- Very difficult approximation to implement in practice for general sources at high post-Minkowskian orders

Linearized multipolar vacuum solution [Thorne 1980]

General solution of linearized vacuum field equations in harmonic coordinates

$$\boxed{\Box h^{\alpha\beta}_{(1)} = \partial_\mu h^{\alpha\mu}_{(1)} = 0}$$

$$h^{00}_{(1)} = -\frac{4}{c^2}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L\left(\frac{1}{r}M_L(u)\right)$$

$$h^{0i}_{(1)} = \frac{4}{c^3}\sum_{\ell=1}^{+\infty}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-1}\left(\frac{1}{r}M^{(1)}_{iL-1}(u)\right) + \frac{\ell}{\ell+1}\epsilon_{iab}\partial_{aL-1}\left(\frac{1}{r}S_{bL-1}(u)\right)\right\}$$

$$h^{ij}_{(1)} = -\frac{4}{c^4}\sum_{\ell=2}^{+\infty}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-2}\left(\frac{1}{r}M^{(2)}_{ijL-2}(u)\right) + \frac{2\ell}{\ell+1}\partial_{aL-2}\left(\frac{1}{r}\epsilon_{ab(i}S^{(1)}_{j)bL-2}(u)\right)\right\}$$

- multipole moments $M_L(u)$ and $S_L(u)$ arbitrary functions of $u = t - r/c$
- mass $M = \text{const}$, center-of-mass position $X_i \equiv M_i/M = \text{const}$, linear momentum $P_i \equiv M^{(1)}_i = 0$, angular momentum $S_i = \text{const}$

Multipolar-post-Minkowskian expansion

[Blanchet & Damour 1986, 1988, 1992; Blanchet 1987, 1993, 1998]

- Starts with the solution of the linearized equations outside an isolated source in the form of multipole expansions
- An explicit MPM algorithm is constructed out of it by induction at any order n in the post-Minkowskian expansion

$$h^{\alpha\beta}_{\text{MPM}} = \sum_{n=1}^{+\infty} G^n \underbrace{h^{\alpha\beta}_{(n)}[M_L, S_L]}_{\text{explicit functional of multipole moments}}$$

- A finite-part (FP) regularization based on analytic continuation is required in order to cope with the divergency of the multipolar expansion when $r \to 0$

Multipolar-post-Minkowskian expansion

[Blanchet & Damour 1986, 1988, 1992; Blanchet 1987, 1993, 1998]

Theorem 1:
The MPM solution is the most general solution of Einstein's vacuum equations outside an isolated matter system

Theorem 2:
The general structure of the PN expansion is

$$h^{\alpha\beta}_{\mathrm{PN}}(\mathbf{x},t,c) = \sum_{\substack{p\geqslant 2\\ q\geqslant 0}} \frac{(\ln c)^q}{c^p} h^{\alpha\beta}_{p,q}(\mathbf{x},t)$$

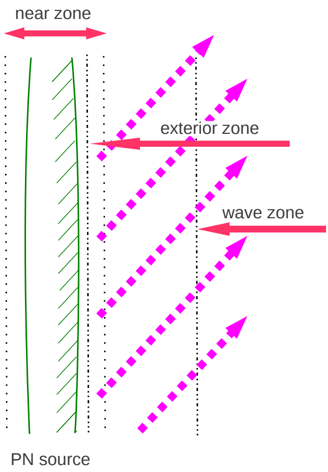
Theorem 3:
The MPM solution is asymptotically simple at future null infinity in the sense of Penrose [1963, 1965] and agrees with the Bondi-Sachs [1962] formalism

$$\underbrace{M_{\mathrm{B}}(u)}_{\text{Bondi mass}} = \underbrace{M}_{\text{ADM mass}} - \frac{G}{5c^5}\int_{-\infty}^{u} \mathrm{d}\tau M^{(3)}_{ij}(\tau) M^{(3)}_{ij}(\tau)$$

$+$ higher multipoles and higher PM computable to any order

The MPM-PN formalism

A multipolar post-Minkowskian (MPM) expansion in the exterior zone is matched to a general post-Newtonian (PN) expansion in the near zone



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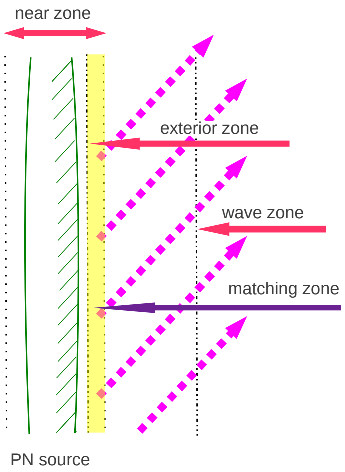


$$\overline{\mathcal{M}(h^{\alpha\beta})} = \mathcal{M}(\bar{h}^{\alpha\beta})$$

The matching equation

[Lagerström *et al.* 1967; Burke & Thorne 1971; Kates 1980; Anderson *et al.* 1982; Blanchet 1998]

1. This is a variant of the theory of matched asymptotic expansions

$$\text{match} \begin{cases} \text{the multipole expansion } \mathcal{M}(h^{\alpha\beta}) \equiv h^{\alpha\beta}_{\text{MPM}} \\ \quad\quad\text{with} \\ \text{the PN expansion } \bar{h}^{\alpha\beta} \equiv h^{\alpha\beta}_{\text{PN}} \end{cases}$$

$$\boxed{\overline{\mathcal{M}(h^{\alpha\beta})} = \mathcal{M}(\bar{h}^{\alpha\beta})}$$

- Left side is the NZ expansion ($r \to 0$) of the exterior MPM field
- Right side is the FZ expansion ($r \to +\infty$) of the inner PN field

2. The matching equation has been implemented at any post-Minkowskian order in the exterior field and any PN order in the inner field
3. It gives a unique (formal) multipolar-post-Newtonian solution valid everywhere inside and outside the source

General solution for the multipolar field [Blanchet 1995, 1998]

$$\mathcal{M}(h^{\mu\nu}) = \text{FP}\Box^{-1}_{\text{ret}}\mathcal{M}(\Lambda^{\mu\nu}) + \underbrace{\sum_{\ell=0}^{+\infty}\partial_L\left\{\frac{M_L^{\mu\nu}(t-r/c)}{r}\right\}}_{\text{homogeneous retarded solution}}$$

where
$$M_L^{\mu\nu}(t) = \text{FP}\int \mathrm{d}^3\mathbf{x}\,\hat{x}_L\int_{-1}^{1}\mathrm{d}z\,\delta_\ell(z)\;\underbrace{\bar{\tau}^{\mu\nu}(\mathbf{x},t-zr/c)}_{\text{PN expansion of the pseudo-tensor}}$$

- The FP procedure plays the role of an UV regularization in the non-linearity term but an IR regularization in the multipole moments
- From this one obtains the multipole moments of the source at any PN order solving the wave generation problem

General solution for the inner PN field

[Poujade & Blanchet 2002; Blanchet, Faye & Nissanke 2005]

$$\bar{h}^{\mu\nu} = \text{FP}\Box^{-1}_{\text{ret}}\bar{\tau}^{\mu\nu} + \underbrace{\sum_{\ell=0}^{+\infty}\partial_L\left\{\frac{R_L^{\mu\nu}(t-r/c) - R_L^{\mu\nu}(t+r/c)}{r}\right\}}_{\text{homogeneous antisymmetric solution}}$$

where $$R_L^{\mu\nu}(t) = \text{FP}\int \mathrm{d}^3\mathbf{x}\,\hat{x}_L\int_1^{+\infty}\mathrm{d}z\,\gamma_\ell(z)\,\underbrace{\mathcal{M}(\tau^{\mu\nu})(\mathbf{x},t-zr/c)}_{\text{multipole expansion of the pseudo-tensor}}$$

- The radiation reaction effects starting at 2.5PN order appropriate to an isolated system are determined to any order
- In particular nonlinear radiation reaction effects associated with tails are contained in the second term and start at 4PN order

Quadrupole observable at future null infinity

[Marchand, Blanchet & Faye 2016]

$$U_{ij}(t) = M_{ij}^{(2)}(t) + \underbrace{\frac{GM}{c^3}\int_0^{+\infty} \mathrm{d}\tau M_{ij}^{(4)}(t-\tau)\left[2\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{11}{6}\right]}_{\text{1.5PN tail integral}}$$

$$+ \frac{G}{c^5}\left\{\underbrace{-\frac{2}{7}\int_0^{+\infty} \mathrm{d}\tau M_{a<i}^{(3)} M_{j>a}^{(3)}(t-\tau)}_{\text{2.5PN memory integral}} + \text{instantaneous terms}\right\}$$

$$+ \underbrace{\frac{G^2M^2}{c^6}\int_0^{+\infty} \mathrm{d}\tau M_{ij}^{(5)}(t-\tau)\left[2\ln^2\left(\frac{\tau}{2\tau_0}\right)+\frac{57}{35}\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{124627}{22050}\right]}_{\text{3PN tail-of-tail integral}}$$

$$+ \underbrace{\frac{G^3M^3}{c^9}\int_0^{+\infty} \mathrm{d}\tau M_{ij}^{(6)}(t-\tau)\left[\frac{4}{3}\ln^3\left(\frac{\tau}{2\tau_0}\right)+\cdots+\frac{129268}{33075}+\frac{428}{315}\pi^2\right]}_{\text{4.5PN tail-of-tail-of-tail integral}}$$

$$+ \mathcal{O}\left(\frac{1}{c^{10}}\right)$$

3.5PN energy flux of compact binaries

[BDIWW 1995; B 1996, 1998; BFIJ 2002; BDEI 2006]

$$\mathcal{F} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ 1 + \overbrace{\left(-\frac{1247}{336} - \frac{35}{12}\nu\right) x}^{\text{1PN}} + \overbrace{4\pi x^{3/2}}^{\text{1.5PN tail}}$$

$$+ \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right) x^2 + \overbrace{\left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2}}^{\text{2.5PN tail}}$$

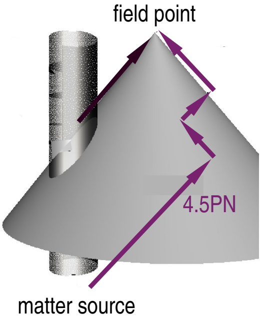
$$+ \Bigg[\frac{6643739519}{69854400} + \overbrace{\frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_\mathsf{E} - \frac{856}{105}\ln(16\,x)}^{\text{3PN tail-of-tail}}$$

$$+ \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3 \Bigg] x^3$$

$$+ \underbrace{\left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2}}_{\text{3.5PN tail}} + \mathcal{O}\left(\frac{1}{c^8}\right) \Bigg\}$$

4.5PN tail interactions between moments

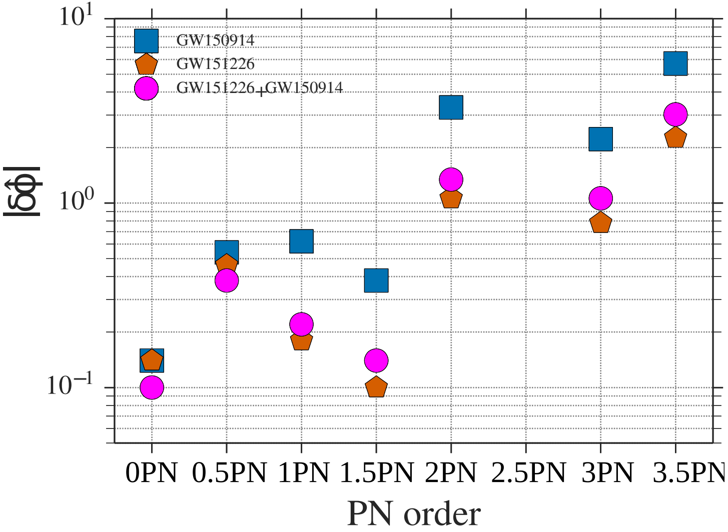
[Marchand, Blanchet & Faye 2016]



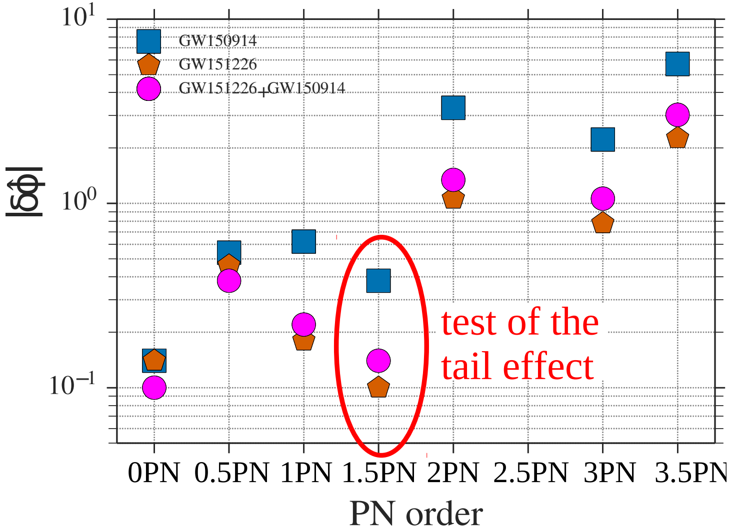
$$\mathcal{F}^{\text{4.5PN}} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ \left(\frac{265978667519}{745113600} - \frac{6848}{105}\gamma_\text{E} - \frac{3424}{105}\ln(16x) + \left[\frac{2062241}{22176} + \frac{41}{12}\pi^2\right]\nu - \frac{133112905}{290304}\nu^2 - \frac{3719141}{38016}\nu^3 \right) \pi x^{9/2} \Bigg\}$$

- Perfect agreement with results from BH perturbation theory in the small mass ratio limit $\nu \to 0$ [Tanaka, Tagoshi & Sasaki 1996]
- However the 4PN term in the flux is still in progress

Measurement of PN parameters [LIGO/VIRGO 2016]

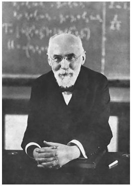


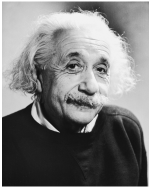
Measuring GW tails [Blanchet & Sathyaprakash 1994, 1995]



The 1PN equations of motion

[Lorentz & Droste 1917; Einstein, Infeld & Hoffmann 1938]



$$\frac{\mathrm{d}^2\boldsymbol{r}_A}{\mathrm{d}t^2} = -\sum_{B\neq A}\frac{Gm_B}{r_{AB}^2}\boldsymbol{n}_{AB}\Bigg[1 - 4\sum_{C\neq A}\frac{Gm_C}{c^2 r_{AC}} - \sum_{D\neq B}\frac{Gm_D}{c^2 r_{BD}}\left(1 - \frac{\boldsymbol{r}_{AB}\cdot\boldsymbol{r}_{BD}}{r_{BD}^2}\right) + \frac{1}{c^2}\left(\boldsymbol{v}_A^2 + 2\boldsymbol{v}_B^2 - 4\boldsymbol{v}_A\cdot\boldsymbol{v}_B - \frac{3}{2}(\boldsymbol{v}_B\cdot\boldsymbol{n}_{AB})^2\right)\Bigg] + \sum_{B\neq A}\frac{Gm_B}{c^2 r_{AB}^2}\boldsymbol{v}_{AB}[\boldsymbol{n}_{AB}\cdot(3\boldsymbol{v}_B - 4\boldsymbol{v}_A)] - \frac{7}{2}\sum_{B\neq A}\sum_{D\neq B}\frac{G^2 m_B m_D}{c^2 r_{AB} r_{BD}^3}\boldsymbol{n}_{BD}$$

4PN: state-of-the-art on equations of motion

$$\frac{\mathrm{d}v_1^i}{\mathrm{d}t} = -\frac{Gm_2}{r_{12}^2} n_{12}^i + \overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2m_1m_2}{r_{12}^3} + \frac{4G^2m_2^2}{r_{12}^3} + \cdots\right] n_{12}^i + \cdots\right\}}^{\text{1PN Lorentz-Droste-Einstein-Infeld-Hoffmann term}} + \underbrace{\frac{1}{c^4}\left[\cdots\right]}_{\text{2PN}} + \underbrace{\frac{1}{c^5}\left[\cdots\right]}_{\substack{\text{2.5PN}\\\text{radiation reaction}}} + \underbrace{\frac{1}{c^6}\left[\cdots\right]}_{\text{3PN}} + \underbrace{\frac{1}{c^7}\left[\cdots\right]}_{\substack{\text{3.5PN}\\\text{radiation reaction}}} + \underbrace{\frac{1}{c^8}\left[\cdots\right]}_{\substack{\text{4PN}\\\text{conservative \& radiation tail}}} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

3PN

[Jaranowski & Schäfer 1999; Damour, Jaranowski & Schäfer 2001ab]	ADM Hamiltonian
[Blanchet-Faye-de Andrade 2000, 2001; Blanchet & Iyer 2002]	Harmonic EOM
[Itoh & Futamase 2003; Itoh 2004]	Surface integral method
[Foffa & Sturani 2011]	Effective field theory

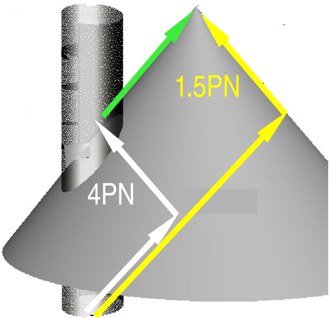
4PN

[Jaranowski & Schäfer 2013; Damour, Jaranowski & Schäfer 2014]	ADM Hamiltonian
[Bernard, Blanchet, Bohé, Faye, Marchand & Marsat 2015, 2016, 2017ab]	Fokker Lagrangian
[Foffa & Sturani 2012, 2013] (partial results)	Effective field theory

Gravitational wave tail effect at the 4PN order

[Blanchet & Damour 1988; Blanchet 1993, 1997]

- At the 4PN order there is an imprint of gravitational wave tails in the local (near-zone) dynamics of the source
- This leads to a non-local-in-time contribution in the Fokker action
- This implies the appearance of IR divergences in the Fokker action at the 4PN order



$$S_\mathrm{F}^\mathrm{tail} = \frac{G^2 M}{5c^8} \mathop{\mathrm{Pf}}_{s_0} \iint \frac{\mathrm{d}t\mathrm{d}t'}{|t-t'|} I_{ij}^{(3)}(t)\, I_{ij}^{(3)}(t')$$

where the Hadamard partie finie (Pf) is parametrized by an arbitrary constant s_0

Problem of the IR divergences

1. Our initial calculation of the Fokker action was based on the Hadamard regularization to treat the IR divergences (FP procedure when $B \to 0$)
2. However computing the conserved energy and periastron advance for circular orbits we found it does not agree with GSF calculations
3. The problem was due to the HR and conjectured that a different IR regularization would give (modulo shifts)

$$L = L^{\mathsf{HR}} + \underbrace{\frac{G^4 m\, m_1^2 m_2^2}{c^8 r_{12}^4}\Big(\delta_1 (n_{12}v_{12})^2 + \delta_2 v_{12}^2\Big)}_{\text{two ambiguity parameters } \delta_1 \text{ and } \delta_2}$$

4. Matching with GSF results for the energy and periastron advance uniquely fixes the two ambiguity parameters and we are in complete agreement with the results from the Hamiltonian formalism [DJS]

Dimensional regularization of the IR divergences

- The Hadamard regularization of IR divergences reads

$$I^{\mathsf{HR}}_{\mathcal{R}} = \mathop{\mathrm{FP}}_{B=0} \int_{r>\mathcal{R}} \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B F(\mathbf{x})$$

- The corresponding dimensional regularization reads

$$I^{\mathsf{DR}}_{\mathcal{R}} = \int_{r>\mathcal{R}} \frac{\mathrm{d}^d\mathbf{x}}{\ell_0^{d-3}} F^{(d)}(\mathbf{x})$$

- The difference between the two regularization is of the type ($\varepsilon = d - 3$)

$$\boxed{\mathcal{D}I = \sum_q \Bigg[\underbrace{\frac{1}{(q-1)\varepsilon}}_{\text{IR pole}} - \ln\left(\frac{r_0}{\ell_0}\right)\Bigg] \int \mathrm{d}\Omega_{2+\varepsilon}\, \varphi^{(\varepsilon)}_{3,q}(\mathbf{n}) + \mathcal{O}(\varepsilon)}$$

Ambiguity-free completion of the 4PN EOM

[Marchand, Bernard, Blanchet & Faye 2017]

1. The tail effect contains a UV pole which cancels the IR pole coming from the instantaneous part of the action

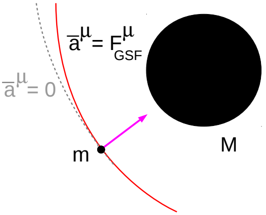
$$g_{00}^{\text{tail}} = -\frac{8G^2M}{5c^8}\,x^{ij}\int_0^{+\infty}\mathrm{d}\tau\left[\ln\left(\frac{c\sqrt{\bar{q}}\,\tau}{2\ell_0}\right)\underbrace{-\frac{1}{2\varepsilon}}_{\text{UV pole}}+\frac{41}{60}\right]I_{ij}^{(7)}(t-\tau)+\mathcal{O}\left(\frac{1}{c^{10}}\right)$$

2. Adding up all contributions we obtain the conjectured form of the ambiguity terms with the correct values of the ambiguity parameters δ_1 and δ_2
3. It is likely that the EFT formalism will also succeed in deriving the full EOM without ambiguities [Porto & Rothstein 2017]
4. The lack of a consistent matching in the ADM Hamiltonian formalism [DJS] forces this formalism to be plagued by one ambiguity parameter

Problem of the gravitational self-force (GSF)

[Mino, Sasaki & Tanaka 1997; Quinn & Wald 1997; Detweiler & Whiting 2003]

- A particle is moving on a background space-time of a massive black hole
- Its stress-energy tensor modifies the background gravitational field
- Because of the back-reaction the motion of the particle deviates from a background geodesic hence the gravitational self force

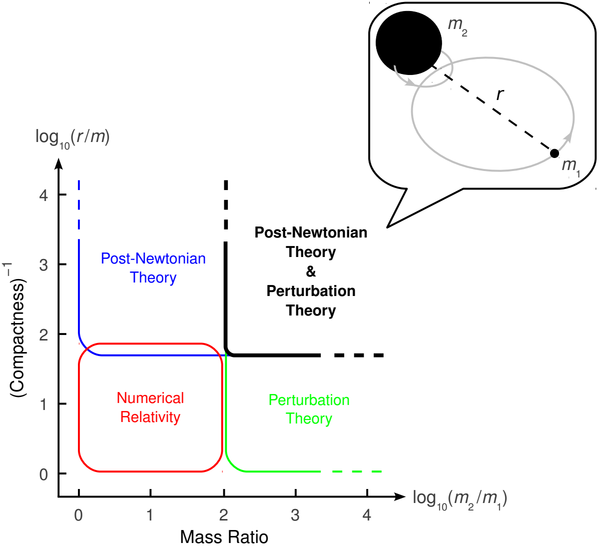


$$\bar{a}^{\mu} = F^{\mu}_{\mathsf{GSF}} = \mathcal{O}\left(\frac{m}{M}\right)$$

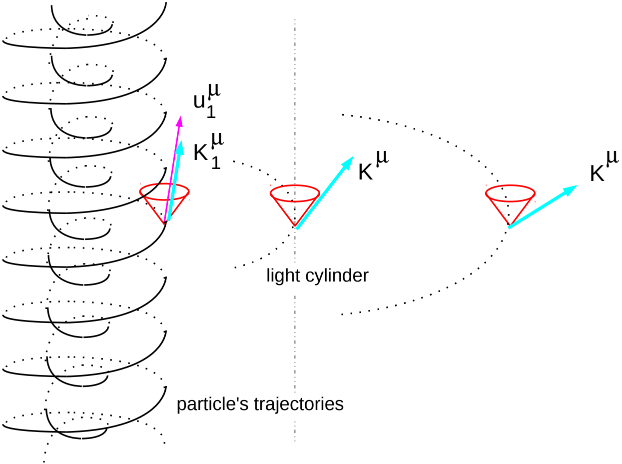
The GSF is computed to high accuracy by

- numerical methods [Sago, Barack & Detweiler 2008; Shah, Friedmann & Whiting 2014]
- analytical ones [Mano, Susuki & Takasugi 1996; Bini & Damour 2013, 2014]

Checking the PN machinery with GSF



Looking at the conservative part of the dynamics



Space-time for exact circular orbits admits a Helical Killing Vector (HKV) K^μ

Choice of a gauge-invariant observable [Detweiler 2008]

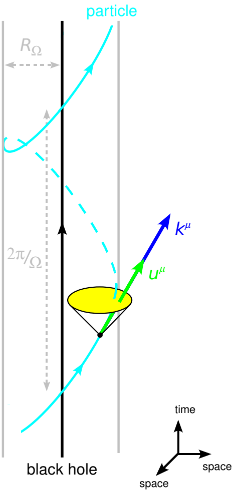
1. For exactly circular orbits the geometry admits a helical Killing vector with

$$K^\mu \partial_\mu = \partial_t + \Omega\, \partial_\varphi \quad \text{(asymptotically)}$$

2. The four-velocity of the particle is necessarily tangent to the Killing vector hence

$$K_1^\mu = z_1\, u_1^\mu$$

3. This z_1 is the Killing energy of the particle associated with the HKV and is also a redshift
4. The relation $z_1(\Omega)$ is well-defined in both PN and GSF approaches and is gauge-invariant

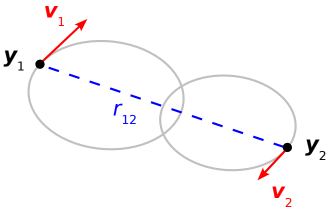


PN calculation of the redshift factor

[Blanchet, Detweiler, Le Tiec & Whiting 2010, 2011; Blanchet, Faye & Whiting 2014, 2015]

1. In a coordinate system such that $K^\mu \partial_\mu = \partial_t + \Omega\, \partial_\varphi$ everywhere this invariant quantity reduces to the zero-th component of the particle's four-velocity,

$$u_1^t = \frac{1}{z_1} = \Bigg(- \underbrace{\mathbf{Reg}_1 \left[g_{\mu\nu} \right]}_{\text{regularized metric}} \frac{v_1^\mu v_1^\nu}{c^2} \Bigg)^{-1/2}$$



2. One needs a self-field regularization
   - Hadamard regularization will yield an ambiguity at 3PN order
   - Dimensional regularization will be free of any ambiguity at 3PN order

Standard PN theory agrees with GSF calculations

$$
\begin{aligned}
u^t_{\rm SF} = &-y - 2y^2 - 5y^3 + \left(-\frac{121}{3} + \frac{41}{32}\pi^2\right) y^4 \\
&+ \left(-\frac{1157}{15} + \frac{677}{512}\pi^2 - \frac{128}{5}\gamma_{\rm E} - \frac{64}{5}\ln(16y)\right) y^5 \\
&- \frac{956}{105} y^6 \ln y - \frac{13696\pi}{525} y^{13/2} - \frac{51256}{567} y^7 \ln y + \frac{81077\pi}{3675} y^{15/2} \\
&+ \frac{27392}{525} y^8 \ln^2 y + \frac{82561159\pi}{467775} y^{17/2} - \frac{27016}{2205} y^9 \ln^2 y \\
&- \frac{11723776\pi}{55125} y^{19/2} \ln y - \frac{4027582708}{9823275} y^{10} \ln^2 y \\
&+ \frac{99186502\pi}{1157625} y^{21/2} \ln y + \frac{23447552}{165375} y^{11} \ln^3 y + \cdots
\end{aligned}
$$

1. Integral PN terms such as 3PN permit checking dimensional regularization
2. Half-integral PN terms starting at 5.5PN order permit checking the non-linear tails (and tail-of-tails)

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Summary of known PN orders

Method	Equations of motion	Energy flux	Waveform
Multipolar-post-Minkowskian & post-Newtonian (MPM-PN)	4PN non-spin 3.5PN (NNL) SO 3PN (NL) SS 3.5PN (NL) SSS	3.5PN non-spin 4PN (NNL) SO 3PN (NL) SS 3.5PN (NL) SSS	3.5PN non-spin 1.5PN (L) SO 2PN (L) SS
Canonical ADM Hamiltonian	4PN non-spin 3.5PN (NNL) SO 4PN (NNL) SS 3.5PN (NL) SSS		
Effective Field Theory (EFT)	3PN non-spin 2.5PN (NL) SO 4PN (NNL) SS	2PN non-spin 3PN (NL) SS	
Direct Integration of Relaxed Equations (DIRE)	2.5PN non-spin 1.5PN (L) SO 2PN (L) SS	2PN non-spin 1.5PN (L) SO 2PN (L) SS	2PN non-spin 1.5PN (L) SO 2PN (L) SS
Surface Integral	3PN non-spin		

- The 4.5PN non-spin coefficient in the energy flux also known
- Many works devoted to spins:
  - Spin effects (SO, SS, SSS) are known in EOM up to 4PN order
  - SO effects are known in radiation field up to 4PN
  - SS in radiation field known to 3PN