Recent Advances in Convex Optimization and Fixed Point Algorithms

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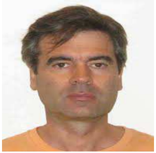
In collaboration with



P. Combettes

N. Pustelnik

A. Repetti

Motivation

- In many areas (machine learning, inverse problems, computer vision,...), iterative solutions are needed to solve variational problems.

- Sequence such that

$$(\forall n \in \mathbb{N}) \qquad x_{n+1} = T x_n$$

where T denotes an operator from H to H, where H real Hilbert space with norm $\|\cdot\|$.

$\rightsquigarrow$ How can we build T so as to solve the problem of interest ?

$\rightsquigarrow$ Which properties are required by T in order to ensure the convergence of $(x_n)_{n\in\mathbb{N}}$ to $\widehat{x} \in$ H ?

Naive answer

Fixed point theorem (E. Picard, 1856-1941)

If

- $\widehat{x}$ is a fixed point of T, i.e. $\widehat{x} = T\widehat{x}$
- T is a strict contraction, i.e. there exists $\rho \in [0, 1[$ such that

$$\big(\forall (x, x') \in \mathsf{H}^2\big) \qquad \|Tx - Tx'\| \leqslant \rho \|x - x'\|$$

then $(x_n)_{n\in\mathbb{N}}$ converges (strongly) to $\widehat{x}$.

Proof: For all $n \in \mathbb{N}$,

$$\begin{aligned} \|x_{n+1} - \widehat{x}\| &= \|Tx_n - T\widehat{x}\| \\ &\leqslant \rho \|x_n - \widehat{x}\|. \end{aligned}$$

Consequently, $\|x_n - \widehat{x}\| \leqslant \rho^n \|x_0 - \widehat{x}\|$. Hence, we have proved that $(x_n)_{n\in\mathbb{N}}$ converges linearly to $\widehat{x}$.

Why do we need to go further ?

Limitations:

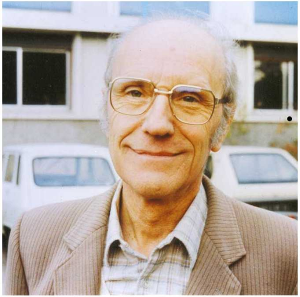
- It is difficult (even sometimes impossible) to build a *strictly* contractive operator T.
- One may prefer iterations built as

$$(\forall n \in \mathbb{N}) \qquad x_{n+1} = T_n x_n$$

where T_n is an operator from H to H.

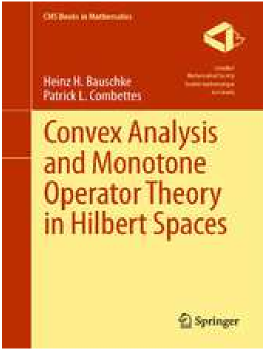
- It is often intricate to build T_n, while it may be easier to write T_n as a composition of simpler operators (splitting techniques).

A pioneer



Jean-Jacques Moreau
(1923–2014)

Reference book



– H.H. Bauschke and P.L. Combettes –

Outline

1. Background on monotone operators

   $\rightarrow$ *Inversion, subdifferential, conjugate of a convex function*

2. Nonexpansive operators

   $\rightarrow$ *Taxonomy, resolvent, and proximity operator*

3. Fixed point algorithms

   $\rightarrow$ *Fejér monotonicity, Douglas-Rachford, Forward-Backward*

4. Duality

   $\rightarrow$ *Main theorems, ADMM, primal-dual methods, stochastic methods*

Part 1: Background

1. Monotone operators
   - Definition
   - Properties
   - Basic operations
   - Inversion
   - Maximality
   - Usefulness for convex optimization (subdifferential)

2. Maximally monotone operators
   - Properties
   - Basic operations
   - Inversion
   - Usefulness of inversion for convex optimization (conjugate)

Hilbert spaces

A (real) **Hilbert space H** is a complete real vector space endowed with an inner product $\langle \cdot \mid \cdot \rangle$ whose associated norm is

$$(\forall x \in \mathsf{H}) \qquad \|x\| = \sqrt{\langle x \mid x \rangle}.$$

- Particular case: $\mathsf{H} = \mathbb{R}^N$ (Euclidean space with dimension N).

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2^{H} is the power set of H, i.e. the family of all subsets of H.

Hilbert spaces

Let H and G be two Hilbert spaces.
A linear operator $L\colon \mathsf{H} \to \mathsf{G}$ is **bounded** (or continuous) if

$$\|L\| = \sup_{\|x\|_{\mathsf{H}} \leqslant 1} \|Lx\|_{\mathsf{G}} < +\infty$$

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$\mathcal{B}(\mathsf{H},\mathsf{G})$: Banach space of bounded linear operators from H to G.

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Let H and G be two Hilbert spaces.
Let $L \in \mathcal{B}(\mathsf{H}, \mathsf{G})$. Its adjoint L^* is the operator in $\mathcal{B}(\mathsf{G}, \mathsf{H})$ defined as

$$(\forall (x, y) \in \mathsf{H} \times \mathsf{G}) \qquad \langle y \mid Lx \rangle_{\mathsf{G}} = \langle L^* y \mid x \rangle_{\mathsf{H}} .$$

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$$(\forall (x,y) \in \mathsf{H} \times \mathsf{G}) \qquad \langle Lx \mid y\rangle = \langle x \mid L^* y\rangle \,.$$

Example:

$$\text{If} \qquad L\colon \mathsf{H} \to \mathsf{H}^n \colon x \mapsto (x,\ldots,x)$$

$$\text{then} \qquad L^*\colon \mathsf{H}^n \to \mathsf{H}\colon y = (y_1,\ldots,y_n) \mapsto \sum_{i=1}^n y_i$$

Proof:

$$\langle Lx \mid y\rangle = \langle (x,\ldots,x) \mid (y_1,\ldots,y_n)\rangle = \sum_{i=1}^n \langle x \mid y_i\rangle = \left\langle x \;\middle|\; \sum_{i=1}^n y_i \right\rangle$$

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$$(\forall (x, y) \in \mathsf{H} \times \mathsf{G}) \qquad \langle Lx \mid y \rangle = \langle x \mid L^* y \rangle \, .$$

- We have $\|L^*\| = \|L\|$.
- If L is bijective (i.e. an **isomorphism**) then $L^{-1} \in \mathcal{B}(\mathsf{G}, \mathsf{H})$ and $(L^{-1})^* = (L^*)^{-1}$.
- If $\mathsf{H} = \mathbb{R}^N$ and $\mathsf{G} = \mathbb{R}^M$ then $L^* = L^\top$.

Hilbert spaces

Let H be a Hilbert space and $L \in \mathcal{B}(\mathsf{H}, \mathsf{H})$.

- L is self-adjoint if $L^* = L$.
- L is positive if $(\forall x \in \mathsf{H})\ \langle x \mid Lx \rangle \geqslant 0$.
- L is strictly positive if L is positive and if $(\forall x \in \mathsf{H})\ \langle x \mid Lx \rangle = 0 \Leftrightarrow x = 0$.
- L is ρ-strongly positive with $\rho \in \left]0, +\infty\right[$ if $(\forall x \in \mathsf{H})\ \langle x \mid Lx \rangle \geqslant \rho \|x\|^2$.

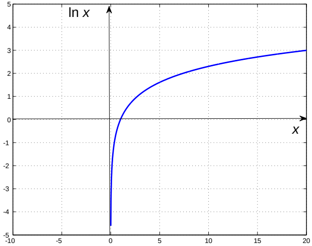
Mappings versus multivalued operators

Let H be a real Hilbert space.
A is an H-valued mapping defined on $D \subset$ H if

$$A\colon D \to \mathrm{H}$$
$$x \mapsto A(x)$$

- Example:

$$A\colon \,]0,+\infty[\to \mathbb{R}$$
$$x \mapsto \ln x$$



Mappings versus multivalued operators

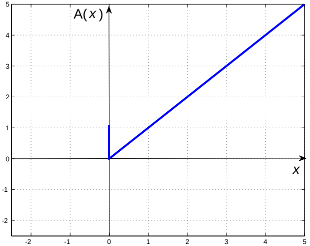
Let H be a real Hilbert space.
A is a (multivalued) operator if

$$\begin{aligned} A\colon \mathsf{H} &\to 2^{\mathsf{H}} \\ x &\mapsto \left\{ A_i(x) \,\middle|\, i \in I_x \subset \mathbb{R} \right\} \end{aligned}$$

- Example:

$$A\colon \mathbb{R} \to 2^{\mathbb{R}}$$

$$x \mapsto \begin{cases} \{x\} & \text{if } x > 0 \\ [0,1] & \text{if } x = 0 \\ \varnothing & \text{if } x < 0 \end{cases}$$



Graph

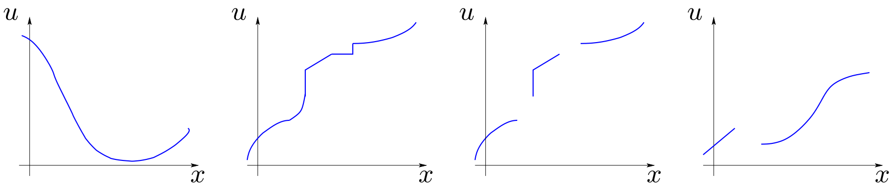
Let H be a real Hilbert space.

Let $A : \mathsf{H} \to 2^{\mathsf{H}}$.

The graph of A is

$$\operatorname{gra} A = \big\{ (x,u) \in \mathsf{H}^2 \mid u \in Ax \big\}.$$

- Graph examples:



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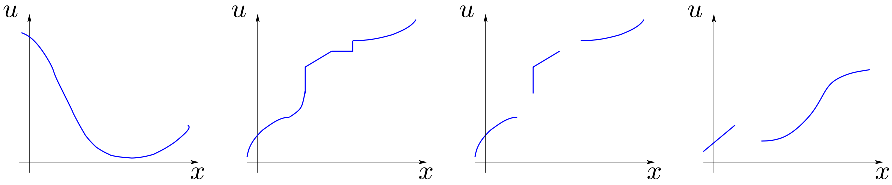


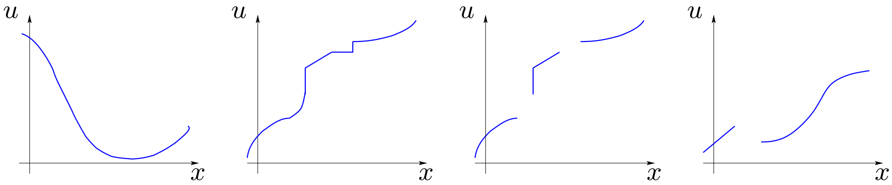
Single-valued Multivalued Multivalued Single-valued

Monotone operator: definition

Let H be a real Hilbert space.
Let $A : \mathsf{H} \to 2^{\mathsf{H}}$.
A is monotone if

$$\big(\forall (x_1, u_1) \in \operatorname{gra} A\big)\big(\forall (x_2, u_2) \in \operatorname{gra} A\big) \qquad \langle u_1 - u_2 \mid x_1 - x_2 \rangle \geqslant 0 .$$

- Monotone operators ?



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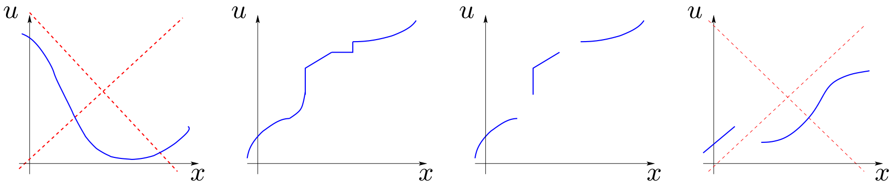

Monotone operator: example

Let H be a real Hilbert space.

Let $A \in \mathcal{B}(\mathsf{H}, \mathsf{H})$.

- A is monotone $\Leftrightarrow$ A is positive
- A monotone $\Leftrightarrow$ $A + A^*$ monotone $\Leftrightarrow$ A^* monotone.

Monotone operator: example

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- A is monotone $\Leftrightarrow$ A is positive
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Proof:

$$
\begin{aligned}
A \text{ monotone} \quad &\Leftrightarrow \quad (\forall (x_1, x_2) \in \mathsf{H}^2) \qquad \langle x_1 - x_2 \mid Ax_1 - Ax_2 \rangle \geqslant 0 \\
&\Leftrightarrow \quad (\forall x \in \mathsf{H}) \qquad 2 \langle x \mid Ax \rangle \geqslant 0 \\
&\Leftrightarrow \quad (\forall x \in \mathsf{H}) \qquad \langle x \mid Ax \rangle + \langle A^* x \mid x \rangle \geqslant 0 \\
&\Leftrightarrow \quad (\forall x \in \mathsf{H}) \qquad \langle x \mid (A + A^*) x \rangle \geqslant 0 \\
&\Leftrightarrow \quad A + A^* \text{ monotone}
\end{aligned}
$$

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Let H be a real Hilbert space.
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- A is monotone $\Leftrightarrow$ A is positive
- A monotone $\Leftrightarrow$ $A + A^*$ monotone $\Leftrightarrow$ A^* monotone.

- For $A \in \mathcal{B}(\mathsf{H},\mathsf{H})$ to be monotone, A is not required to be self-adjoint.

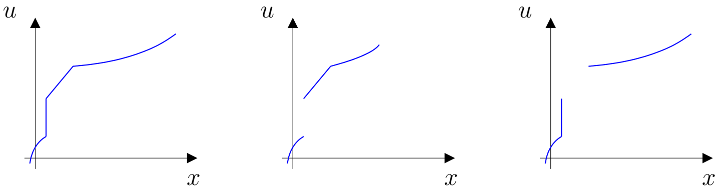
  Example : $A \in \mathcal{B}(\mathsf{H},\mathsf{H})$ skewed (i.e. $A^* = -A$) is monotone.

Domain

Let H be a real Hilbert space.
Let $A : \mathsf{H} \to 2^{\mathsf{H}}$.
The domain of A is

$$\operatorname{dom} A = \left\{ x \in \mathsf{H} \;\middle|\; Ax \neq \varnothing \right\}.$$

- Which domain ?



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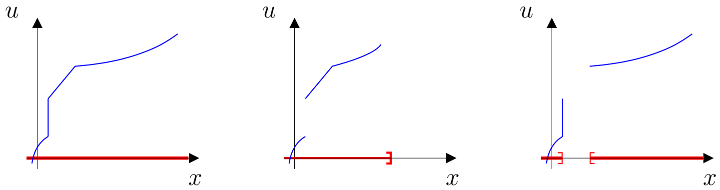

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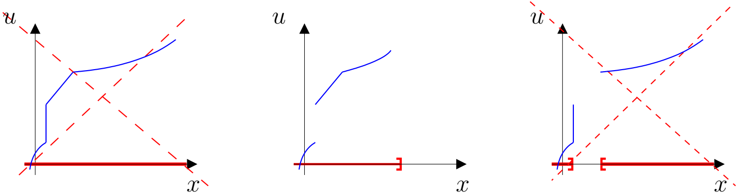
- Let $C \subset \mathsf{H}$. If $\operatorname{dom} A = C$ and for every $x \in C$, Ax is a singleton, we view A as a mapping from C to H.



Monotone operator: properties

Let H and G be two Hilbert spaces.
Let $A\colon \mathsf{H} \to 2^{\mathsf{H}}$ and $B\colon \mathsf{G} \to 2^{\mathsf{G}}$ be two monotone operators.
The following operators are monotone:

- $x \mapsto y + \gamma\rho A(\rho x + z) = \big\{y + \gamma\rho u \;\big|\; u \in A(\rho x + z)\big\}$
  where $(y, z) \in \mathsf{H}^2$, $\gamma \in [0, +\infty[$ and $\rho \in \mathbb{R}$.
- $A \times B$: $\mathsf{H} \times \mathsf{G} \to 2^{\mathsf{H}\times\mathsf{G}}$
  $(x, y) \mapsto Ax \times Ay = \big\{(u, v) \;\big|\; u \in Ax, v \in Bx\big\}$.
- $A + B$: $x \mapsto \big\{u + v \;\big|\; u \in Ax, v \in Bx\big\}$ if $\mathsf{G} = \mathsf{H}$.
- L^*BL : $x \mapsto \big\{L^*v \;\big|\; v \in B(Lx)\big\}$ if $L \in \mathcal{B}(\mathsf{H}, \mathsf{G})$.

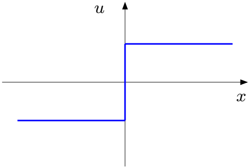
Monotone operator: inversion

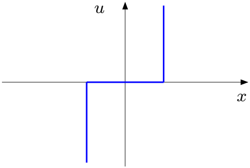
Let H be a Hilbert space.
Let $A\colon \mathsf{H} \to 2^{\mathsf{H}}$.
A^{-1} is the operator from H to 2^{H} the graph of which is

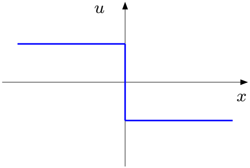
$$\operatorname{gra}(A^{-1}) = \big\{(u,x) \;\big|\; (x,u) \in \operatorname{gra} A\big\}.$$

Graph of A

Graph of A^{-1} ?







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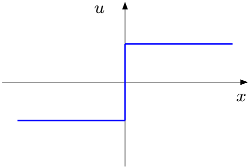
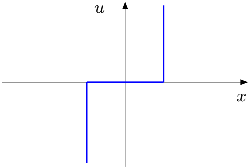
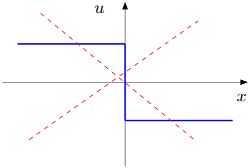


Graph of A^{-1} ?





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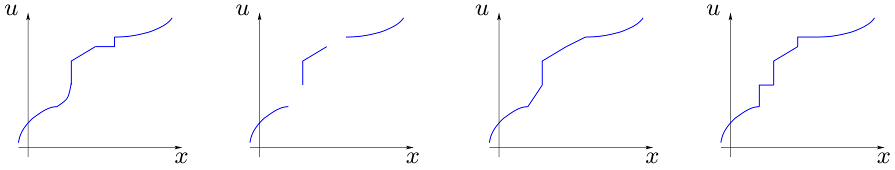
Let H be a Hilbert space.
Let $A\colon \mathsf{H} \to 2^{\mathsf{H}}$ be a monotone operator.
A^{-1} is monotone.

Maximally monotone operator: definition

Let H be a Hilbert space.
Let $A : \mathsf{H} \to 2^{\mathsf{H}}$.
A is **maximally monotone** if A is monotone and if there exists no monotone operator $B\colon \mathsf{H} \to 2^{\mathsf{H}}$ (different from A) such that $\operatorname{gra} B$ properly contains $\operatorname{gra} A$.

Maximally monotone operator ?



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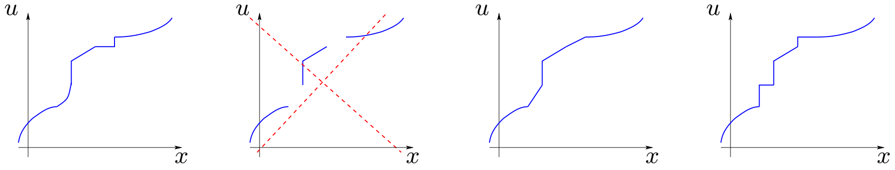

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Maximally monotone operator: second definition

Let H be a Hilbert space.
$A : \mathsf{H} \to 2^{\mathsf{H}}$ is maximally monotone if one if the following equivalent conditions is satisfied:

(*i*) A is monotone and there exists no monotone operator $B\colon \mathsf{H} \to 2^{\mathsf{H}}$ such that $\operatorname{gra} B$ properly contains $\operatorname{gra} A$.

(*ii*) For every $(x_1, u_1) \in \mathsf{H}^2$,

$$(x_1, u_1) \in \operatorname{gra} A \quad \Leftrightarrow \quad \big(\forall (x_2, u_2) \in \operatorname{gra} A\big) \quad \langle x_1 - x_2 \mid u_1 - u_2 \rangle \geqslant 0.$$

Continuous functions

Let H be a Hilbert space.
Let $A\colon \mathrm{H} \to \mathrm{H}$ be monotone and continuous. Then A is maximally monotone.

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<u>Proof</u> :
Let $(x_1, u_1) \in \mathsf{H}^2$.
Assume that, for every $x_2 \in \mathsf{H}$, $\langle x_1 - x_2 \mid u_1 - Ax_2 \rangle \geqslant 0$.
Let $x_2^\alpha = x_1 + \alpha(u_1 - Ax_1)$ where $\alpha > 0$.
We have $\langle u_1 - Ax_1 \mid u_1 - Ax_2^\alpha \rangle = -\alpha^{-1} \langle x_1 - x_2^\alpha \mid u_1 - Ax_2^\alpha \rangle \leqslant 0$.
As $\alpha \to 0$, $x_2^\alpha \to x_1$ and $\|u_1 - Ax_1\|^2 \leqslant 0$. Then, $u_1 = Ax_1$.

Continuous functions

Let H be a Hilbert space.
Let $A \colon \mathsf{H} \to \mathsf{H}$ be monotone and continuous. Then A is maximally monotone.

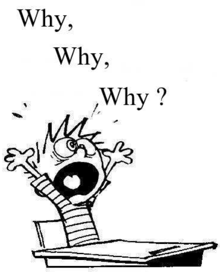
Example :
If $L \in \mathcal{B}(\mathsf{H}, \mathsf{H})$ is positive, then L is maximally monotone.

Maximally monotone operator

Maximally

monotone

operator



Maximally monotone operator

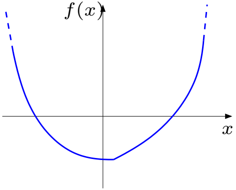
Usefulness in convex optimization

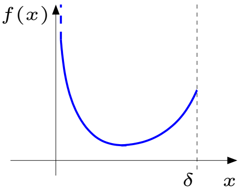
Convex analysis: definitions

Let $f : \mathsf{H} \to \left]-\infty, +\infty\right]$ where H is a Hilbert space.

- The **domain** of f is $\operatorname{dom} f = \{x \in \mathsf{H} \mid f(x) < +\infty\}$.
- The function f is **proper** if $\operatorname{dom} f \neq \varnothing$.

Domains of the functions ?





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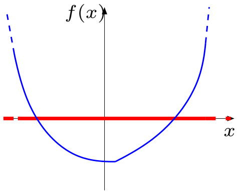
Domains of the functions ?



$\operatorname{dom} f = \mathbb{R}$
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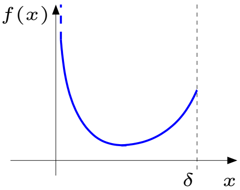

Convex analysis: definitions

Let $f : \mathsf{H} \to]-\infty, +\infty]$ where H is a Hilbert space.

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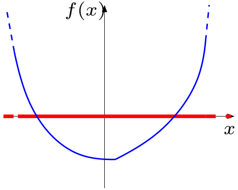

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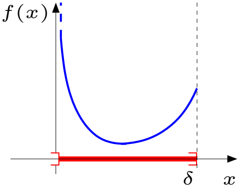

$\operatorname{dom} f =]0, \delta]$
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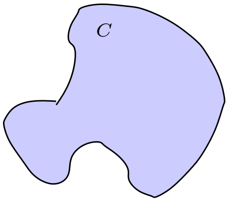
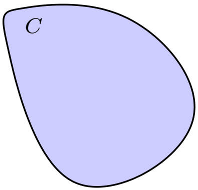
Convex analysis: definitions

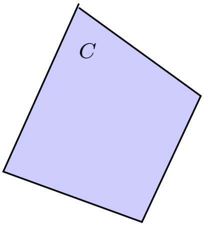
$C \subset$ H is a **convex set** if

$$(\forall (x, y) \in C^2)(\forall \alpha \in]0, 1[) \qquad \alpha x + (1 - \alpha) y \in C$$

Convex sets ?







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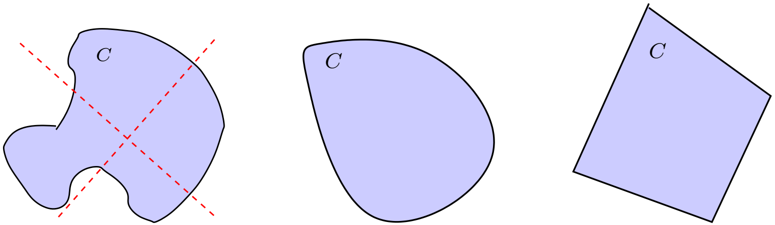

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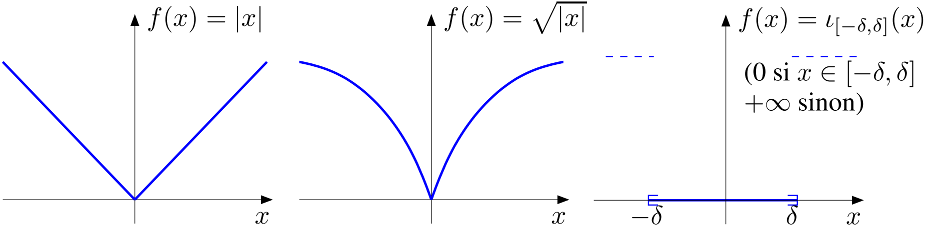

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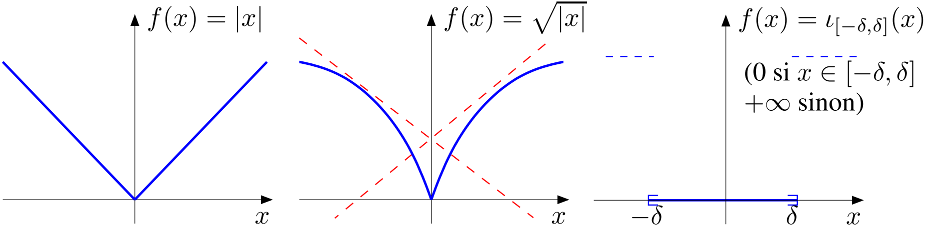

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$f : \mathsf{H} \to]-\infty, +\infty]$ is convex $\Leftrightarrow$ the epigraph of f, i.e.

$$\operatorname{epi} f = \left\{(x, \zeta) \in \operatorname{dom} f \times \mathbb{R} \;\middle|\; f(x) \leqslant \zeta\right\}$$

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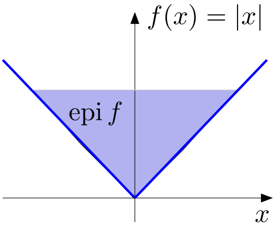

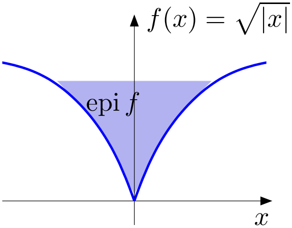

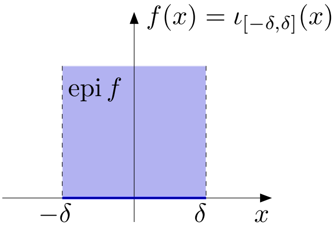

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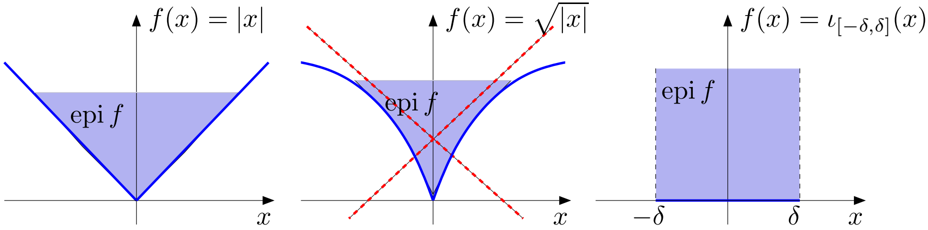

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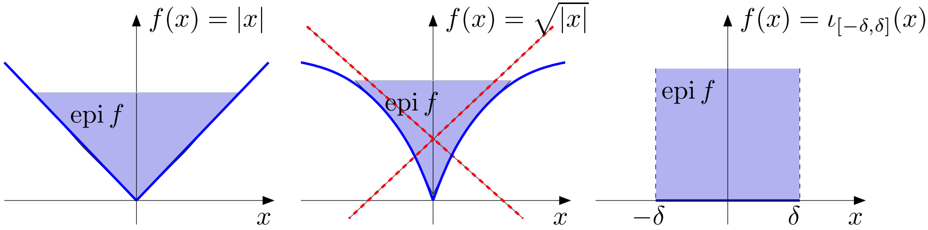

- $f : \mathcal{H} \to [-\infty, +\infty[$ is concave if $-f$ is convex.

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Let $f : \mathcal{H} \to]-\infty, +\infty]$.
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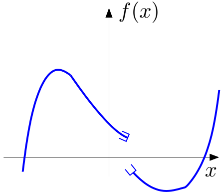
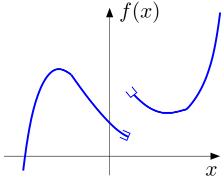
- l.s.c. functions ?





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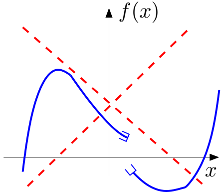

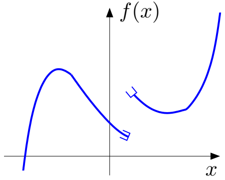

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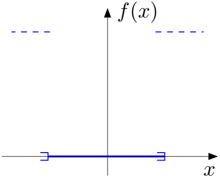

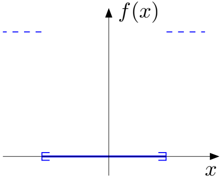

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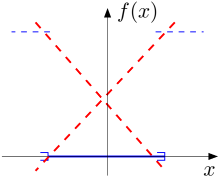

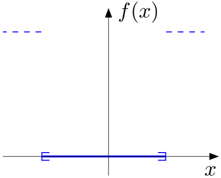

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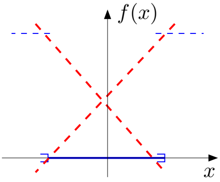

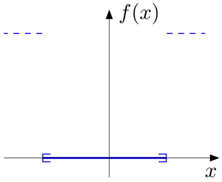

Convex analysis: definitions/properties

- $\Gamma_0(\mathsf{H})$: class of convex, l.s.c., and proper functions from H to $]-\infty,+\infty]$.
- Every continuous function on H is l.s.c.
- Every finite sum of l.s.c. (convex) functions is l.s.c. (convex).
- Let $(f_i)_{i\in I}$ be a family of l.s.c. (convex) functions. $\sup_{i\in I} f_i$ is l.s.c. (convex).

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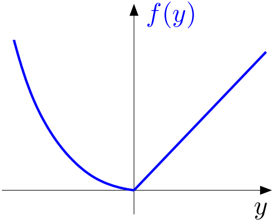

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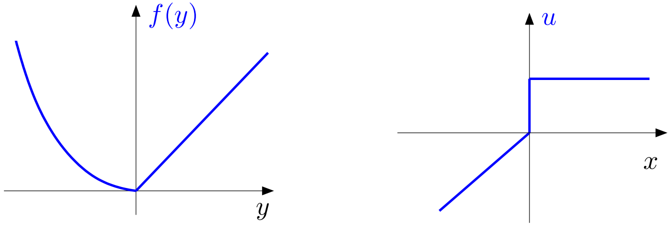

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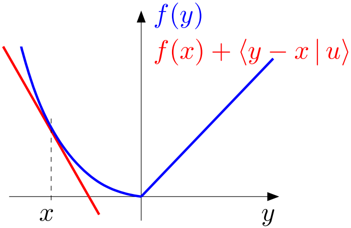

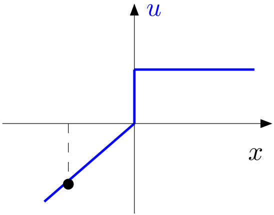

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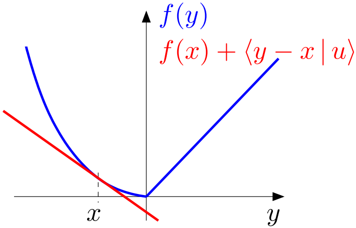

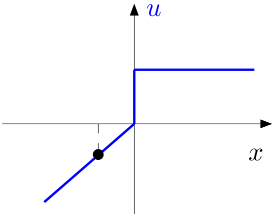

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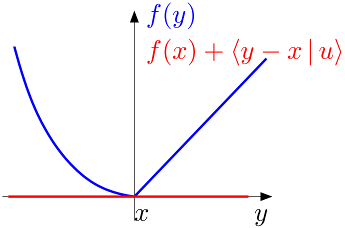

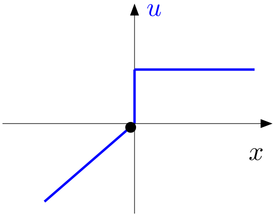

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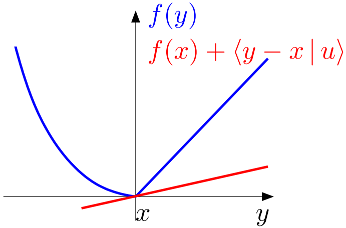

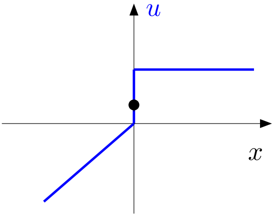

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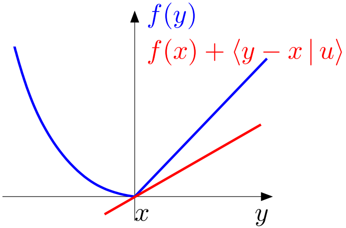

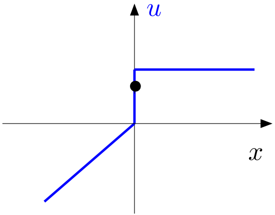

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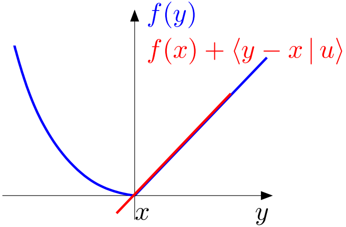

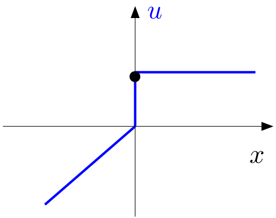

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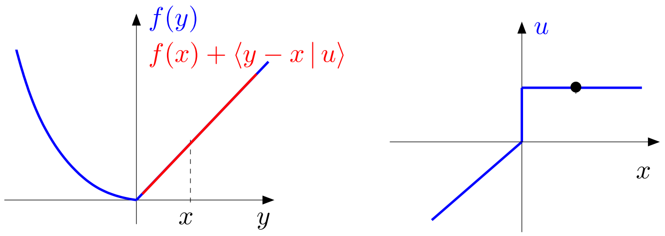

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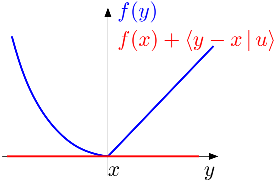

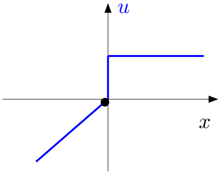

- **Fermat's rule:** $$\begin{aligned}0 \in \partial f(x) &\Leftrightarrow (\forall y \in \mathsf{H})\ \langle y - x \mid 0\rangle + f(x) \leqslant f(y)\\ &\Leftrightarrow x \in \operatorname{Argmin} f\end{aligned}$$

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- For every $x \in \operatorname{dom} f$, $\partial f(x)$ is a closed and convex set.

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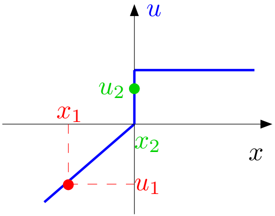

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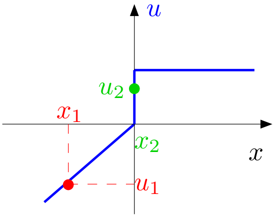
  Let $u_1 \in \partial f(x_1)$ and $u_2 \in \partial f(x_2)$.



Subdifferential of a convex function: properties

Let $f : \mathsf{H} \to]-\infty, +\infty]$ be a proper function.
The (Moreau) subdifferential of f, denoted ∂f, is such that

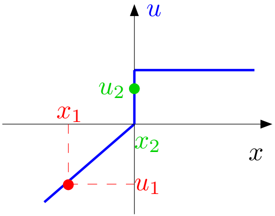
$$\partial f : \mathsf{H} \to 2^{\mathsf{H}}$$
$$x \to \{u \in \mathsf{H} \,|\, (\forall y \in \mathsf{H})\ \langle y - x | u \rangle + f(x) \leqslant f(y)\}$$

- ∂f is a monotone operator :

  Let $u_1 \in \partial f(x_1)$ and $u_2 \in \partial f(x_2)$.

  By using the subdifferential definition:

  $$\langle x_2 - x_1 | u_1 \rangle + f(x_1) \leqslant f(x_2)$$
  $$\langle x_1 - x_2 | u_2 \rangle + f(x_2) \leqslant f(x_1)$$



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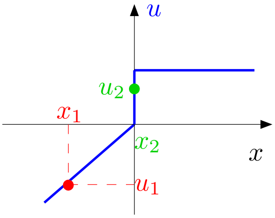
  $$\langle x_1 - x_2 | u_2 \rangle + f(x_2) \leqslant f(x_1)$$
  and thus $\langle x_1 - x_2 \,|\, u_1 - u_2 \rangle \geqslant 0$.



Subdifferential of a convex function: properties

- The subdifferential of a convex and proper function is:
  - Monotone
  - If f is Gâteaux differentiable at x, then $\partial f(x) = \{\nabla f(x)\}$

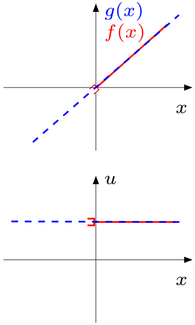
Subdifferential of a convex function: properties

- The subdifferential of a convex and proper function is:
  - Monotone
  - If f is Gâteaux differentiable at x, then $\partial f(x) = \{\nabla f(x)\}$
  - Non necessarily maximally monotone

Counterexample: For every $x \in \mathrm{H}$,

$$f(x) = \begin{cases} x & \text{if } x > 0 \\ +\infty & \text{otherwise} \end{cases}, \qquad g(x) = x$$

$$\Rightarrow \quad \partial f(x) = \begin{cases} \{1\} & \text{if } x > 0 \\ \varnothing & \text{otherwise} \end{cases}, \qquad \partial g(x) = \{1\}.$$



Consequently, $\operatorname{gra} \partial f =]0, +\infty[\times \{1\} \subset \mathbb{R} \times \{1\}$
$\subset \operatorname{gra} \partial g$

.

Subdifferential of a convex function: properties

- The subdifferential of a convex, proper and l.s.c. function is
  - Maximally monotone

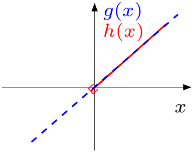
Subdifferential of a convex function: properties

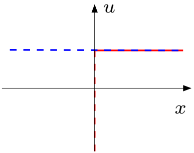
- The subdifferential of a convex, proper and l.s.c. function is
  - Maximally monotone

Example: For every $x \in$ H,

$$h(x) = \begin{cases} x & \text{if } x \geqslant 0 \\ +\infty & \text{otherwise} \end{cases}, \quad g(x) = x$$

$$\Rightarrow \quad \partial h(x) = \begin{cases} \{1\} & \text{if } x > 0 \\ ]-\infty, 1] & \text{if } x = 0 \\ \varnothing & \text{otherwise} \end{cases}, \quad \partial g(x) = \{1\}.$$





Consequently, $\mathrm{gra}\, \partial h \not\subset \mathrm{gra}\, \partial g$.

Subdifferential of a convex function: properties

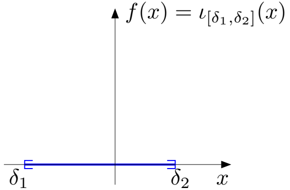
- The subdifferential of a convex, proper and l.s.c. function is
  - Maximally monotone
  - If $H = \mathbb{R}$, equivalence between both properties.

Subdifferential of a convex function: example

Soit $C \subset \mathsf{H}$.
The indicator function of C is

$$(\forall x \in \mathsf{H}) \qquad \iota_C(x) = \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{otherwise.} \end{cases}$$

Example : $C = [\delta_1, \delta_2]$

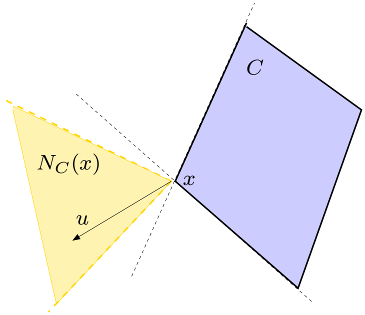


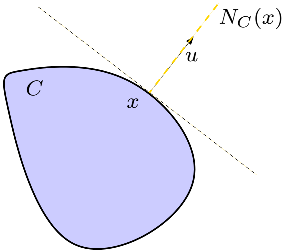
- $\iota_C \in \Gamma_0(\mathsf{H}) \Leftrightarrow C$ is a nonempty closed convex set.
  Proof: $\mathrm{epi}_{\iota_C} = C \times [0, +\infty[$.

Subdifferential of a convex function: example

For every $x \in \mathsf{H}$, $\partial\iota_C(x)$ is the normal cone to C at x defined by

$$N_C(x) = \begin{cases} \{u \in \mathsf{H} \mid (\forall y \in C) \quad \langle u \mid y - x \rangle \leqslant 0\} & \text{if } x \in C \\ \varnothing & \text{otherwise.} \end{cases}$$





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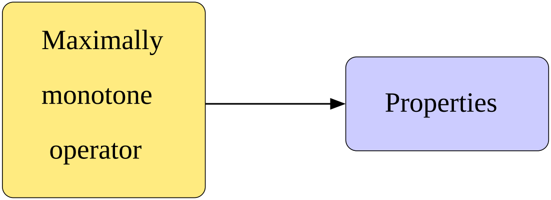
- If $x \in \mathrm{int} C$ then $N_C(x) = \{0\}$.
- If C is a vector space then, for every $x \in C$, $N_C(x) = C^{\perp}$.


Maximally monotone operator
Properties

Maximally monotone operator: properties

Let H be a Hilbert space.
Let $A\colon \mathsf{H} \to 2^{\mathsf{H}}$ be a maximally monotone operator.
For every $x \in \mathsf{H}$, Ax is a closed convex set.

<u>Proof</u>:

$$Ax = \bigcap_{(x',u')\in \operatorname{gra} A} \big\{u \in \mathsf{H} \;\big|\; \big\langle x - x' \mid u - u' \big\rangle \geqslant 0\big\}.$$

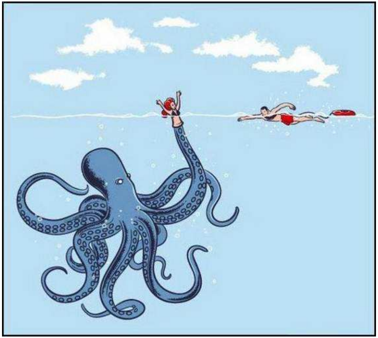
Consequently, Ax is an intersection of closed convex sets.

Maximally monotone operator: properties

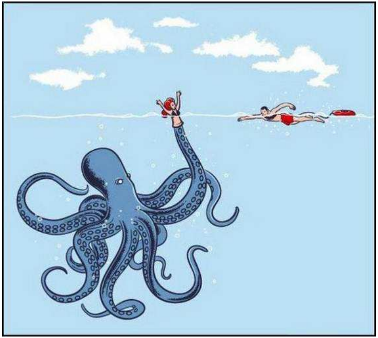
Let H and G be two Hilbert spaces.
Let $A\colon \mathsf{H} \to 2^{\mathsf{H}}$ and $B\colon \mathsf{G} \to 2^{\mathsf{G}}$ be two maximally monotone operators.
The following operators are maximally monotone:

- $y + \gamma\rho A(\rho\cdot + z)$ where $(y, z) \in \mathsf{H}^2$, $\gamma \in [0, +\infty[$ and $\rho \in \mathbb{R}$,
- $A \times B$,
- A^{-1}.

Inverse of a maximally monotone operator



Inverse of a maximally monotone operator → Usefulness ?



Conjugate: definition

Let H be a Hilbert space and $f\colon \mathsf{H} \to \,]-\infty, +\infty]$.
The conjugate of f is the function $f^*\colon \mathsf{H} \to [-\infty, +\infty]$ such that

$$(\forall u \in \mathsf{H}) \qquad f^*(u) = \sup_{x \in \mathsf{H}} \big(\langle x \mid u \rangle - f(x) \big) .$$

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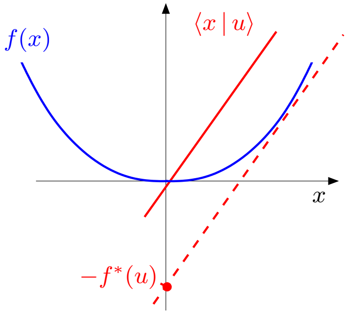

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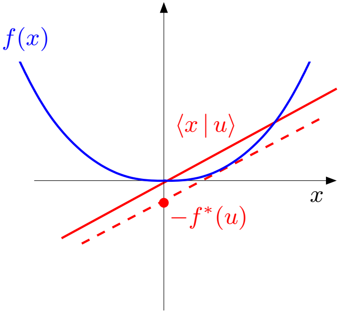

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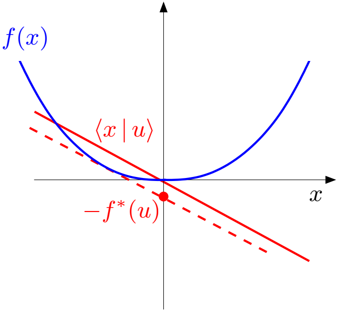

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Example :

- $(\forall x \in \mathbb{R}^N)\ f(x) = \frac{1}{q}\|x\|_q^q$ with $q \in]1, +\infty[$
  $\Rightarrow (\forall u \in \mathbb{R}^N)\ f^*(u) = \frac{1}{q^*}\|u\|_{q^*}^{q^*}$ with $\frac{1}{q} + \frac{1}{q^*} = 1$

Conjugate: definition

Let H be a Hilbert space and $f\colon \mathsf{H} \to \,]-\infty, +\infty]$.
The conjugate of f is the function $f^*\colon \mathsf{H} \to [-\infty, +\infty]$ such that

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- f^* is l.s.c. and convex.

Conjugate: definition

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Moreau-Fenchel theorem

Let H be a Hilbert space and $f\colon \mathsf{H} \to \left]-\infty, +\infty\right]$ be a proper function.

$$f \text{ is l.s.c. and convex} \Leftrightarrow f^{**} = f.$$

- Consequence: If $f \in \Gamma_0(\mathsf{H})$, then $f^* \in \Gamma_0(\mathsf{H})$.

Conjugate: definition

Let H be a Hilbert space and $f\colon \mathsf{H} \to \,]-\infty,+\infty]$.
The conjugate of f is the function $f^*\colon \mathsf{H} \to [-\infty,+\infty]$ such that

$$(\forall u \in \mathsf{H}) \qquad f^*(u) = \sup_{x\in\mathsf{H}} \big(\langle x \mid u\rangle - f(x)\big).$$

Let $f \in \Gamma_0(\mathsf{H})$.
$(\partial f)^{-1} = \partial f^*$

Consequence: $(\forall (x,u) \in \mathsf{H}^2)\ u \in \partial f(x) \Leftrightarrow x \in \partial f^*(u)$.

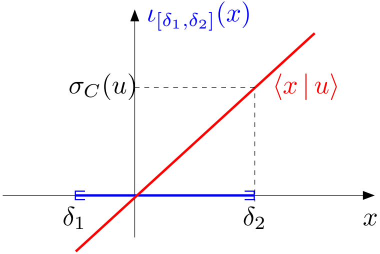
Conjugate versus Fourier transform

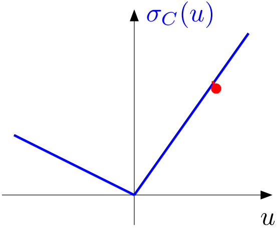
	conjugate		Fourier transform	
Property	$h(x)$	$h^*(u)$	$h(x)$	$\widehat{h}(\nu)$
invariant function	$\frac{1}{2}\\|x\\|^2$	$\frac{1}{2}\\|u\\|^2$	$\exp(-\pi\\|x\\|^2)$	$\exp(-\pi\\|\nu\\|^2)$
translation $c \in \mathsf{H}$	$f(x-c)$	$f^*(u) + \langle u \mid c\rangle$	$f(x-c)$	$\exp(-\jmath 2\pi\langle \nu \mid c\rangle)\widehat{f}(\nu)$
dual translation $c \in \mathsf{H}$	$f(x) + \langle x \mid c\rangle$	$f^*(u-c)$	$\exp(\jmath 2\pi\langle x \mid c\rangle) f(x-c)$	$\widehat{f}(\nu - c)$
scalar multiplication $\alpha \in]0,+\infty[$	$\alpha f(x)$	$\alpha f^*\left(\frac{u}{\alpha}\right)$	$\alpha f(x)$	$\alpha \widehat{f}(\nu)$
scaling $\alpha \in \mathbb{R}^*$	$f\left(\frac{x}{\alpha}\right)$	$f^*(\alpha u)$	$f\left(\frac{x}{\alpha}\right)$	$\|\alpha\|\widehat{f}(\alpha\nu)$
isomorphism $L \in \mathcal{B}(\mathsf{G},\mathsf{H})$	$f(Lx)$	$f^*\big((L^{-1})^\top u\big)$	$f(Lx)$	$\frac{1}{\|\det(L)\|}\widehat{f}\big((L^{-1})^\top \nu\big)$
reflection	$f(-x)$	$f^*(-u)$	$f(-x)$	$\widehat{f}(-\nu)$
separability	$\sum_{n=1}^{N} \varphi_n(x^{(n)})$ $x = (x^{(n)})_{1\leqslant n\leqslant N}$	$\sum_{n=1}^{N} \varphi_n^*(u^{(n)})$ $u = (u^{(n)})_{1\leqslant n\leqslant N}$	$\prod_{n=1}^{N} \varphi_n(x^{(n)})$ $x = (x^{(n)})_{1\leqslant n\leqslant N}$	$\prod_{n=1}^{N} \widehat{\varphi}_n(\nu^{(n)})$ $\nu = (\nu^{(n)})_{1\leqslant n\leqslant N}$
isotropy	$\psi(\\|x\\|)$	$\psi^*(\\|u\\|)$	$\psi(\\|x\\|)$	$\widetilde{\psi}(\\|\nu\\|)$
identity element of convolution	$\iota_{\{0\}}(x)$	0	$\delta(x)$	1
identity element of addition/product	0	$\iota_{\{0\}}(u)$	1	$\delta(\nu)$

Conjugate: example

Let H be a Hilbert space and $C \subset$ H.
σ_C is the support function of C if

$$(\forall u \in \mathsf{H}) \qquad \sigma_C(u) = \sup_{x \in C} \langle x \mid u \rangle = \iota_C^*(u).$$





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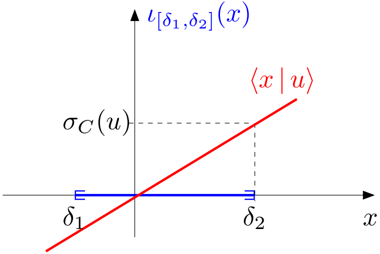

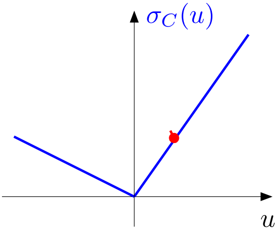

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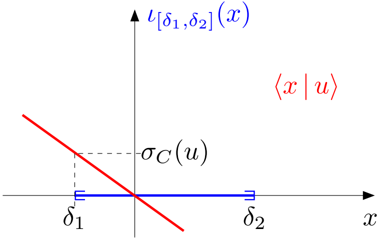

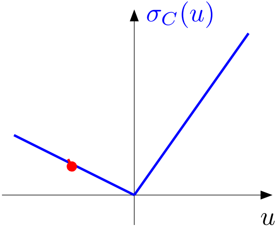

Conjugate: example

Let H be a Hilbert space.
$f\colon \mathsf{H} \to \left]-\infty,+\infty\right]$ is positively homogeneous if

$$(\forall x \in \mathsf{H})(\forall \alpha \in \left]0,+\infty\right[) \qquad f(\alpha x) = \alpha f(x).$$

f is positively homogeneous and belongs to $\Gamma_0(\mathsf{H})$

$$\Updownarrow$$

$f = \sigma_C$ where C is a nonempty closed convex subset of H.

Conjugate: example

Let H be a Hilbert space.
$f\colon \mathsf{H} \to\,]-\infty,+\infty]$ is positively homogeneous if

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$$\Updownarrow$$

$f = \sigma_C$ where C is a nonempty closed convex subset of H.

- Example 1: Let $f\colon \mathbb{R} \to\,]-\infty,+\infty] : x \mapsto \begin{cases} \delta_1 x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \delta_2 x & \text{if } x > 0 \end{cases}$

  with $-\infty \leqslant \delta_1 < \delta_2 \leqslant +\infty$. Then, $f = \sigma_C$ where C is the closed real interval such that $\inf C = \delta_1$ and $\sup C = \delta_2$.

Conjugate: example

Let H be a Hilbert space.
$f : \mathsf{H} \to]-\infty, +\infty]$ is positively homogeneous if

$$(\forall x \in \mathsf{H})(\forall \alpha \in]0, +\infty[) \qquad f(\alpha x) = \alpha f(x).$$

f is positively homogeneous and belongs to $\Gamma_0(\mathsf{H})$

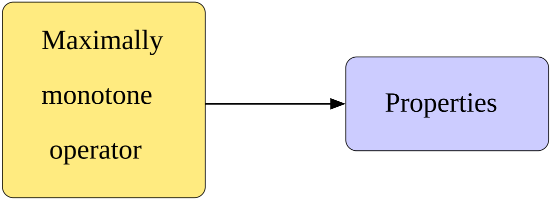
$\Updownarrow$

$f = \sigma_C$ where C is a nonempty closed convex subset of H.

- Example 2: Let f be a ℓ^q norm of $\mathbb{R}^N$ with $q \in [1, +\infty]$.
  We have $f = \sigma_C$ where

$$C = \{y \in \mathbb{R}^N \mid \|y\|_{q^*} \leqslant 1\} \qquad \text{with } \frac{1}{q} + \frac{1}{q^*} = 1.$$

Particular case : ℓ^1 norm of $\mathbb{R}^N \Rightarrow C = [-1, 1]^N$.


Maximally
monotone
operator
Properties

Maximally monotone operator: sum

Let A and B be two maximally monotone operators.
$A + B$ is monotone but may not be maximally monotone.

Maximally monotone operator: sum

Let H be a Hilbert space.
Let A and B be two maximally monotone operators from H to 2^{H} such that one of the following assumptions is satisfied:

- $\operatorname{dom} B = \mathsf{H}$
- $\operatorname{dom} A \cap \operatorname{int}(\operatorname{dom} B) \neq \varnothing$
- $0 \in \operatorname{int}(\operatorname{dom} A - \operatorname{dom} B)$

then $A + B$ is maximally monotone.

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- $0 \in \operatorname{int}(\operatorname{dom} A - \operatorname{dom} B)$

then $A + B$ is maximally monotone.

Consequence: Let $\alpha \in [0, +\infty[$. If A is maximally monotone, then $A + \alpha \mathrm{Id}$ is maximally monotone.

Maximally monotone operator: linear transform

Let H and G two Hilbert spaces.
Let $B\colon \mathsf{G} \to 2^{\mathsf{G}}$ be a maximally monotone operator and $L \in \mathcal{B}(\mathsf{H},\mathsf{G})$ such that one of the following assumptions is satisfied:

- L surjective
- $0 \in \mathrm{int}(\mathrm{dom}\, B - \mathrm{ran}\, L)$

then $L^* B L$ is maximally monotone.

Maximally monotone operator: linear transform

Let H and G two Hilbert spaces.
Let $B\colon \mathsf{G} \to 2^{\mathsf{G}}$ be a maximally monotone operator and $L \in \mathcal{B}(\mathsf{H},\mathsf{G})$ such that one of the following assumptions is satisfied:

- L surjective
- $0 \in \operatorname{int}(\operatorname{dom} B - \operatorname{ran} L)$

then L^*BL is maximally monotone.

Consequence: Let $\mu \in]0,+\infty[$.
If B is maximally monotone and $LL^* = \mu\mathrm{Id}$, then L^*BL is maximally monotone.

Proof: $LL^* = \mu\mathrm{Id} \Rightarrow \operatorname{ran} L = \mathsf{H}$.

Part 2: Nonexpansive operators

1. Background on nonexpansive operators
   - Definition
   - Properties
   - Examples
   - Resolvent

2. Proximal operator
   - Definition
   - Properties
   - Examples

Nonexpansive operator: definition

Let H be a Hilbert space and let C be a nonempty subset of H.
Let $A\colon C \to \mathsf{H}$.
A is **nonexpansive** if $\big(\forall (x,y) \in C^2\big) \quad \|Ax - Ay\| \leqslant \|x - y\|$.

Nonexpansive operator: definition

Let H be a Hilbert space and let C be a nonempty subset of H.
Let $A\colon C \to$ H and $\nu \in \left]0, +\infty\right[$

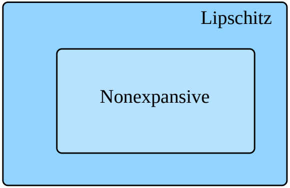
$\nu^{-1}A$ is nonexpansive if $\big(\forall (x,y) \in C^2\big) \quad \|Ax - Ay\| \leqslant \nu \|x - y\|$.

Nonexpansive operator: definition

Let H be a Hilbert space and let C be a nonempty subset of H.
Let $A\colon C \to \mathrm{H}$ and $\nu \in \left]0,+\infty\right[$

$\nu^{-1}A$ is nonexpansive if $\big(\forall (x,y) \in C^2\big) \quad \|Ax - Ay\| \leqslant \nu \|x - y\|$.

$\nu^{-1}A$ is nonexpansive $\Leftrightarrow$ A is ν-Lipschitzian.



Nonexpansive operator: definition

Let H be a real Hilbert space.
Let $A\colon \mathsf{H} \to 2^{\mathsf{H}}$
A is firmly nonexpansive if

$$(\forall (x,u) \in \operatorname{gra} A)(\forall (y,v) \in \operatorname{gra} A) \quad \|u-v\|^2 \leqslant \langle u-v \mid x-y \rangle .$$

Nonexpansive operator: definition

Let H be a real Hilbert space and let C be a nonempty subset of H.
Let $A\colon C \to$ H.
A is firmly nonexpansive if

$$(\forall x \in C)(\forall y \in C) \quad \|Ax - Ay\|^2 \leqslant \langle Ax - Ay \mid x - y\rangle .$$

Nonexpansive operator: definition

Let H be a real Hilbert space and let C be a nonempty subset of H.
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$$\big(\forall (x,y) \in C^2\big) \quad \|Ax - Ay\|^2 + \|(\mathrm{Id} - A)x - (\mathrm{Id} - A)y\|^2 \leqslant \|x - y\|^2 .$$

Nonexpansive operator: definition

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- A is firmly nonexpansive $\Leftrightarrow$ $\mathrm{Id} - A$ is firmly nonexpansive.
- A is firmly nonexpansive $\Leftrightarrow$ $2A - \mathrm{Id}$ is nonexpansive.

Nonexpansive operator: definition

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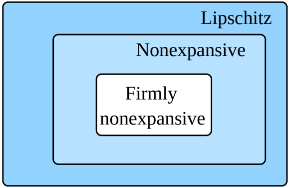
- A is firmly nonexpansive $\Leftrightarrow$ $\underbrace{2A - \mathrm{Id}}_{\text{Reflection of A}}$ is nonexpansive.

Nonexpansive operator: definition

Let $\mathcal{H}$ be a real Hilbert space and let C be a nonempty subset of $\mathcal{H}$.
Let $A\colon C \to \mathcal{H}$.
A is **firmly nonexpansive** if

$$(\forall (x,y) \in C^2) \quad \|Ax - Ay\|^2 + \|(\mathrm{Id} - A)x - (\mathrm{Id} - A)y\|^2 \leqslant \|x - y\|^2.$$

A is firmly nonexpansive $\Rightarrow$ A is nonexpansive.



Nonexpansive operator: definition

Let H be a real Hilbert space and let C be a nonempty subset of H.
Let $A\colon C \to$ H and $\beta \in \,]0, +\infty[$.
A is **β-cocoercive** if βA is firmly nonexpansive, i.e.,

$$(\forall x \in C)(\forall y \in C) \quad \beta\|Ax - Ay\|^2 \leqslant \langle x - y \mid Ax - Ay\rangle .$$

Nonexpansive operator: definition

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- Let H and G be two real Hilbert spaces, $L \in \mathcal{B}(\mathsf{G}, \mathsf{H})$ nonzero, and $A\colon$ H $\to$ H. A is β-cocoercive $\Rightarrow L^*AL$ is $\|L\|^{-2}\beta$-cocoercive.

Nonexpansive operator: definition

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- A is β-cocoercive $\Rightarrow$ A is β^{-1}-Lipschitzian.

Nonexpansive operator: definition

Let H be a real Hilbert space and let C be a nonempty subset of H.
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- A is β-cocoercive $\Rightarrow$ A is β^{-1}-Lipschitzian.
- $A\colon \mathsf{H} \to \mathsf{H}$ is β-cocoercive $\Rightarrow$ A is maximally monotone.

Nonexpansive operator: definition

Let H be a real Hilbert space and let C be a nonempty subset of H.
Let $A\colon C \to$ H and $\beta \in \,]0,+\infty[$.
A is β-cocoercive if βA is firmly nonexpansive, i.e.,

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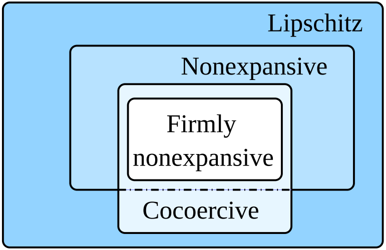

Nonexpansive operator: definition

Let H be a real Hilbert space and let C be a nonempty subset of H.
Let $A : C \to$ H and let $\alpha \in]0, 1[$.
A is **α-averaged** if there exists a nonexpansive operator $R\colon C \to$ H such that

$$A = (1-\alpha)\mathrm{Id} + \alpha R.$$

Nonexpansive operator: definition

Let H be a real Hilbert space and let C be a nonempty subset of H.
Let $A : C \to$ H and let $\alpha \in]0, 1[$.
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- A is α-averaged $\Rightarrow$ A is nonexpansive.
- A is $\frac{1}{2}$-averaged $\Leftrightarrow$ A is firmly nonexpansive.
- A is α-averaged $\Rightarrow$ A is α'-averaged for every $\alpha' \in [\alpha, 1[$.
- Let $\lambda \in]0, 1/\alpha[$. A is α-averaged $\Rightarrow$ $(1-\lambda)\mathrm{Id} + \lambda A$ is $\lambda\alpha$-averaged.

Nonexpansive operator: definition

Let H be a real Hilbert space and let C be a nonempty subset of H.
Let $A : C \to$ H and let $\alpha \in]0,1[$.
A is α-averaged if

$$(\forall (x,y) \in C^2) \quad \|Ax - Ay\|^2 + \frac{1-\alpha}{\alpha}\|(\mathbf{Id} - A)x - (\mathbf{Id} - A)y\|^2 \leqslant \|x - y\|^2.$$

- Let $(\omega_i)_{1 \leqslant i \leqslant n} \in]0,1]^n$ be such that $\sum_{i=1}^n \omega_i = 1$ and let $(\alpha_i)_{1 \leqslant i \leqslant n} \in]0,1[^n$. If, for every $i \in \{1,\ldots,n\}$, $A_i : C \to$ H is α_i-averaged, then $\sum_{i=1}^n \omega_i A_i$ is α-averaged with $\alpha = \max_{1 \leqslant i \leqslant n} \alpha_i$.

- Let $(\alpha_i)_{1 \leqslant i \leqslant n} \in]0,1[^n$. If, for every $i \in \{1,\ldots,n\}$, $A_i : C \to C$ is α_i-averaged, then $A_1 \cdots A_n$ is α-averaged with
$$\alpha = \frac{n}{n - 1 + \frac{1}{\max_{1 \leqslant i \leqslant n} \alpha_i}}.$$

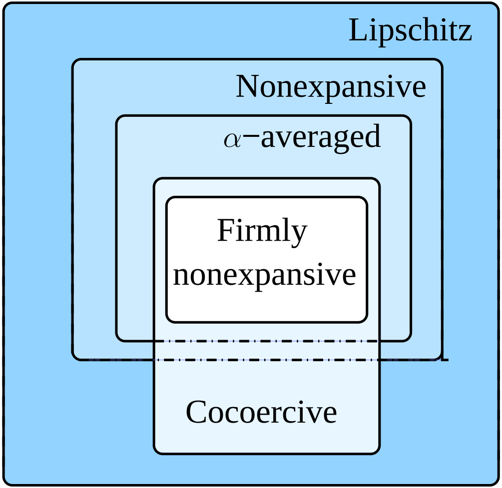
Nonexpansive operator: definition

Let H be a real Hilbert space and let C be a nonempty subset of H.
Let $A : C \to$ H and let $\alpha \in]0,1[$.
A is **α-averaged** if

$$(\forall (x,y) \in C^2) \quad \|Ax - Ay\|^2 + \frac{1-\alpha}{\alpha}\|(\mathrm{Id} - A)x - (\mathrm{Id} - A)y\|^2 \leqslant \|x-y\|^2.$$

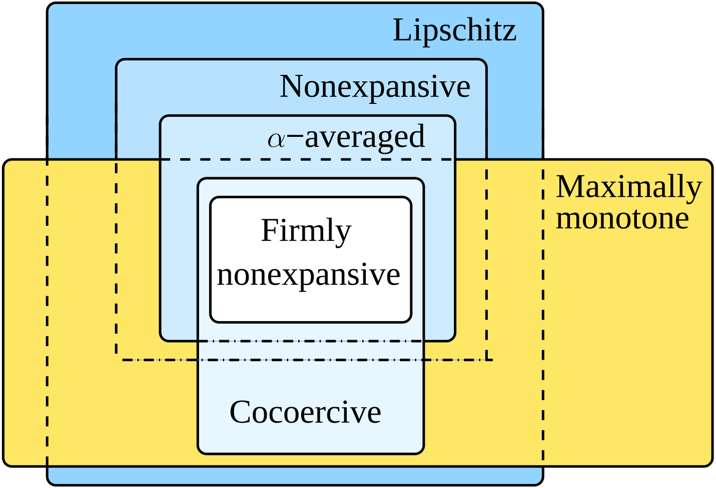
$A :$ H $\to$ H is α-averaged with $\alpha \in]0,1/2] \Rightarrow A$ is maximally monotone.

Nonexpansive operator: recap



Nonexpansive operator: recap

(if the domain C is equal to H)



Nonexpansive operator: properties

Let H be a real Hilbert space and let C be a nonempty subset of H.
Let $A\colon C \to$ H.
Let $\beta \in \,]0,+\infty[$ and $\gamma \in]0, 2\beta[$.
If A is β-cocoercive, then $\mathrm{Id} - \gamma A$ is $\gamma/(2\beta)$-averaged.

Proof :
A β-cocoercive $\Leftrightarrow$ βA firmly nonexpansive.
There exists a nonexpansive operator $R\colon C \to$ H such that
$\beta A = (\mathrm{Id} + R)/2$.
Thus

$$\mathrm{Id} - \gamma A = \Big(1 - \frac{\gamma}{2\beta}\Big)\mathrm{Id} + \frac{\gamma}{2\beta}(-R).$$

$(-R)$ being nonexpansive, $\mathrm{Id} - \gamma A$ is $\gamma/(2\beta)$-averaged.

Nonexpansive operators

Nonexpansive operators $\longrightarrow$ What is their use ?

Nonexpansive operator: example

Descent lemma

Let H be a real Hilbert space, $f\colon \mathsf{H} \to \mathbb{R}$ and $\nu \in \left]0,+\infty\right[$.
If f is differentiable and its gradient is ν-Lipschitzian, then

$$\big(\forall (x,y) \in \mathsf{H}^2\big) \quad f(y) \leqslant f(x) + \langle y-x \mid \nabla f(x)\rangle + \frac{\nu}{2}\|y-x\|^2.$$

Baillon-Haddad theorem

Let H be a real Hilbert space, $f \in \Gamma_0(\mathsf{H})$ and $\nu \in \left]0,+\infty\right[$.
If f is differentiable, then ∇f ν-Lipschitzian $\Leftrightarrow$ ∇f ν^{-1}-cocoercive.

Nonexpansive operator: example

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Let H be a Hilbert space, $f \in \Gamma_0(\mathsf{H})$, $\nu \in \,]0,+\infty[$ and $\gamma \in]0, 2\nu^{-1}[$.
f differentiable and ∇f ν-Lipschitzian $\Rightarrow$ $\mathrm{Id} - \gamma\nabla f$ is $\gamma\nu/2$-averaged.

Nonexpansive operator: example

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Nonexpansive operator: example

Descent lemma

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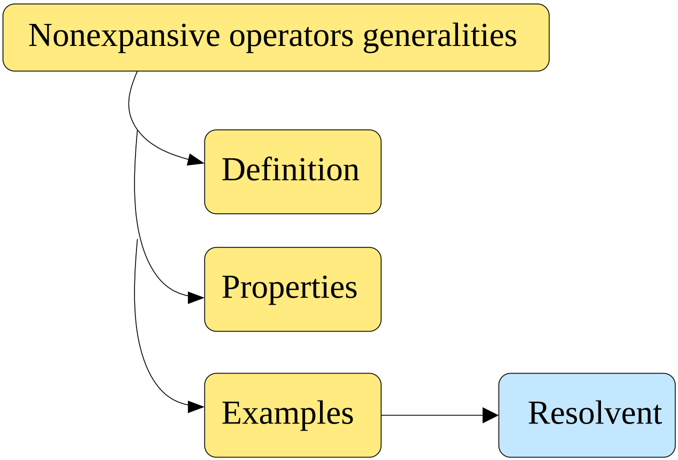
Let H be a Hilbert space, $f \in \Gamma_0(\mathsf{H})$, and $\nu \in\,]0,+\infty[$
f differentiable and ∇f ν-Lipschitzian $\Leftrightarrow$ f^* is ν^{-1}-strongly convex.

Remark : f^* is ν^{-1}-strongly convex if $f^* - \nu^{-1}\|\cdot\|^2/2$ is convex.



Nonexpansive operators generalities
Definition
Properties
Examples
Resolvent

Resolvent: definition

Let H be a Hilbert space.
Let $A : \mathsf{H} \to 2^{\mathsf{H}}$.
The revolvent of A is

$$J_A = (\mathrm{Id} + A)^{-1}.$$

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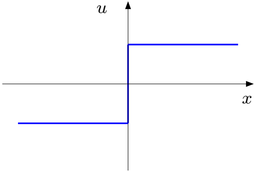

A $\qquad$ $A + \mathrm{Id}$? $\qquad$ J_A ?

Resolvent: definition

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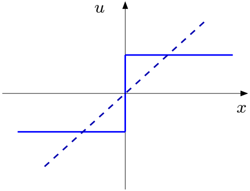

A and Id $\qquad A + \mathrm{Id}$? $\qquad J_A$?

Resolvent: definition

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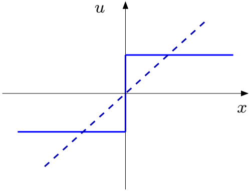
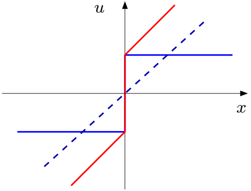
- Example :



A and Id



$A + \mathrm{Id}$

J_A ?

Resolvent: definition

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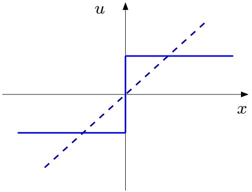
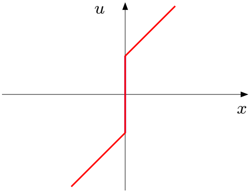


A and Id



$A + \mathrm{Id}$

J_A ?

Resolvent: definition

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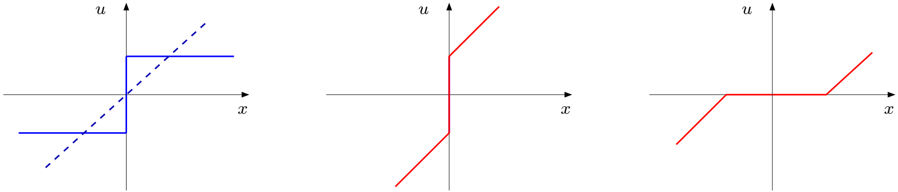

A and Id $\qquad$ $A + \mathrm{Id}$ $\qquad$ J_A

Resolvent: definition

The range of an operator $B\colon \mathsf{H} \to 2^{\mathsf{H}}$ is

$$\operatorname{ran} B = \big\{u \in \mathsf{H} \;\big|\; \exists x \in \mathsf{H}, u \in Bx\big\}.$$

Minty theorem

Let H be a Hilbert space.
Let $A\colon \mathsf{H} \to 2^{\mathsf{H}}$ be a monotone operator.

$$\operatorname{ran}(\mathrm{Id} + A) = \mathsf{H} \quad \Leftrightarrow \quad A \text{ is maximally monotone.}$$

Resolvent: properties

Let H be a real Hilbert space. Let $A : \mathsf{H} \to 2^{\mathsf{H}}$.
A is monotone $\Leftrightarrow$ J_A is firmly nonexpansive.

Proof : A is monotone if and only if

$$
\begin{aligned}
&\big(\forall (x,u) \in \operatorname{gra} A\big)\big(\forall (y,v) \in \operatorname{gra} A\big) \quad \langle x-y \mid u-v\rangle \geqslant 0\\
\Leftrightarrow&\big(\forall (x,u) \in \operatorname{gra} A\big)\big(\forall (y,v) \in \operatorname{gra} A\big) \quad \langle x-y \mid x-y+u-v\rangle \geqslant \|x-y\|^2\\
\Leftrightarrow&\big(\forall (x,u') \in \operatorname{gra}(\mathrm{Id}\ +A)\big)\big(\forall (y,v') \in \operatorname{gra}(\mathrm{Id}\ +A)\big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad \big\langle x-y \mid u'-v'\big\rangle \geqslant \|x-y\|^2\\
\Leftrightarrow&\big(\forall (u',x) \in \operatorname{gra} J_A\big)\big(\forall (v',y) \in \operatorname{gra} J_A\big) \quad \big\langle u'-v' \mid x-y\big\rangle \geqslant \|x-y\|^2\\
\Leftrightarrow& J_A \text{ is firmly nonexpansive}
\end{aligned}
$$

Resolvent: properties

Let H be a real Hilbert space. Let $A : \mathsf{H} \to 2^{\mathsf{H}}$.
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\begin{aligned}
&\big(\forall (x,u)\in \operatorname{gra} A\big)\big(\forall (y,v)\in \operatorname{gra} A\big) \quad \langle x-y \mid u-v\rangle \geqslant 0\\
\Leftrightarrow &\big(\forall (x,u)\in \operatorname{gra} A\big)\big(\forall (y,v)\in \operatorname{gra} A\big) \quad \langle x-y \mid x-y+u-v\rangle \geqslant \|x-y\|^2\\
\Leftrightarrow &\big(\forall (x,u')\in \operatorname{gra}(\mathrm{Id}+A)\big)\big(\forall (y,v')\in \operatorname{gra}(\mathrm{Id}+A)\big)\\
&\qquad\qquad \big\langle x-y \mid u'-v'\big\rangle \geqslant \|x-y\|^2\\
\Leftrightarrow &\big(\forall (u',x)\in \operatorname{gra} J_A\big)\big(\forall (v',y)\in \operatorname{gra} J_A\big) \quad \big\langle u'-v' \mid x-y\big\rangle \geqslant \|x-y\|^2\\
\Leftrightarrow &J_A \text{ is firmly nonexpansive}
\end{aligned}
$$

Remark : $J_A : \operatorname{ran}(\mathrm{Id}+A) \to \mathsf{H}$.

Resolvent: properties

Let H be a real Hilbert space. Let $A : \mathsf{H} \to 2^{\mathsf{H}}$.
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Let H be a real Hilbert space. Let $A : \mathsf{H} \to 2^{\mathsf{H}}$.
A is maximally monotone $\Leftrightarrow$ J_A : H $\to$ H is firmly nonexpansive.

Resolvent: properties

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Let H be a real Hilbert space. Let $A : \mathsf{H} \to 2^{\mathsf{H}}$.
A is maximally monotone $\Leftrightarrow$ $J_A :$ H $\to$ H is firmly nonexpansive.

Proof: A monotone $\Leftrightarrow$ $J_A : \operatorname{ran}(\mathrm{Id} + A) \to$ H firmly nonexpansive
+ Minty theorem.

Resolvent: properties

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Let H be a real Hilbert space. Let $A : \mathsf{H} \to 2^{\mathsf{H}}$.
A is maximally monotone $\Leftrightarrow$ $J_A : \mathsf{H} \to \mathsf{H}$ is firmly nonexpansive.

Let H be a Hilbert space. Let $A : \mathsf{H} \to 2^{\mathsf{H}}$ maximally monotone and $\gamma \in]0, +\infty[$. For every $x \in \mathsf{H}$, there exists a unique $p \in \mathsf{H}$ such that $x - p \in \gamma A p$ and thus $p = J_{\gamma A} x$.

<u>Proof</u>: $x \in (\mathrm{Id} + \gamma A)(p) \Leftrightarrow p \in (\mathrm{Id} + \gamma A)^{-1} x \Leftrightarrow p = J_{\gamma A} x$

Resolvent: properties

Let H be a real Hilbert space.
Let $A : \mathsf{H} \to 2^{\mathsf{H}}$ be a maximally monotone and $\gamma \in]0, +\infty[$.

- $J_{\gamma A}$ and $\mathrm{Id} - J_{\gamma A}$ are firmly nonexpansive.
- The reflected resolvent $R_{\gamma A} = 2J_{\gamma A} - \mathrm{Id}$ is nonexpansive.
- ${}^{\gamma}A$ is γ-cocoercive.

Let H be a Hilbert space.
Let $A : \mathsf{H} \to 2^{\mathsf{H}}$ and $\gamma \in]0, +\infty[$.
The Yosida approximation of A of index γ is

$$ {}^{\gamma}A = \frac{1}{\gamma}(\mathrm{Id} - J_{\gamma A}). $$

Resolvent: properties

Let H be a Hilbert space.
Let $A\colon \mathsf{H} \to 2^{\mathsf{H}}$ be a maximally monotone operator.

- Let $z \in \mathsf{H}$ and $B = A(\cdot - z)$. Then $J_B = z + J_A(\cdot - z)$.
- Let $z \in \mathsf{H}$ and $B = z + A$. Then $J_B = J_A(\cdot - z)$.
- Let $\alpha \in [0, +\infty[$ and $B = A + \alpha \mathrm{Id}$. Then $J_B = J_{\frac{A}{1+\alpha}}\left(\frac{\cdot}{1+\alpha}\right)$

Resolvent: properties

For every $i \in \{1, \ldots, n\}$, let H_i be a Hilbert space and $A_i \colon \mathsf{H}_i \to 2^{\mathsf{H}_i}$ be a maximally monotone operator.

$$J_{A_1 \times \cdots \times A_n} = J_{A_1} \times \cdots \times J_{A_n} \colon \mathsf{H}_1 \times \cdots \times \mathsf{H}_n \to \mathsf{H}_1 \times \cdots \times \mathsf{H}_n$$
$$(x_1, \ldots, x_n) \mapsto (J_{A_1} x_1, \ldots, J_{A_n} x_n).$$

Resolvent: properties

Let H be a Hilbert space.
Let $A : \mathsf{H} \to 2^{\mathsf{H}}$ be a maximally monotone operator and $\gamma \in]0, +\infty[$.

$$J_{\gamma A^{-1}} = \mathrm{Id} - \gamma J_{\gamma^{-1} A}(\gamma^{-1} \cdot)$$

Preuve: Pour tout $x \in \mathsf{H}$,

$$\begin{aligned}
p = J_{\gamma A^{-1}} x \quad &\Leftrightarrow \quad x \in (\mathrm{Id} + \gamma A^{-1})(p) \\
&\Leftrightarrow \quad \gamma^{-1}(x - p) \in A^{-1} p \\
&\Leftrightarrow \quad p \in A\big(\gamma^{-1}(x - p)\big) \\
&\Leftrightarrow \quad \gamma^{-1} p \in \gamma^{-1} A\big(\gamma^{-1}(x - p)\big) \\
&\Leftrightarrow \quad \gamma^{-1} x \in (\mathrm{Id} + \gamma^{-1} A)\big(\gamma^{-1}(x - p)\big) \\
&\Leftrightarrow \quad \gamma^{-1}(x - p) = J_{\gamma^{-1} A}\big(\gamma^{-1} x\big) \\
&\Leftrightarrow \quad p = x - \gamma J_{\gamma^{-1} A}\big(\gamma^{-1} x\big).
\end{aligned}$$

Resolvent: properties

Let H be a Hilbert space.
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$$J_{\gamma A^{-1}} = \mathrm{Id} - \gamma J_{\gamma^{-1} A}(\gamma^{-1} \cdot)$$

Remarque: $J_A + J_{A^{-1}} = \mathrm{Id}$.

Resolvent: properties

Let H and G be two Hilbert spaces.
Let $A : \mathsf{H} \to 2^{\mathsf{H}}$ be a maximally monotone operator and $L \in \mathcal{B}(\mathsf{G}, \mathsf{H})$ such that $LL^* = \mu \mathrm{Id}$ where $\mu \in]0, +\infty[$. Then

$$J_{L^*AL} = \mathrm{Id} - L^* \circ {}^{\mu}A \circ L \, .$$

Resolvent: properties

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Let H be a Hilbert space.
Let $A : \mathsf{H} \to 2^{\mathsf{H}}$ be a maximally monotone operator and $L \in \mathcal{B}(\mathsf{H}, \mathsf{H})$ be a unitary operator. Then

$$J_{L^*AL} = L^* J_A L.$$

Resolvent: properties

Let H be a Hilbert space.
Let $A : \mathsf{H} \to 2^{\mathsf{H}}$ be a maximally monotone operator and $B = \rho A(\rho\cdot)$ where $\rho \in \mathbb{R}^*$. Then

$$J_B = \rho^{-1} J_{\rho^2 A}(\rho\cdot).$$

<u>Preuve</u>: Set $L = \rho\mathrm{Id}$ and apply formula

$$J_{L^*AL} = \mathrm{Id} - L^* \circ {}^{\mu}A \circ L.$$

Resolvent: properties

Let H be a Hilbert space.
Let $A : \mathsf{H} \to 2^{\mathsf{H}}$ be a maximally monotone operator and $B = \rho A(\rho\cdot)$ where $\rho \in \mathbb{R}^*$. Then

$$J_B = \rho^{-1} J_{\rho^2 A}(\rho\cdot).$$

Let H be a Hilbert space.
Let $A : \mathsf{H} \to 2^{\mathsf{H}}$ be a maximally monotone operator and $B = -A(-\cdot)$. Then

$$J_B = -J_A(-\cdot).$$

Resolvent

Resolvent

Proximity operator

Convex analysis

Let H be a Hilbert space. Let $f\colon \mathsf{H} \to]-\infty, +\infty]$.
f is coercive if $\lim_{\|x\|\to+\infty} f(x) = +\infty$.

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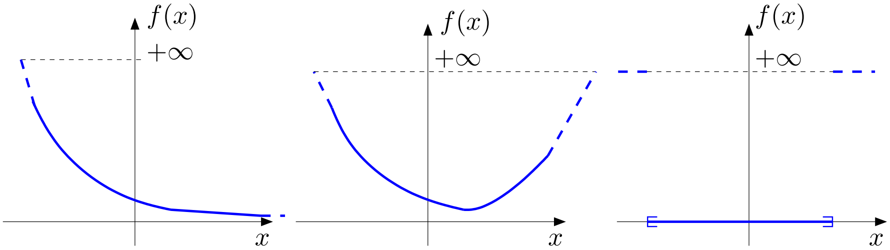
Coercive functions ?



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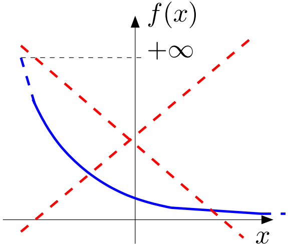

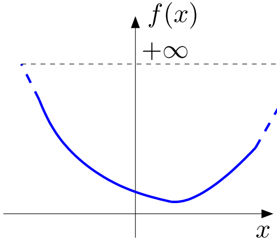

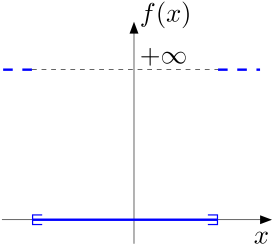

Convex analysis

Let H be a Hilbert space. Let $f\colon \mathsf{H} \to\,]-\infty, +\infty]$.

f is strictly convex if

$$(\forall x \in \operatorname{dom} f)(\forall y \in \operatorname{dom} f)(\forall \alpha \in]0,1[)$$
$$x \neq y \quad \Rightarrow \quad f(\alpha x + (1-\alpha)y) < \alpha f(x) + (1-\alpha) f(y).$$

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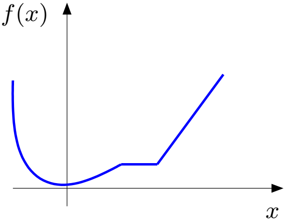
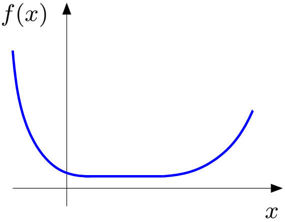
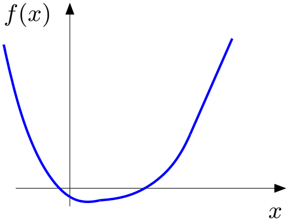
Strictly convex functions ?







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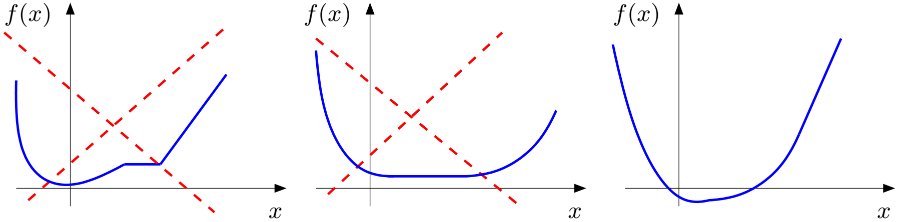

Convex analysis

Let H be a Hilbert space and C be a closed convex set of H.
Let $f \in \Gamma_0(\mathsf{H})$ such that $\operatorname{dom} f \cap C \neq \varnothing$.
If f is coercive or C is bounded, then there exists $p \in C$ such that

$$f(p) = \inf_{x \in C} f(x).$$

Moreover, if f is strictly convex, this minimizer p is unique.

Proximity operator: definition

Let H be a Hilbert space. Let $f \in \Gamma_0(\mathsf{H})$ and $\gamma \in]0,+\infty[$.
For every $x \in \mathsf{H}$, there exists a unique $p \in \mathsf{H}$ such that

$$f(p) + \frac{1}{2\gamma}\|p - x\|^2 = \inf_{y \in \mathsf{H}} f(y) + \frac{1}{2\gamma}\|y - x\|^2 .$$

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Let H be a Hilbert space. Let $f \in \Gamma_0(\mathsf{H})$.

- The Moreau envelope of f of parameter $\gamma \in]0, +\infty[$ is

$$^{\gamma}f \colon \mathsf{H} \to \mathbb{R} \colon x \mapsto \inf_{y \in \mathsf{H}} f(y) + \frac{1}{2\gamma}\|y - x\|^2.$$

- The proximity operator of f is

$$\operatorname{prox}_f \colon \mathsf{H} \to \mathsf{H} \colon x \mapsto \underset{y \in \mathsf{H}}{\operatorname{argmin}}\; f(y) + \frac{1}{2}\|y - x\|^2.$$

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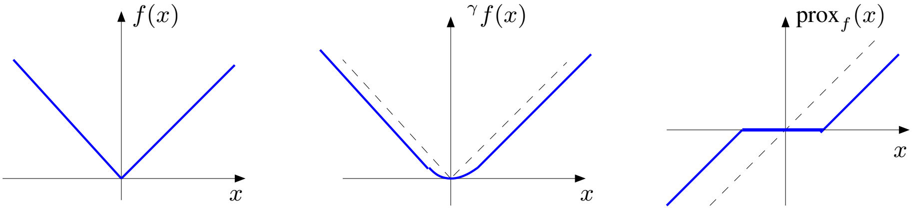

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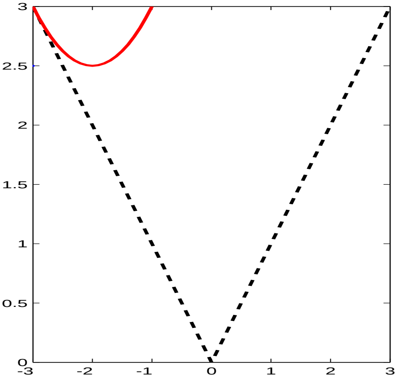

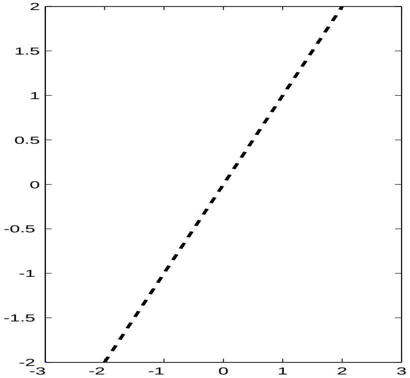

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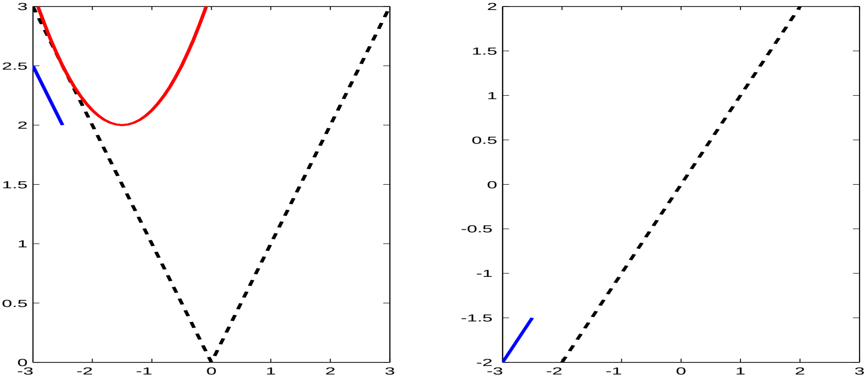

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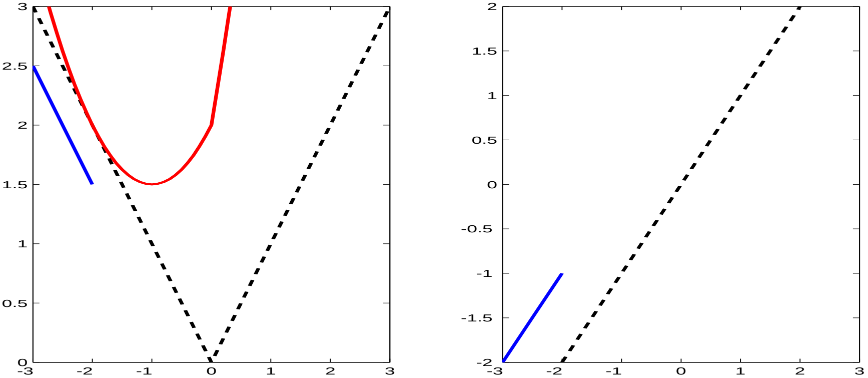

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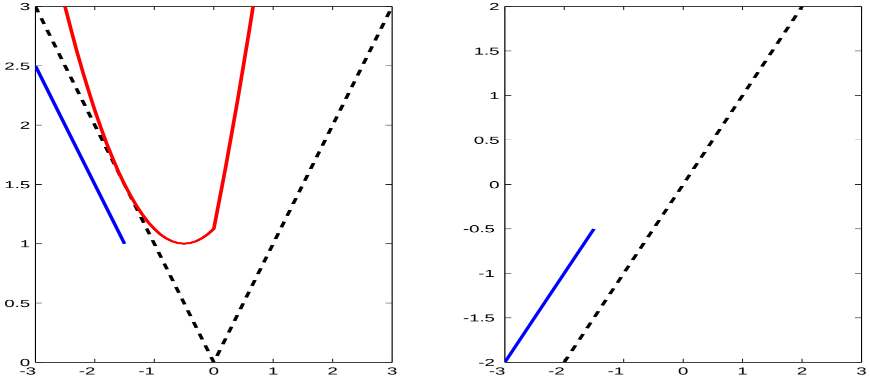

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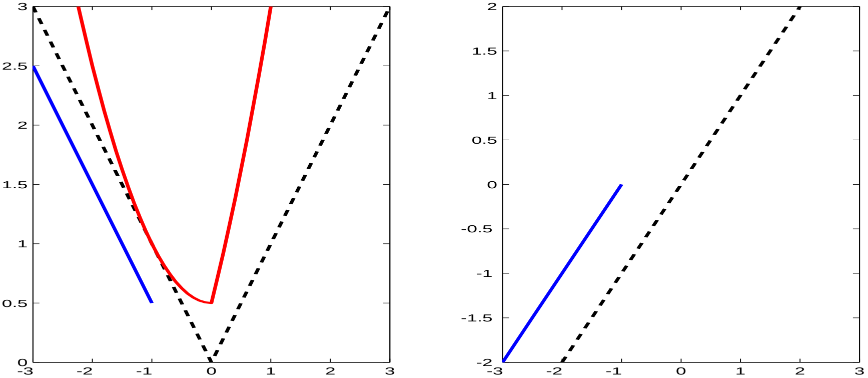

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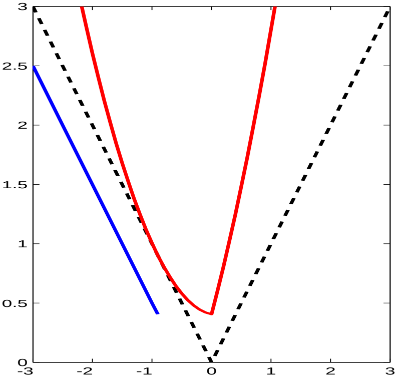

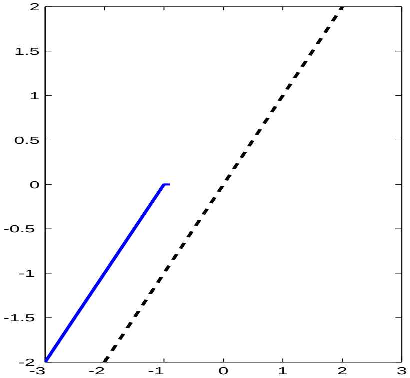

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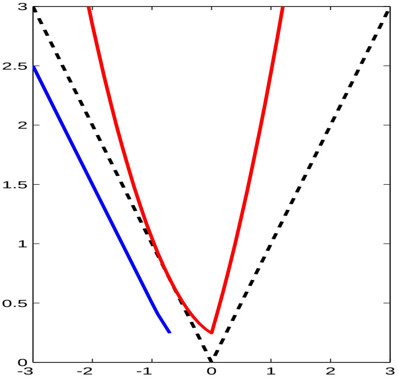

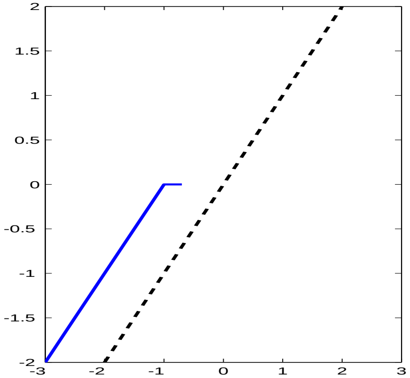

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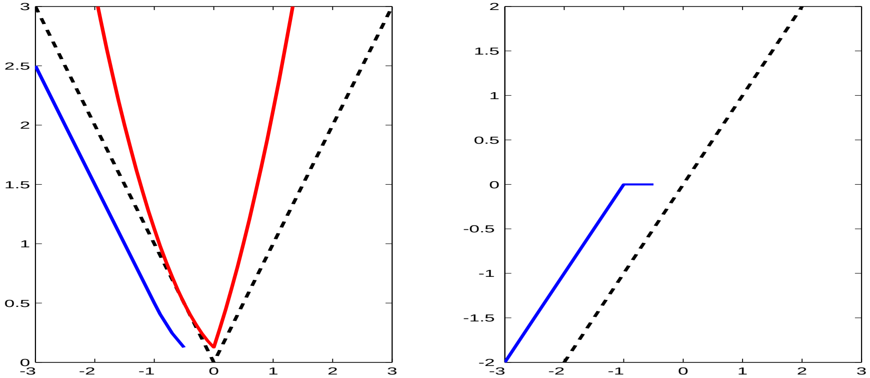

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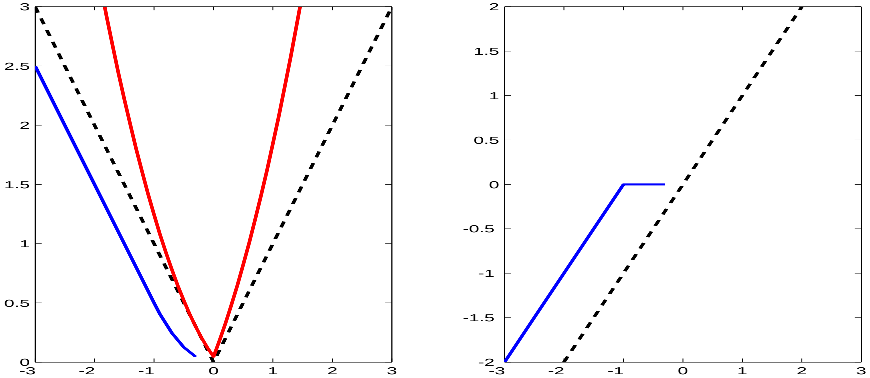

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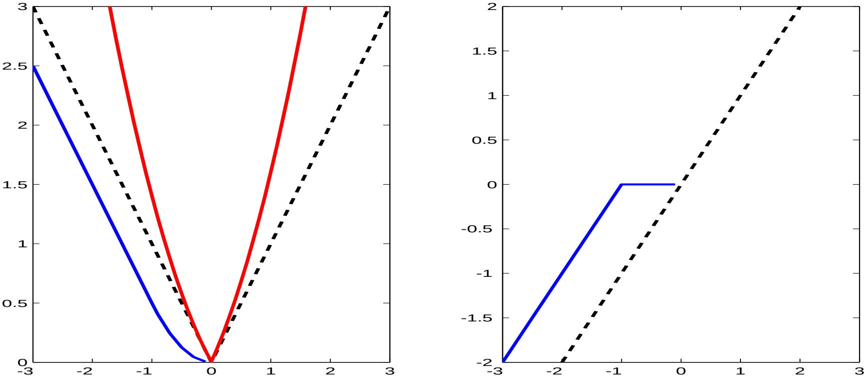

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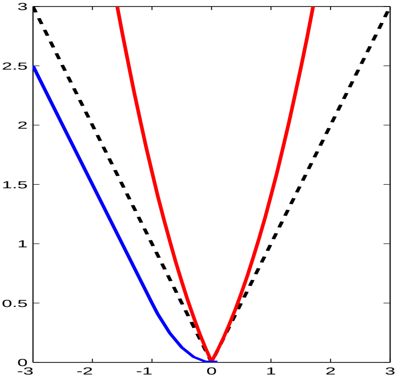

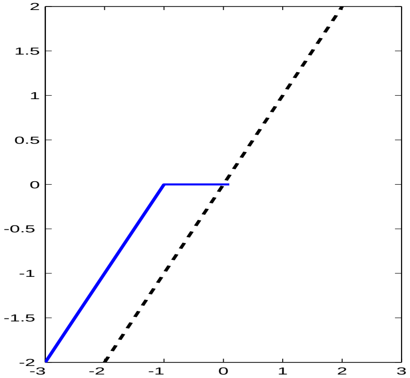

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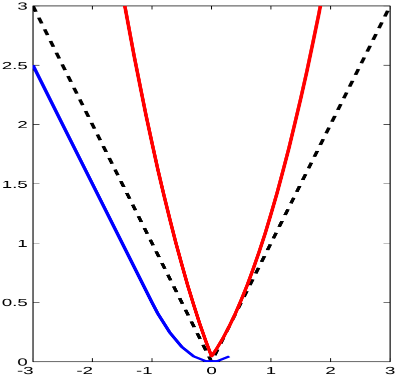

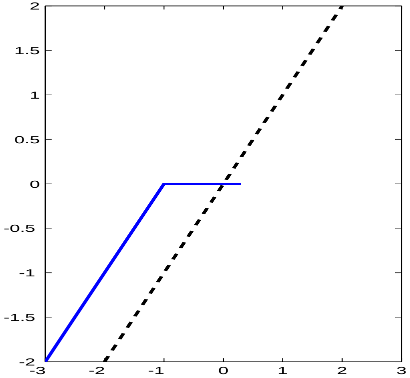

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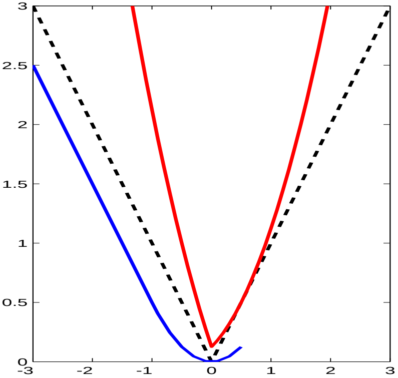

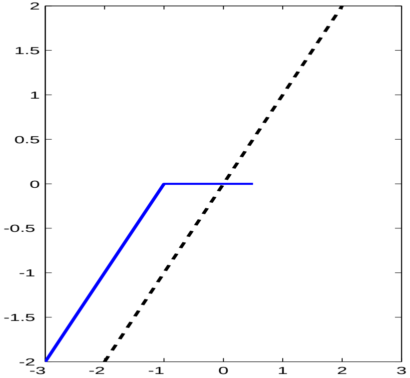

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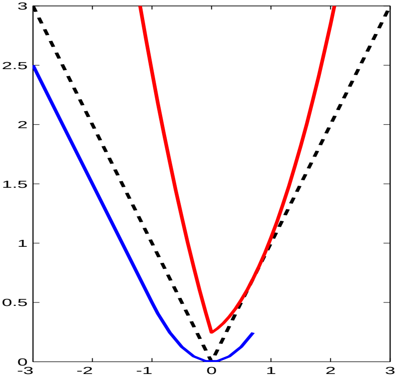

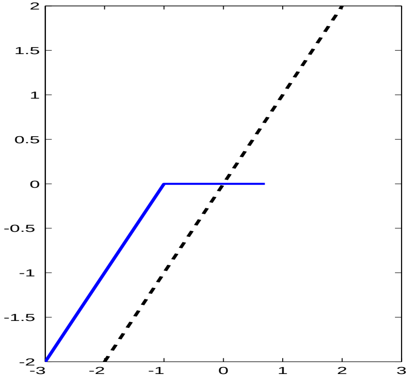

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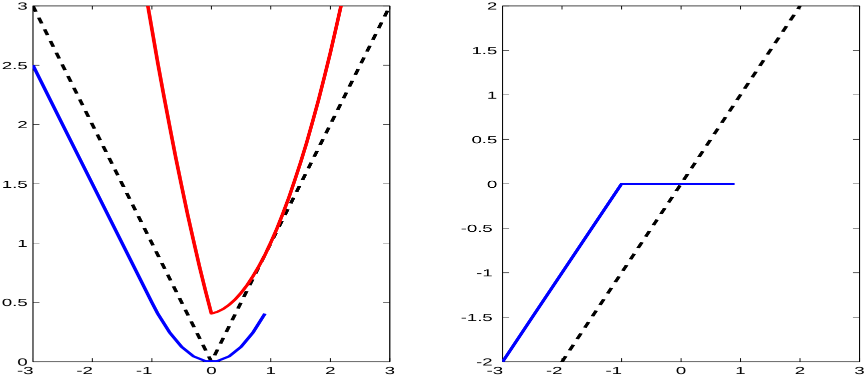

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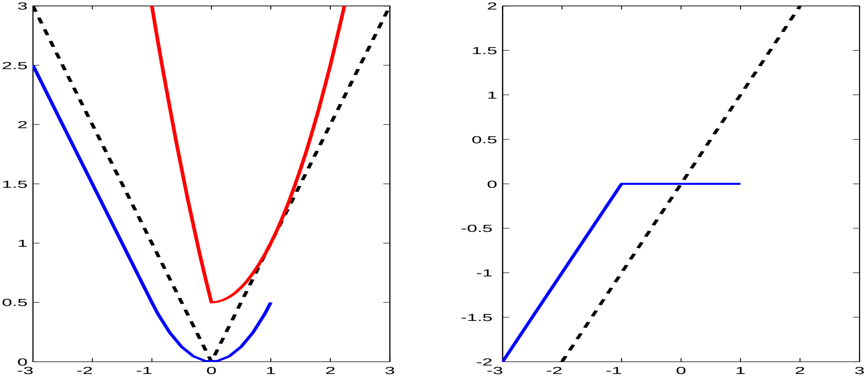

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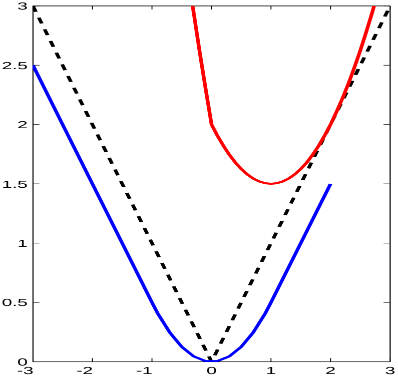

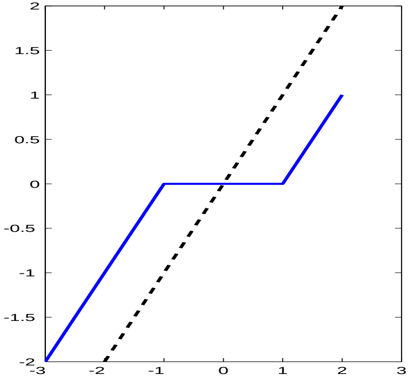

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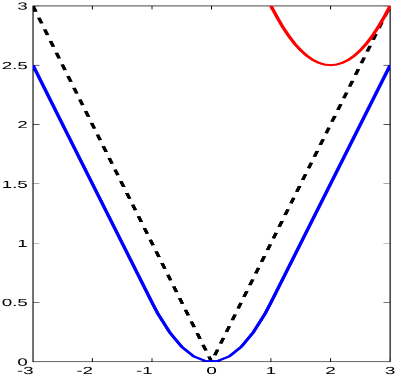

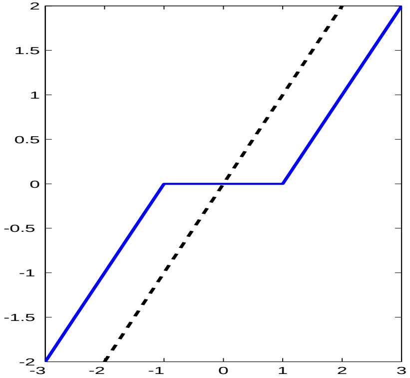

Proximity operator: definition

Let H be a Hilbert space. Let $f \in \Gamma_0(\mathsf{H})$ and $g \in \Gamma_0(\mathsf{H})$.
If $\operatorname{dom} f \cap \operatorname{int}(\operatorname{dom} g) \neq \varnothing$ then $\partial(f+g) = \partial f + \partial g$.

Let H be a Hilbert space and $f \in \Gamma_0(\mathsf{H})$.

$$\operatorname{prox}_f = J_{\partial f} .$$

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Let H be a Hilbert space and $f \in \Gamma_0(\mathsf{H})$.

$$\boxed{\operatorname{prox}_f = J_{\partial f}}\,.$$

Proof: By using Fermat's rule, for every $x \in \mathsf{H}$,

$$
\begin{aligned}
p = \arg\min\ f + \frac{1}{2}\|\cdot - x\|^2 \Leftrightarrow \quad & 0 \in \partial\Big(f + \frac{1}{2}\|\cdot - x\|^2\Big)(p) \\
\Leftrightarrow \quad & 0 \in \partial f(p) + p - x \\
\Leftrightarrow \quad & x \in (\mathrm{Id} + \partial f)(p) \\
\Leftrightarrow \quad & p = (\mathrm{Id} + \partial f)^{-1}(x).
\end{aligned}
$$

Proximity operator: definition

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If $\operatorname{dom} f \cap \operatorname{int}(\operatorname{dom} g) \neq \varnothing$ then $\partial(f+g) = \partial f + \partial g$.

Let H be a Hilbert space and $f \in \Gamma_0(\mathsf{H})$.

$$\boxed{\operatorname{prox}_f = J_{\partial f}}\,.$$

<u>Remark</u>: As $\operatorname{dom}(\operatorname{prox}_f) = \mathsf{H}$, this provides a proof that ∂f is maximally monotone !

Proximity operator: properties

Let H be a Hilbert space, $f \in \Gamma_0(\mathsf{H})$ and $(x, p) \in \mathsf{H}^2$.

$$p = \operatorname{prox}_{\gamma f} x \quad \Leftrightarrow \quad (\forall y \in \mathsf{H}) \ \ \langle y - p \mid x - p \rangle + f(p) \leqslant f(y).$$

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Let H be a Hilbert space, $f \in \Gamma_0(\mathsf{H})$ and $\gamma \in \,]0,+\infty[$.
${}^{\gamma}f$ is differentiable and $\nabla\, {}^{\gamma}f$ is γ^{-1}-Lipschitzian

$$(\forall x \in \mathsf{H}) \quad \nabla \underbrace{{}^{\gamma}f}_{\substack{\text{Moreau}\\ \text{envelope}}} = \gamma^{-1}(\mathrm{Id} - \operatorname{prox}_{\gamma f}) = \underbrace{{}^{\gamma}\partial f}_{\substack{\text{Yosida}\\ \text{approximation}}}.$$

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<u>Proof</u>: Previous property + ... calculations.

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Interpretation: ${}^{\gamma}f$ is a smooth approximation of f.

Proximity operator: properties

Let H be a Hilbert space, $x \in \mathsf{H}$ and $f \in \Gamma_0(\mathsf{H})$.

Properties	$g(x)$	$\operatorname{prox}_g x$
Translation	$f(x-z),\, z \in \mathsf{H}$	$z + \operatorname{prox}_f(x-z)$
Quadratic perturbation	$f(x) + \alpha \parallel x \parallel^2 /2 + \langle z \mid x \rangle + \gamma$ $z \in \mathsf{H}, \alpha > 0, \gamma \in \mathbb{R}$	$\operatorname{prox}_{\frac{f}{\alpha+1}}(\frac{x-z}{\alpha+1})$
Scale change	$f(\rho x), \rho \in \mathbb{R}^*$	$\frac{1}{\rho}\operatorname{prox}_{\rho^2 f}(\rho x)$
Reflection	$f(-x)$	$-\operatorname{prox}_f(-x)$
Moreau envelope	${}^{\gamma}f(x) = \inf_{y \in \mathsf{H}} f(y) + \frac{1}{2\gamma}\|x-y\|^2$ $\gamma > 0$	$\frac{1}{1+\gamma}\Big(\gamma x + \operatorname{prox}_{(1+\gamma)f}(x)\Big)$

Proximity operator: properties

For every $i \in \{1,\ldots,n\}$, let H_i be a Hilbert space and $f_i \in \Gamma_0(\mathsf{H}_i)$.
For all $(x_1,\ldots,x_n) \in \mathsf{H}_1 \times \cdots \times \mathsf{H}_n$,
if

$$f(x_1,\ldots,x_n) = \sum_{i=1}^{n} f_i(x_i).$$

then

$$\boxed{\operatorname{prox}_f(x_1,\ldots,x_n) = \big(\operatorname{prox}_{f_i}(x_i)\big)_{1\leqslant i\leqslant n}}.$$

Proximity operator: properties

Let H be a separable Hilbert space.
Let $(b_i)_{i\in I}$ be an orthonormal basis of H.
For every $i \in I$, let $\varphi_i \in \Gamma_0(\mathbb{R})$ such that $\varphi_i \geqslant 0$. For every $x \in$ H, if

$$f(x) = \sum_{i\in I} \varphi_i(\langle x \mid b_i\rangle)$$

then

$$\boxed{\operatorname{prox}_f(x) = \textstyle\sum_{i\in I} \operatorname{prox}_{\varphi_i}(\langle x \mid b_i\rangle) b_i.}$$

Remark: The assumption $(\forall i \in I)\ \varphi_i \geqslant 0$ can be relaxed if H is finite dimensional.

Proximity operator: properties

Let H be a separable Hilbert space.
Let $(b_i)_{i\in I}$ be an orthonormal basis of H.
For every $i \in I$, let $\varphi_i \in \Gamma_0(\mathbb{R})$ such that $\varphi_i \geqslant 0$. For every $x \in$ H, if

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then

$$\operatorname{prox}_f(x) = \textstyle\sum_{i\in I} \operatorname{prox}_{\varphi_i}(\langle x \mid b_i\rangle) b_i.$$

Example: H $= \mathbb{R}^N$, $(b_i)_{1\leqslant i\leqslant N}$ canonical basis of $\mathbb{R}^N$, $f = \lambda\|\cdot\|_1$ with $\lambda \in [0, +\infty[$.

$$(\forall x = (x^{(i)})_{1\leqslant i\leqslant N}) \in \mathbb{R}^N) \qquad \operatorname{prox}_{\lambda\|\cdot\|_1}(x) = \big(\operatorname{prox}_{\lambda|\cdot|}(x^{(i)})\big)_{1\leqslant i\leqslant N}$$

Proximity operator: properties

Moreau decomposition formula

Let H be a Hilbert space, $f \in \Gamma_0(\mathsf{H})$ and $\gamma \in \,]0,+\infty[$.

$$(\forall x \in \mathsf{H}) \qquad \boxed{J_{\gamma(\partial f)^{-1}} x = x - \gamma J_{\gamma^{-1}\partial f}(\gamma^{-1} x)}\,.$$

Proximity operator: properties

Moreau decomposition formula

Let H be a Hilbert space, $f \in \Gamma_0(\mathsf{H})$ and $\gamma \in \,]0,+\infty[$.

$$(\forall x \in \mathsf{H}) \qquad \boxed{\operatorname{prox}_{\gamma f^*} x = x - \gamma \operatorname{prox}_{\gamma^{-1} f}(\gamma^{-1} x)}\,.$$

Proximity operator: properties

Moreau decomposition formula

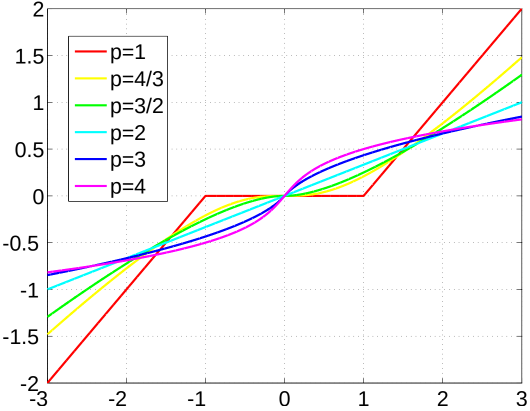
Let H be a Hilbert space, $f \in \Gamma_0(\mathsf{H})$ and $\gamma \in \,]0,+\infty[$.

$$(\forall x \in \mathsf{H}) \qquad \boxed{\mathrm{prox}_{\gamma f^*} x = x - \gamma \mathrm{prox}_{\gamma^{-1} f}(\gamma^{-1} x)}\,.$$

Example: If $\mathsf{H} = \mathbb{R}^N$, $f = \frac{1}{q}\|\cdot\|_q^q$ with $q \in]1,+\infty[$, then $f^* = \frac{1}{q^*}\|\cdot\|_{q^*}^{q^*}$ with $1/q + 1/q^* = 1$, and

$$(\forall x \in \mathbb{R}^N) \qquad \mathrm{prox}_{\frac{\gamma}{q^*}\|\cdot\|_{q^*}^{q^*}} x = x - \gamma \mathrm{prox}_{\frac{1}{\gamma q}\|\cdot\|_q^q}(\gamma^{-1} x).$$

Proximity operator: properties



Proximity operator: properties

Let H and G be two Hilbert spaces. Let $f \in \Gamma_0(\mathsf{H})$ and $L \in \mathcal{B}(\mathsf{G}, \mathsf{H})$ such that $\operatorname{ran} L = \mathsf{H}$. Then

$$\partial(f \circ L) = L^* \, \partial f \, L.$$

Proximity operator: properties

Let H and G be two Hilbert spaces. Let $f \in \Gamma_0(\mathsf{H})$ and $L \in \mathcal{B}(\mathsf{G},\mathsf{H})$ such that $\operatorname{ran} L = \mathsf{H}$. Then

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Let H and G be two Hilbert spaces. Let $f \in \Gamma_0(\mathsf{H})$ and $L \in \mathcal{B}(\mathsf{G},\mathsf{H})$ such that $LL^* = \mu \mathrm{Id}$ where $\mu \in \,]0,+\infty[$. Then

$$\operatorname{prox}_{f \circ L} = \mathrm{Id} - \mu^{-1} L^* \circ (\mathrm{Id} - \operatorname{prox}_{\mu f}) \circ L.$$

Proximity operator: properties

Let H and G be two Hilbert spaces. Let $f \in \Gamma_0(\mathsf{H})$ and $L \in \mathcal{B}(\mathsf{G}, \mathsf{H})$ such that $\operatorname{ran} L = \mathsf{H}$. Then

$$\partial(f \circ L) = L^* \, \partial f \, L.$$

Let H and G be two Hilbert spaces. Let $f \in \Gamma_0(\mathsf{H})$ and $L \in \mathcal{B}(\mathsf{G}, \mathsf{H})$ such that $LL^* = \mu \mathrm{Id}$ where $\mu \in \,]0, +\infty[$. Then

$$\operatorname{prox}_{f \circ L} = \mathrm{Id} - \mu^{-1} L^* \circ (\mathrm{Id} - \operatorname{prox}_{\mu f}) \circ L.$$

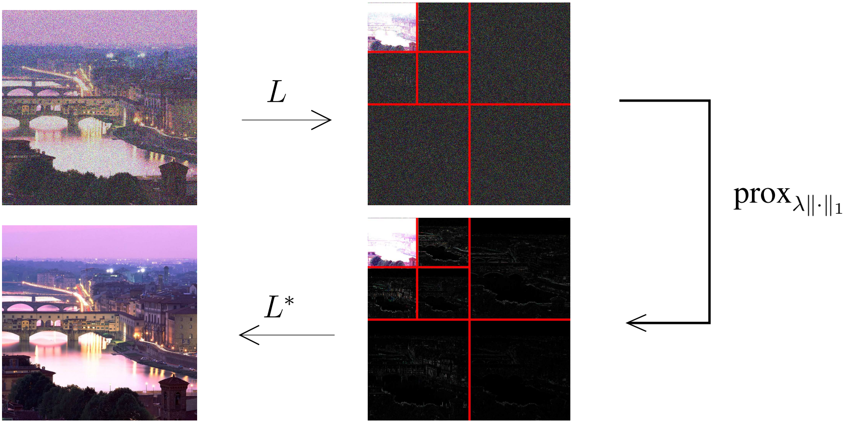
Remark :

Useful property for data fidelity terms involving a neg-log-likelihood f and a synthesis tight frame operator L.

Proximity operator: properties

Particular case : $L \in \mathcal{B}(\mathsf{H}, \mathsf{H})$ unitary, $\operatorname{prox}_{f \circ L} = L^* \operatorname{prox}_f L$.

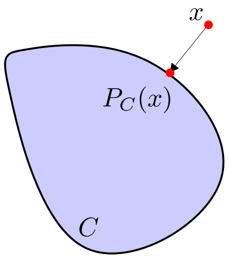
- Illustration: denoising using an ℓ_1 penalty on the coefficients resulting from an orthogonal wavelet transform L.



Proximity operator: examples

Projection :
Let H be a Hilbert space. Let C be a nonempty closed convex subset of H.

$$(\forall x \in \mathsf{H}) \qquad \operatorname{prox}_{\iota_C}(x) = \underset{y \in C}{\operatorname{argmin}} \frac{1}{2}\|y - x\|^2 = P_C(x).$$



Proximity operator: examples

Projection :
Let H be a Hilbert space. Let C be a nonempty closed convex subset of H.

$$(\forall x \in \mathsf{H}) \qquad \operatorname{prox}_{\iota_C}(x) = \underset{y \in C}{\operatorname{argmin}} \frac{1}{2}\|y-x\|^2 = P_C(x).$$

Remark :

- $p = P_C(x) \Leftrightarrow x - p \in \partial \iota_C(p) = N_C(p)$
  $$\Leftrightarrow (\forall y \in C)\ \langle y-p \mid x-p\rangle \leqslant 0.$$
  Particular case: if C is a vector space: $p = P_C(x) \Leftrightarrow \begin{cases} x-p \in C^\perp \\ p \in C. \end{cases}$
- ${}^{\gamma}\iota_C = (2\gamma)^{-1} d_C^2$ where d_C distance to the convex set C is defined by $d_C \colon x \mapsto \inf_{y \in C}\|y-x\| = \|x - P_C x\|$. We have then $\nabla d_C^2 = \nabla(\,{}^{\frac{1}{2}}\iota_C) = 2(\mathrm{Id} - P_C)$.

Proximity operator: examples

Quadratic function :
Let H and G be two Hilbert spaces.
Let $L \in \mathcal{B}(\mathsf{G}, \mathsf{H})$, $\gamma \in]0, +\infty[$ and $z \in \mathsf{G}$.

$$f = \gamma \left\| L \cdot - z \right\|^2 / 2 \quad \Rightarrow \quad \operatorname{prox}_f = (\mathrm{Id} + \gamma L^* L)^{-1}(\cdot + \gamma L^* z).$$

Proximity operator: examples

Support function :
Let H be a Hilbert space and $C \subset$ H be nonempty closed convex.

$$(\forall x \in \mathsf{H}) \qquad \operatorname{prox}_{\sigma_C} = \operatorname{Id} - P_C.$$

Soft-thresholding : $\mathsf{H} = \mathbb{R}$, $\delta_1 = \inf C$ and $\delta_2 = \sup C$. For every $x \in \mathbb{R}$,

$$\sigma_C(x) = \begin{cases} \delta_1 x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \delta_2 x & \text{if } x > 0 \end{cases} \;\Rightarrow\; \operatorname{prox}_{\sigma_C}(x) = \operatorname{soft}_C(x) = \begin{cases} x - \delta_1 & \text{if } x < \delta_1 \\ 0 & \text{if } x \in C \\ x - \delta_2 & \text{if } x > \delta_2. \end{cases}$$

Proximity operator: Bayesian interpretation

- If $\mathsf{H} = \mathbb{R}^N$ and
$$x = \overline{y} + w$$
where $\overline{y}$ is a realization of a random vector with probability density function $\exp(-f)$ and w is a realization of a $\mathcal{N}(0, \mathbb{I})$ noise, then $\operatorname{prox}_f(x)$ is a Maximum A Posteriori estimate of $\overline{y}$.
- Explicit form for objective functions associated with usual log-concave probability densities [Chaux *et al.* - 2007].

➤ Laplace	➤ Gaussian
➤ Generalized Gaussian	➤ Huber
➤ maximum entropy	➤ Smoothed Laplace
➤ gamma	➤ chi
➤ uniform	➤ triangular
➤ Weibull	➤ Pearson type I
➤ Generalized inverse Gaussian	...

- And many other functions !

Part 3: Fixed point algorithms

1. Convergence
   - Definition
   - Fejér monotonicity
   - Demiclosedness principle

2. Algorithms
   - Krasnosel'skii Mann
   - Douglas-Rachford
   - PPXA
   - Forward-Backward

Fixed point algorithms

Fixed point algorithms: convergence

Let H be a Hilbert space.
Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in H and $\widehat{x} \in$ H.

- $(x_n)_{n\in\mathbb{N}}$ **converges strongly** to $\widehat{x}$ if

$$\lim_{n\to+\infty} \|x_n - \widehat{x}\| = 0.$$

  It is denoted by $x_n \to \widehat{x}$.

- $(x_n)_{n\in\mathbb{N}}$ **converges weakly** to $\widehat{x}$ if

$$(\forall y \in \mathsf{H}) \qquad \lim_{n\to+\infty} \langle y \mid x_n - \widehat{x}\rangle = 0.$$

  It is denoted by $x_n \rightharpoonup \widehat{x}$.

Remark: In a finite dimensional Hilbert space, strong and weak convergences are equivalent.

Fixed point algorithms: convergence

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of H.
$(x_n)_{n\in\mathbb{N}}$ converges weakly if and only if

- $(x_n)_{n\in\mathbb{N}}$ is bounded

  and
- $(x_n)_{n\in\mathbb{N}}$ possesses at most one sequential cluster point in the weak topology.

- $\widehat{x}$ is a sequential cluster point of $(x_n)_{n\in\mathbb{N}}$ in the weak topology if there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ that converges weakly to $\widehat{x}$.

Fixed point algorithms: convergence

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of H.

$(x_n)_{n\in\mathbb{N}}$ converges weakly if and only if

- $(x_n)_{n\in\mathbb{N}}$ is bounded

  and

- $(x_n)_{n\in\mathbb{N}}$ possesses at most one sequential cluster point in the weak topology.

Illustration:

x_0	x_1	x_2	x_3	x_4	x_5	...
1	-1	1	-1	1	-1	...

$\rightarrow$ $(x_n)_{n\in\mathbb{N}}$ is bounded but it has 2 sequential cluster points: -1 and 1.

$\rightarrow$ $(x_n)_{n\in\mathbb{N}}$ does not converge.

Fixed point algorithms: convergence

Lemma 1

Let D be a nonempty subset of H.
Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in H.
$(x_n)_{n\in\mathbb{N}}$ weakly converges to a point in D if

- for every $x \in D$, $(\|x_n - x\|)_{n\in\mathbb{N}}$ converges

  and
- every weak sequential cluster point of $(x_n)_{n\in\mathbb{N}}$ lies in D.

Fixed point algorithms: Fejér-monotone sequence

Let D be a nonempty subset of a Hilbert space H.
Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in H.
$(x_n)_{n\in\mathbb{N}}$ is Fejér-monotone with respect to D if

$$(\forall x \in D)(\forall n \in \mathbb{N}) \qquad \|x_{n+1} - x\| \leqslant \|x_n - x\|.$$

Let $D \subset$ H.
Let $(x_n)_{n\in\mathbb{N}}$ be Fejér-monotone with respect to D then

- for every $x \in D$, $(\|x_n - x\|)_{n\in\mathbb{N}}$ converges,
- $(x_n)_{n\in\mathbb{N}}$ is bounded.

Fixed point algorithms: Fejér-monotone sequence

Fejér-monotone convergence

Let D be a nonempty subset of a Hilbert space H.
Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in H.
$(x_n)_{n\in\mathbb{N}}$ converges weakly to a point in D if

- $(x_n)_{n\in\mathbb{N}}$ is Fejér-monotone with respect to D

  and
- every weak sequential cluster point of $(x_n)_{n\in\mathbb{N}}$ lies in D.

Fixed point algorithms: Fejér-monotone sequence

Lemma 2

Let C be a nonempty closed convex subset of H.
If $(x_n)_{n\in\mathbb{N}}$ denotes a sequence in C that weakly converges to $\widehat{x}$ then $\widehat{x} \in C$.

Fixed point algorithms: Fejér-monotone sequence

Lemma 2

Let C be a nonempty closed convex subset of H.
If $(x_n)_{n\in\mathbb{N}}$ denotes a sequence in C that weakly converges to $\widehat{x}$ then $\widehat{x} \in C$.

Let C be a nonempty set of a Hilbert space H. Let $T\colon C \to$ H.
The set of fixed points of T is

$$\operatorname{Fix} T = \{x \in C \mid x = Tx\}.$$

Let $S\colon C \to 2^{\mathsf{H}}$. The set of zeros of S is

$$\operatorname{zer} S = \{x \in C \mid 0 \in Sx\}.$$

Nonexpansive operator: fixed point algorithm

Demiclosedness principle

Let C be a nonempty closed convex subset of a Hilbert space H.
Let $T\colon C \to$ H be a nonexpansive operator.
If $(x_n)_{n\in\mathbb{N}}$ is a sequence in C that converges weakly to $\widehat{x}$ and if $Tx_n - x_n \to 0$, then $\widehat{x} \in \operatorname{Fix} T$.

Nonexpansive operator: fixed point algorithm

Let C be a nonempty closed convex subset of a Hilbert space H.
Let $T: C \to C$ be a nonexpansive operator

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = Tx_n.$$

Nonexpansive operator: fixed point algorithm

Let C be a nonempty closed convex subset of a Hilbert space H.
Let $T : C \to C$ be a nonexpansive operator

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = Tx_n.$$

If $x_n - Tx_n \to 0$,

Nonexpansive operator: fixed point algorithm

Let C be a nonempty closed convex subset of a Hilbert space H.
Let $T\colon C \to C$ be a nonexpansive operator such that $\operatorname{Fix} T \neq \varnothing$.
Let $x_0 \in C$,

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = Tx_n.$$

If $x_n - Tx_n \to 0$, then $(x_n)_{n\in\mathbb{N}}$ converges weakly to a point in $\operatorname{Fix} T$.

Proof : For every $n \in \mathbb{N}$ and $y \in \operatorname{Fix} T$,
$\|x_{n+1} - y\| \leqslant \|Tx_n - Ty\| \leqslant \|x_n - y\|$.
$(x_n)_{n\in\mathbb{N}}$ is Fejér-monotone with respect to $\operatorname{Fix} T$.

Let $(x_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(x_n)_{n\in\mathbb{N}}$ such that $x_{n_k} \rightharpoonup \widehat{x}$ where $\widehat{x} \in$ H.
By assumption $x_{n_k} - Tx_{n_k} \to 0$ and thus, according to the demiclosedness principle, $\widehat{x} \in \operatorname{Fix} T$.

This shows the weak convergence of $(x_n)_{n\in\mathbb{N}}$.

Fixed point algorithms: Fejér-monotone sequence

Krasnosel'skii-Mann algorithm

Let C be a nonempty closed convex subset of a Hilbert space H.
Let $T\colon C \to C$ be a nonexpansive operator such that $\operatorname{Fix} T \neq \varnothing$.
Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[0,1]$ such that

$$\sum_{n\in\mathbb{N}} \lambda_n(1-\lambda_n) = +\infty.$$

Let $x_0 \in C$ and $(\forall n \in \mathbb{N})\ x_{n+1} = x_n + \lambda_n(Tx_n - x_n)$. Then,

- $(x_n)_{n\in\mathbb{N}}$ is Fejér-monotone with respect to $\operatorname{Fix} T$.
- $(Tx_n - x_n)_{n\in\mathbb{N}}$ converges strongly to 0.
- $(x_n)_{n\in\mathbb{N}}$ converges weakly to a point in $\operatorname{Fix} T$.

Typical choice: $(\forall n \in \mathbb{N})\ \lambda_n = \lambda \in]0,1[$.

α-averaged operator: recall

Let $C \subset \mathrm{H}$ be a nonempty set of a Hilbert space H.
Let $A : C \to \mathrm{H}$ and let $\alpha \in]0,1[$.
A is an **α-averaged** operator if there exists a nonexpansive operator $R\colon C \to \mathrm{H}$ such that

$$A = (1-\alpha)\mathrm{Id} + \alpha R.$$

Remark : When $\alpha = 1/2$, A is **firmly nonexpansive**.

Fixed point algorithms: α-averaged operator

Let $T\colon \mathsf{H} \to \mathsf{H}$ be an α-averaged operator with $\alpha \in]0,1[$ such that $\operatorname{Fix} T \neq \varnothing$.
Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[0, 1/\alpha]$ such that

$$\sum_{n\in\mathbb{N}} \lambda_n(1-\alpha\lambda_n) = +\infty.$$

Let $x_0 \in \mathsf{H}$ and $(\forall n \in \mathbb{N})$ $x_{n+1} = x_n + \lambda_n(Tx_n - x_n)$. The following properties are satisfied

- $(x_n)_{n\in\mathbb{N}}$ is Fejér-monotone with respect to $\operatorname{Fix} T$.
- $(Tx_n - x_n)_{n\in\mathbb{N}}$ converges strongly to 0.
- $(x_n)_{n\in\mathbb{N}}$ converges weakly to a point in $\operatorname{Fix} T$.

Fixed point algorithms: α-averaged operator

Proof :
Since T is α-averaged, there exists a non expansive operator R such that $T = (1-\alpha)\mathrm{Id} + \alpha R$.
Let $(\forall n \in \mathbb{N})\ \mu_n = \alpha\lambda_n \in [0,1]$.
The iterations can be written as

$$
\begin{aligned}
(\forall n \in \mathbb{N}) \qquad x_{n+1} &= x_n + \lambda_n(Tx_n - x_n) \\
&= x_n + \mu_n(Rx_n - x_n).
\end{aligned}
$$

Moreover, $\mathrm{Fix}\, R = \mathrm{Fix}\, T$.
+ Krasnosel'skii-Mann algorithm.

Optimization algorithms: *Forward-Backward*

Let $f \in \Gamma_0(\mathsf{H})$.
Let $g \in \Gamma_0(\mathsf{H})$ be differentiable with a ν-Lipschitzian gradient where $\nu \in]0,+\infty[$.
Let $\gamma \in]0, 2/\nu[$ and $\delta = 2 - \gamma\nu/2 \in]1,2[$.
Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[0,\delta]$ such that $\sum_{n\in\mathbb{N}} \lambda_n(\delta - \lambda_n) = +\infty$.
We assume that $\operatorname{Argmin}(f+g) \neq \varnothing$. Let $x_0 \in \mathsf{H}$ and

$$(\forall n \in \mathbb{N}) \qquad \begin{cases} y_n = x_n - \gamma \nabla g(x_n) \\ x_{n+1} = x_n + \lambda_n(\operatorname{prox}_{\gamma f} y_n - x_n). \end{cases}$$

Then, $(x_n)_{n\in\mathbb{N}}$ converges weakly to a minimizer of $f+g$.

Optimization algorithms: *Forward-Backward*

Proof: Let $T = \mathrm{prox}_{\gamma f} \circ (\mathrm{Id} - \gamma \nabla g)$. For every $x \in$ H,

$$x \in \mathrm{Fix}\, T \Leftrightarrow (\mathrm{Id} - \gamma \nabla g)x \in (\mathrm{Id} + \gamma \partial f)x \Leftrightarrow 0 \in \nabla g(x) + \partial f(x).$$

Consequently, $\mathrm{Fix}\, T = \mathrm{zer}\,(\nabla g + \partial f) = \mathrm{zer}\,\big(\partial(g+f)\big) \neq \varnothing$.
Moreover, for every $n \in \mathbb{N}$,

$$x_{n+1} = x_n + \lambda_n (T x_n - x_n).$$

$\mathrm{prox}_{\gamma f}$ is $1/2$-averaged and $\mathrm{Id} - \gamma \nabla g$ is $\gamma\nu/2$-averaged.
It follows that T is α-averaged with

$$\alpha = \frac{\frac{1}{2} + \frac{\gamma\nu}{2} - 2\frac{1}{2}\frac{\gamma\nu}{2}}{1 - \frac{1}{2}\frac{\gamma\nu}{2}} \quad \Leftrightarrow \quad \alpha^{-1} = \delta.$$

Optimization algorithms: *Forward-Backward* with varying stepsize

Let $f \in \Gamma_0(\mathsf{H})$.
Let $g \in \Gamma_0(\mathsf{H})$ be differentiable with a ν-Lipschitzian gradient where $\nu \in]0, +\infty[$.
Let $(\gamma_n)_{n\in\mathbb{N}}$ in $[\underline{\gamma}, \overline{\gamma}]$ where $0 < \underline{\gamma} < \overline{\gamma} < 2/\nu$ and
let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[\underline{\lambda}, 1]$ with $0 < \underline{\lambda} \leqslant 1$.
We assume that $\mathrm{Argmin}(f+g) \neq \varnothing$. Let $x_0 \in \mathsf{H}$ and

$$(\forall n \in \mathbb{N}) \qquad \begin{cases} y_n = x_n - \gamma_n \nabla g(x_n) \\ x_{n+1} = x_n + \lambda_n(\mathrm{prox}_{\gamma_n f} y_n - x_n). \end{cases}$$

Then, $(x_n)_{n\in\mathbb{N}}$ converges weakly to a minimizer of $f+g$.

Optimization algorithms: projected gradient

Let C be a nonempty closed convex subset of H.
Let $g \in \Gamma_0(\mathsf{H})$ be differentiable with a ν-Lipschitzian gradient where $\nu \in]0,+\infty[$.
Let $\gamma \in]0, 2/\nu[$ and $\delta = 2 - \gamma\nu/2 \in]1,2[$.
Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[0,\delta]$ such that $\sum_{n\in\mathbb{N}} \lambda_n(\delta-\lambda_n) = +\infty$.
We assume that $\underset{x\in C}{\operatorname{Argmin}}\; g(x) \neq \varnothing$. Let $x_0 \in \mathsf{H}$ and

$$(\forall n \in \mathbb{N}) \qquad \begin{cases} y_n = x_n - \gamma \nabla g(x_n) \\ x_{n+1} = x_n + \lambda_n (P_C y_n - x_n). \end{cases}$$

Then, $(x_n)_{n\in\mathbb{N}}$ converges weakly to a minimizer of g over C.

Optimization algorithms: gradient descent

Let $g \in \Gamma_0(\mathsf{H})$ be a differentiable function with a ν-Lipschitzian gradient where $\nu \in]0, +\infty[$.
Let $(\gamma_n)_{n\in\mathbb{N}}$ in $[\underline{\gamma}, \overline{\gamma}]$ where $0 < \underline{\gamma} < \overline{\gamma} < 2/\nu$.
We assume that $\operatorname{Argmin} g \neq \varnothing$. Let $x_0 \in \mathsf{H}$ and

$$(\forall n \in \mathbb{N}) \qquad x_{n+1} = x_n - \gamma_n \nabla g(x_n)$$

Then, $(x_n)_{n\in\mathbb{N}}$ converges weakly to a minimizer of g.

Optimization algorithms: proximal point algorithm

Let $f \in \Gamma_0(\mathsf{H})$.
Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $]0,+\infty[$ such that $\sum_{n\in\mathbb{N}} \gamma_n = +\infty$. We assume that $\operatorname{Argmin} f \neq \varnothing$. Let $x_0 \in \mathsf{H}$ and

$$(\forall n \in \mathbb{N}) \qquad x_{n+1} = \operatorname{prox}_{\gamma_n f} x_n.$$

Then, $(x_n)_{n\in\mathbb{N}}$ converges weakly to a minimizer of f.

Optimization algorithms: Douglas-Rachford

Motivation

Let $f \in \Gamma_0(\mathsf{H})$ and $g \in \Gamma_0(\mathsf{H})$. We want to

$$\underset{x \in \mathsf{H}}{\text{minimize}} \;\; f(x) + g(x).$$

Possible solutions :

- gradient descent algorithm $\Rightarrow$ $f + g$ needs to be smooth
- proximal point algorithm $\Rightarrow$ $f + g$ needs to be “proximable”
- Forward-Backward algorithm $\Rightarrow$ g needs to be smooth

Can we find a splitting algorithm when both f and g are nonsmooth?

Optimization algorithms: Douglas-Rachford

Let $\gamma \in]0, +\infty[$ and let $f \in \Gamma_0(\mathrm{H})$.
The reflection of the proximal operator defined as

$$\mathrm{rprox}_{\gamma f} = 2\mathrm{prox}_{\gamma f} - \mathrm{Id}$$

is nonexpansive.

Let $\gamma \in]0, +\infty[$ and let $f \in \Gamma_0(\mathrm{H})$ and $g \in \Gamma_0(\mathrm{H})$.
We have

$$\mathrm{zer}\,(\partial f + \partial g) = \mathrm{prox}_{\gamma g}(\mathrm{Fix}\, T)$$

where $T = \mathrm{rprox}_{\gamma f} \circ \mathrm{rprox}_{\gamma g}$.

Optimization algorithms: Douglas-Rachford

Let H be a Hilbert space.
Let $f \in \Gamma_0(\mathsf{H})$ and $g \in \Gamma_0(\mathsf{H})$.

$$(\forall n \in \mathbb{N}) \qquad \begin{cases} y_n = \operatorname{prox}_{\gamma g}(x_n) \\ z_n = \operatorname{prox}_{\gamma f}(2y_n - x_n) \\ x_{n+1} = x_n + \lambda_n(z_n - y_n). \end{cases}$$

Optimization algorithms: Douglas-Rachford

Let H be a Hilbert space.

Let $f \in \Gamma_0(\mathsf{H})$ and $g \in \Gamma_0(\mathsf{H})$.

Let $\gamma \in \,]0,+\infty[$ and let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[0,2]$ such that $\sum_{n\in\mathbb{N}} \lambda_n(2-\lambda_n) = +\infty$.

We assume that $\operatorname{zer}(\partial f + \partial g) \neq \varnothing$. Let $x_0 \in \mathsf{H}$ and

$$(\forall n \in \mathbb{N}) \qquad \begin{cases} y_n = \operatorname{prox}_{\gamma g}(x_n) \\ z_n = \operatorname{prox}_{\gamma f}(2y_n - x_n) \\ x_{n+1} = x_n + \lambda_n(z_n - y_n). \end{cases}$$

The following properties are satisfied:

- $x_n \rightharpoonup \widehat{x}$
- $z_n - y_n \to 0$, $y_n \rightharpoonup \widehat{y}$, $z_n \rightharpoonup \widehat{y}$ where $\widehat{y} = \operatorname{prox}_{\gamma g}\widehat{x} \in \operatorname{Argmin}(f+g)$.

Optimization algorithms: Douglas-Rachford

Proof: For simplicity, assume that H is finite dimensional.
Let $T = \mathrm{rprox}_{\gamma f} \circ \mathrm{rprox}_{\gamma g}$. T is nonexpansive and $\varnothing \neq \mathrm{zer}\,(\partial f + \partial g) = \mathrm{prox}_{\gamma g}(\mathrm{Fix}\,T) \Rightarrow \mathrm{Fix}\,T \neq \varnothing$. Moreover, for every $n \in \mathbb{N}$,

$$\begin{aligned}
x_{n+1} &= x_n + \lambda_n\big(\mathrm{prox}_{\gamma f}(2\mathrm{prox}_{\gamma g}(x_n) - x_n) - \mathrm{prox}_{\gamma g}(x_n)\big)\\
&= x_n + \frac{\lambda_n}{2}\big(2\mathrm{prox}_{\gamma f}(2\mathrm{prox}_{\gamma g}(x_n) - x_n) - 2\mathrm{prox}_{\gamma g}(x_n) + x_n - x_n\big)\\
&= x_n + \frac{\lambda_n}{2}\big(2\mathrm{prox}_{\gamma f}(\mathrm{rprox}_{\gamma g}(x_n)) - \mathrm{rprox}_{\gamma g}(x_n) - x_n\big)\\
&= x_n + \frac{\lambda_n}{2}\big(Tx_n - x_n\big).
\end{aligned}$$

$\Rightarrow$ Krasnosel'skii-Mann algorithm with relaxation factors $(\lambda_n/2)_{n\in\mathbb{N}}$.
We deduce that $Tx_n - x_n \to 0$ and $x_n \to \widehat{x} \in \mathrm{Fix}\,T$.

Optimization algorithms: Douglas-Rachford

Proof: For every $n \in \mathbb{N}$,

$$z_n - y_n = \mathrm{prox}_{\gamma f}(2\mathrm{prox}_{\gamma g}(x_n) - x_n) - \mathrm{prox}_{\gamma g}(x_n) = \frac{1}{2}(Tx_n - x_n) \to 0.$$

Moreover, $\mathrm{prox}_{\gamma g}$ being continuous (since nonexpansive), we have $y_n \to \mathrm{prox}_{\gamma g}\widehat{x} \in \mathrm{zer}\,(\partial f + \partial g)$.
Because $z_n - y_n \to 0$, we have: $z_n \to \mathrm{prox}_{\gamma g}\widehat{x} \in \mathrm{zer}\,(\partial f + \partial g)$.
In addition,

$$\begin{aligned} & \varnothing \neq \partial f + \partial g \subset \partial(f+g) \\ \Rightarrow \quad & \widehat{y} = \mathrm{prox}_{\gamma g}(\widehat{x}) \in \mathrm{zer}\,\big(\partial(f+g)\big) = \mathrm{Argmin}(f+g). \end{aligned}$$

Optimization algorithms: Douglas-Rachford

Proof: For every $n \in \mathbb{N}$,

$$z_n - y_n = \mathrm{prox}_{\gamma f}(2\mathrm{prox}_{\gamma g}(x_n) - x_n) - \mathrm{prox}_{\gamma g}(x_n) = \frac{1}{2}(Tx_n - x_n) \to 0.$$

Moreover, $\mathrm{prox}_{\gamma g}$ being continuous (since nonexpansive), we have $y_n \to \mathrm{prox}_{\gamma g}\widehat{x} \in \mathrm{zer}\,(\partial f + \partial g)$.
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In addition,

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Remark: generalization of FB and DR [Raguet *et al.* - 2013].

Optimization algorithms: Parallel form of Douglas-Rachford

Let H and G be two Hilbert spaces.
Let $g \in \Gamma_0(\mathsf{H})$ and $L \in \mathcal{B}(\mathsf{G}, \mathsf{H})$ be such that $\operatorname{ran} L$ is closed and L^*L is an isomorphism.
Let $\gamma \in]0, +\infty[$ and let $(\lambda_n)_{n\in\mathbb{N}}$ a sequence in $[0, 2]$ such that $\sum_{n\in\mathbb{N}} \lambda_n(2 - \lambda_n) = +\infty$.
We assume that $\operatorname{zer}(L^* \circ \partial g \circ L) \neq \varnothing$. Let $x_0 \in \mathsf{H}$, $v_0 = (L^*L)^{-1}L^*x_0$ and

$$(\forall n \in \mathbb{N}) \qquad \begin{cases} y_n = \operatorname{prox}_{\gamma g}(x_n) \\ c_n = (L^*L)^{-1}L^*y_n \\ x_{n+1} = x_n + \lambda_n\big(L(2c_n - v_n) - y_n\big) \\ v_{n+1} = v_n + \lambda_n(c_n - v_n). \end{cases}$$

We have then
$v_n \rightharpoonup \widehat{v}$ where $\widehat{v} \in \operatorname{Argmin}(g \circ L)$.

Optimization algorithms: Parallel form of Douglas-Rachford

Sketch of proof:

$$\underset{v\in\mathsf{G}}{\text{minimize}}\;\; g(Lv) \quad\Leftrightarrow\quad \underset{x\in\mathsf{H}}{\text{minimize}}\;\; \iota_E(x)+g(x)$$

where $E = \operatorname{ran} L$.
We apply Douglas-Rachford algorithm with
$f = \iota_E \Rightarrow \operatorname{prox}_{\gamma f} = P_E$ by setting

$$(\forall n\in\mathbb{N}) \qquad P_E y_n = L c_n \;\text{ and }\; P_E x_n = L v_n$$

where $c_n = \underset{c\in\mathsf{H}}{\operatorname{argmin}}\;\; \|y_n - Lc\|^2 = (L^*L)^{-1}L^*y_n$.

Optimization algorithms: Parallel form of DR

Particular case of Douglas-Rachford algorithm:

$\mathsf{H} = \mathsf{H}_1 \times \cdots \times \mathsf{H}_m$ where $\mathsf{H}_1, \ldots, \mathsf{H}_m$ Hilbert spaces

$\big(\forall x = (x_1, \ldots, x_m) \in \mathsf{H}\big)\ g(x) = \sum_{i=1}^m g_i(x_i)$

where $(\forall i \in \{1, \ldots, m\})\ g_i \in \Gamma_0(\mathsf{H}_i)$

$L\colon v \mapsto (L_1 v, \ldots, L_m v)$ where $(\forall i \in \{1, \ldots, m\})\ L_i \in \mathcal{B}(\mathsf{G}, \mathsf{H}_i)$.

PPXA+ algorithm

Let $(x_{0,i})_{1\leqslant i\leqslant m} \in \mathsf{H}$, $v_0 = (\sum_{i=1}^m L_i^* L_i)^{-1} \sum_{i=1}^m L_i^* x_{0,i}$ and

$$(\forall n \in \mathbb{N}) \qquad \begin{cases} y_{n,i} = \mathrm{prox}_{\gamma g_i}(x_{n,i}), \quad i \in \{1, \ldots, m\} \\ c_n = (\sum_{i=1}^m L_i^* L_i)^{-1} \sum_{i=1}^m L_i^* y_{n,i} \\ x_{n+1,i} = x_{n,i} + \lambda_n \big(L_i(2c_n - v_n) - y_{n,i}\big), \quad i \in \{1, \ldots, m\} \\ v_{n+1} = v_n + \lambda_n (c_n - v_n). \end{cases}$$

We have then $v_n \rightharpoonup \widehat{v} \in \operatorname{Argmin} \sum_{i=1}^m g_i \circ L_i$.

Part 4: Duality

1. General duality concepts
   - Inf-convolution
   - Primal and dual problems
   - Duality theorems
2. Augmented Lagrangian algorithms
3. Primal-dual algorithms
   - FB-based PD algorithm
   - Random block coordinate algorithms

Inf-convolution

Let H be a Hilbert space.
Let $f\colon \mathsf{H} \to \,]-\infty, +\infty]$ and $g\colon \mathsf{H} \to \,]-\infty, +\infty]$.
The inf-convolution of f and g is

$$f \,\square\, g\colon \mathsf{H} \to [-\infty, +\infty]\colon x \mapsto \inf_{y \in \mathsf{H}} f(y) + g(x-y)$$

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- $f \,\Box\, \iota_{\{0\}} = f$
- $f \,\Box\, g = g \,\Box\, f$
- $\operatorname{dom}(f \,\Box\, g) = \operatorname{dom} f + \operatorname{dom} g$
- ${}^{\gamma}f = f \,\Box\, \frac{1}{2\gamma}\|\cdot\|^2$, $\gamma \in\,]0,+\infty[$.

Inf-convolution

Let H be a Hilbert space.
Let $f\colon \mathsf{H} \to\,]-\infty, +\infty]$ and $g\colon \mathsf{H} \to\,]-\infty, +\infty]$.
The inf-convolution of f and g is

$$f \,\square\, g\colon \mathsf{H} \to [-\infty, +\infty]\colon x \mapsto \inf_{y\in\mathsf{H}} f(y) + g(x-y)$$

If $f\colon \mathsf{H} \to\,]-\infty, +\infty]$ and $g\colon \mathsf{H} \to\,]-\infty, +\infty]$ are convex,
then $f \,\square\, g$ is convex.

Inf-convolution

	conjugate		Fourier transform (H finite dimensional)	
Property	$h(x)$	$h^*(u)$	$h(x)$	$\widehat{h}(\nu)$
inf-convolution /convolution	$(f \,\square\, g)(x)$ $= \inf_{y\in\mathsf{H}} f(y) + g(x-y)$	$f^*(u) + g^*(u)$	$(f \star g)(x)$ $= \int_{\mathsf{H}} f(y) g(x-y) dy$	$\widehat{f}(\nu)\widehat{g}(\nu)$
sum/product	$f(x) + g(x)$ $f \in \Gamma_0(\mathsf{H})$ $g \in \Gamma_0(\mathsf{H})$ $\operatorname{dom} f \cap \operatorname{dom} g \neq \varnothing$	$(f^* \,\square\, g^*)(u)$	$f(x)g(x)$	$(\widehat{f} \star \widehat{g})(\nu)$

Proximity operator in an arbitrary metric

Let $f \in \Gamma_0(\mathsf{H})$. Let $\mathsf{U} : \mathsf{H} \to \mathsf{H}$ be a strongly positive self-adjoint linear operator.

The proximity operator $\operatorname{prox}_f^{\mathsf{U}}(x)$ of f at $x \in \mathsf{H}$ relative to the metric induced by U is the unique vector $\widehat{y} \in \mathsf{H}$ such that

$$f(\widehat{y}) + \frac{1}{2}\|\widehat{y} - x\|_{\mathsf{U}}^2 = \inf_{y \in \mathsf{H}} f(y) + \frac{1}{2}\langle y - x \mid \mathsf{U}(y - x)\rangle.$$

Proximity operator in an arbitrary metric

Let $f \in \Gamma_0(\mathsf{H})$. Let $\mathsf{U} : \mathsf{H} \to \mathsf{H}$ be a strongly positive self-adjoint linear operator.
The proximity operator $\operatorname{prox}_f^{\mathsf{U}}(x)$ of f at $x \in \mathsf{H}$ relative to the metric induced by U is the unique vector $\widehat{y} \in \mathsf{H}$ such that

$$f(\widehat{y}) + \frac{1}{2}\|\widehat{y} - x\|_{\mathsf{U}}^2 = \inf_{y \in \mathsf{H}} f(y) + \frac{1}{2}\langle y - x \mid \mathsf{U}(y - x)\rangle.$$

Remark:

- Moreau's decomposition formula:

$$(\forall x \in \mathsf{H}) \quad \operatorname{prox}_{f^*}^{\mathsf{U}}(x) = x - \mathsf{U}^{-1}\operatorname{prox}_f^{\mathsf{U}^{-1}}(\mathsf{U}x).$$

Principles of primal-dual methods

Fenchel-Rockafellar duality

Let H and G be two Hilbert spaces and let

- $f \in \Gamma_0(\mathsf{H})$, $g \in \Gamma_0(\mathsf{G})$
- $h \colon \mathsf{H} \to \mathbb{R}$ convex, μ-Lipschitz differentiable function with $\mu \in]0,+\infty[$
- $l \in \Gamma_0(\mathsf{G})$ ν-strongly convex with $\nu \in]0,+\infty[$
  $\Leftrightarrow l^* \in \Gamma_0(\mathsf{G})$ ν-Lipschitz differentiable
- $L \colon \mathsf{H} \to \mathsf{G}$ linear and bounded.

Primal problem

We want to

$$\underset{x \in \mathsf{H}}{\text{minimize}} \;\; f(x) + h(x) + (g \,\square\, l)(Lx).$$

Dual problem

We want to

$$\underset{v \in \mathsf{G}}{\text{minimize}} \;\; (f^* \,\square\, h^*)(-L^* v) + g^*(v) + l^*(v).$$

Fenchel-Rockafellar duality

Primal problem

We want to

$$\underset{x\in\mathsf{H}}{\text{minimize}} \;\; f(x) + h(x) + (g \,\Box\, l)(Lx).$$

Dual problem

We want to

$$\underset{v\in\mathsf{G}}{\text{minimize}} \;\; (f^* \,\Box\, h^*)(-L^*v) + g^*(v) + l^*(v).$$

Strong duality

If $\operatorname{int}(\operatorname{dom} g + \operatorname{dom} l) \cap L(\operatorname{dom} f) \neq \varnothing$ or
$(\operatorname{dom} g + \operatorname{dom} l) \cap \operatorname{int}(L(\operatorname{dom} f)) \neq \varnothing$, then

$$\inf_{x\in\mathsf{H}} f(x) + h(x) + (g \,\Box\, l)(Lx) = -\min_{v\in\mathsf{G}} (f^* \,\Box\, h^*)(-L^*v) + g^*(v) + l^*(v)$$

Fenchel-Rockafellar duality

Primal problem

We want to

$$\underset{x \in \mathsf{H}}{\text{minimize}} \;\; f(x) + h(x) + (g \,\square\, l)(Lx).$$

Dual problem

We want to

$$\underset{v \in \mathsf{G}}{\text{minimize}} \;\; (f^* \,\square\, h^*)(-L^* v) + g^*(v) + l^*(v).$$

If $(\widehat{x}, \widehat{v}) \in \mathsf{H} \times \mathsf{G}$ is such that

$$-L^* \widehat{v} - \nabla h(\widehat{x}) \in \partial f(\widehat{x}) \;\; \text{and} \;\; L\widehat{x} - \nabla l^*(\widehat{v}) \in \partial g^*(\widehat{v}),$$

then $(\widehat{x}, \widehat{v})$ is called a Kuhn-Tucker point.

- If $(\widehat{x}, \widehat{v})$ is a Kuhn-Tucker point then $\widehat{x}$ (resp. $\widehat{v}$) is a solution to the primal (resp. dual) problem.

Augmented Lagrangian method

- Augmented Lagrange function

  Let $\gamma \in]0,+\infty[$, define, for every $(x,y,z) \in \mathsf{H} \times \mathsf{G}^2$,

$$\mathcal{L}(x,y,z) = f(x) + h(x) + (g \,\square\, l)(y) + \gamma \langle z \mid Lx - y \rangle$$

  The Lagrange multiplier is $v = \gamma z$.

Augmented Lagrangian method

- **Augmented Lagrange function**

  Let $\gamma \in]0,+\infty[$, define, for every $(x,y,z) \in \mathsf{H} \times \mathsf{G}^2$,

$$\mathcal{L}(x,y,z) = f(x) + h(x) + (g \,\square\, l)(y) + \gamma \langle z \mid Lx - y\rangle + \frac{\gamma}{2}\|Lx-y\|^2$$

  The Lagrange multiplier is $v = \gamma z$.

Let $(\widehat{x},\widehat{y},\widehat{z}) \in \mathsf{H} \times \mathsf{G}^2$.
Assume that $\operatorname{int}(\operatorname{dom} g + \operatorname{dom} l) \cap L(\operatorname{dom} f) \neq \varnothing$
or $(\operatorname{dom} g + \operatorname{dom} l) \cap \operatorname{int}\big(L(\operatorname{dom} f)\big) \neq \varnothing$.

$(\widehat{x},\widehat{y},\widehat{z})$ is a saddle point of the augmented Lagrange function

$$\Updownarrow$$

$(\widehat{x},\gamma\widehat{z})$ is a Kuhn-Tucker point and $\widehat{y} = L\widehat{x}$.

Augmented Lagrangian method

ADMM (*Alternating-direction method of multipliers*)

Let L be such that L^*L is an isomorphism and let $\gamma \in]0,+\infty[$.
Assume that $\operatorname{int}(\operatorname{dom} g + \operatorname{dom} l) \cap L(\operatorname{dom} f) \neq \varnothing$
or $(\operatorname{dom} g + \operatorname{dom} l) \cap \operatorname{int}\big(L(\operatorname{dom} f)\big) \neq \varnothing$, and that $\operatorname{Argmin}(f + h + (g \,\square\, l) \circ L) \neq \varnothing$.
Let $(y_0, z_0) \in \mathsf{G}^2$ and

$$(\forall n \in \mathbb{N}) \quad \begin{cases} x_n = \underset{x \in \mathsf{H}}{\operatorname{argmin}} \ \frac{1}{2}\|Lx - y_n + z_n\|^2 + \frac{1}{\gamma}(f(x) + h(x)) \\ s_n = Lx_n \\ y_{n+1} = \operatorname{prox}_{g \,\square\, l}^{\gamma \mathrm{Id}}(z_n + s_n) \\ z_{n+1} = z_n + s_n - y_{n+1}. \end{cases}$$

We have:

- $x_n \rightharpoonup \widehat{x}$ where $\widehat{x}$ is a solution to the primal problem
- $\gamma z_n \rightharpoonup \widehat{v}$ where $\widehat{v}$ is a solution to the dual problem.

Limitations:

- Computation of x_n at iteration $n \in \mathbb{N}$ may be complicated.
- Convergence requires L^*L to be invertible.
- The smoothness of h and l^* is not exploited.

Primal-dual approach

- The optimization problem is reformulated as finding

$$\inf_{x\in\mathsf{H}}\sup_{v\in\mathsf{G}} f(x)+h(x)+\langle v \mid Lx\rangle - g^*(v) - l^*(v).$$

- Arrow-Hurwitz method : Let $(\tau_n)_{n\in\mathbb{N}}$ and $(\sigma_n)_{n\in\mathbb{N}}$ be sequences in $]0,+\infty[$.

$$(\forall n\in\mathbb{N})\qquad \begin{cases} t_n \in \partial f(x_n)\\ x_{n+1} = x_n - \tau_n(t_n+\nabla h(x_n)+L^*v_n)\\ s_n\in\partial g^*(v_n)\\ v_{n+1} = v_n - \sigma_n(s_n+\nabla l^*(v_n) - Lx_{n+1})\end{cases}$$

$\rightsquigarrow$ requires stringent conditions on the choice of the step-sizes (e.g. decaying to zero)
... but it can be modified.

Primal-dual approach

- First modification: Use implicit updates

$$(\forall n \in \mathbb{N}) \qquad \begin{cases} t_n \in \partial f(x_{n+1}) \\ x_{n+1} = x_n - \tau_n(t_n + \nabla h(x_n) + L^* v_n) \\ s_n \in \partial g^*(v_{n+1}) \\ v_{n+1} = v_n - \sigma_n(s_n + \nabla l^*(v_n) - L x_{n+1}) \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} x_{n+1} = \mathrm{prox}_f^{\mathrm{Id}/\tau_n}\big(x_n - \tau_n(L^* v_n + \nabla h(x_n))\big) \\ v_{n+1} = \mathrm{prox}_{g^*}^{\mathrm{Id}/\sigma_n}\big(v_n + \sigma_n(L x_{n+1} - \nabla l^*(v_n))\big) \end{cases}$$

$\rightsquigarrow$ still does not converge for constant values of the step-size.

- Second modification: Use the approximation $x_{n+1} \simeq 2x_{n+1} - x_n$

$$(\forall n \in \mathbb{N}) \qquad \begin{cases} x_{n+1} = \mathrm{prox}_f^{\mathrm{Id}/\tau_n}\big(x_n - \tau_n(L^* v_n + \nabla h(x_n))\big) \\ v_{n+1} = \mathrm{prox}_{g^*}^{\mathrm{Id}/\sigma_n}\Big(v_n + \sigma_n\big(L(2x_{n+1} - x_n) - \nabla l^*(v_n)\big)\Big). \end{cases}$$

Primal-dual optimization algorithm

Modified PD algorithm

Let $\tau \in]0,+\infty[$ and $\sigma \in]0,+\infty[$ be such that

$$1 - \sqrt{\tau\sigma}\|L\| > \max\{\mu\tau, \nu\sigma\}/2.$$

Assume that $\operatorname{zer}\big(\partial f + \nabla h + L^*\partial(g \,\square\, l)L\big) \neq \varnothing$.
Let $x_0 \in \mathsf{H}$, $v_0 \in \mathsf{G}$, and

$$(\forall n \in \mathbb{N}) \qquad \begin{cases} x_{n+1} = \operatorname{prox}_f^{\mathrm{Id}/\tau}\big(x_n - \tau\big(L^*v_n + \nabla h(x_n)\big)\big) \\ v_{n+1} = \operatorname{prox}_{g^*}^{\mathrm{Id}/\sigma}\big(v_n + \sigma(L(2x_{n+1} - x_n) - \nabla l^*(v_n))\big). \end{cases}$$

We have:

- $x_n \rightharpoonup \widehat{x}$ where $\widehat{x}$ is a solution to the primal problem
- $v_n \rightharpoonup \widehat{v}$ where $\widehat{v}$ is a solution to the dual problem.

Remark: If $l = \iota_{\{0\}}$, a more general convergence condition is

$$\tau^{-1} - \sigma\|L\|^2 \geqslant \mu/2.$$

Primal-dual optimization algorithm

Modified PD algorithm (symmetric form)

Let $\tau \in]0,+\infty[$ and $\sigma \in]0,+\infty[$ be such that

$$1-\sqrt{\tau\sigma}\|L\| > \max\{\mu\tau,\nu\sigma\}/2.$$

Assume that $\operatorname{zer}\big(\partial f + \nabla h + L^*\partial(g \,\square\, l)L\big) \neq \varnothing$.
Let $x_0 \in \mathsf{H}$, $v_0 \in \mathsf{G}$, and

$$(\forall n \in \mathbb{N}) \qquad \begin{cases} v_{n+1} = \operatorname{prox}_{g^*}^{\mathrm{Id}/\sigma}\big(v_n + \sigma\big(Lx_n - \nabla l^*(v_n)\big)\big) \\ x_{n+1} = \operatorname{prox}_{f}^{\mathrm{Id}/\tau}\big(x_n - \tau(L(2v_{n+1}-v_n) + \nabla h(x_n))\big). \end{cases}$$

We have:

- $x_n \rightharpoonup \widehat{x}$ where $\widehat{x}$ is a solution to the primal problem
- $v_n \rightharpoonup \widehat{v}$ where $\widehat{v}$ is a solution to the dual problem.

Proximal primal-dual algorithm

Advantages:

- No linear operator inversion.
- Use of proximable or/and differentiable functions.
- Special cases: Forward-Backward and Douglas-Rachford algorithms.

Proximal primal-dual algorithm

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Bibliographical remarks:

- Methods based on Forward-Backward iteration
  - type I: [Vu - 2013][Condat - 2013]
    (extensions of [Esser *et al.* - 2010][Chambolle and Pock - 2011])
  - type II: [Combettes *et al.* - 2014]
    (extensions of [Loris and Verhoeven - 2011][Chen *et al.* - 2014])
- Methods based on Forward-Backward-Forward iteration
  [Combettes and Pesquet - 2012] [Boţ and Hendrich,2014]
- Projection based methods
  [Alotaibi *et al.* - 2013]
- ...

Random block-coordinate strategy

Acceleration via block alternation

- Assumption: f and h are additively block separable functions.

Acceleration via block alternation

- Assumption: f and h are additively block separable functions.

$$\begin{bmatrix} x \end{bmatrix} \in \mathsf{H} \quad \longrightarrow \quad \begin{bmatrix} x_1 \end{bmatrix} \in \mathsf{H}_1, \quad \begin{bmatrix} x_2 \end{bmatrix} \in \mathsf{H}_2, \quad \vdots \quad \begin{bmatrix} x_p \end{bmatrix} \in \mathsf{H}_p$$

$$\times_{j=1}^{p} \mathsf{H}_j = \mathsf{H}$$

where $\mathsf{H}_1, \ldots, \mathsf{H}_p$ separable real Hilbert spaces.

Acceleration via block alternation

- Assumption: f and h are additively block separable functions.

$$f\left(\begin{bmatrix} \\ \\ x \\ \\ \\ \end{bmatrix}\right) = f\left(\begin{matrix} \begin{bmatrix} x_1 \end{bmatrix} \\ \begin{bmatrix} x_2 \end{bmatrix} \\ \vdots \\ \begin{bmatrix} x_p \end{bmatrix} \end{matrix}\right) = \sum_{j=1}^{p} f_j(x_j)$$

where, for every $j \in \{1, \dots, p\}$,
$f_j \in \Gamma_0(\mathsf{H}_j)$

Acceleration via block alternation

- Assumption: f and h are additively block separable functions.

$$h\left(\begin{bmatrix} \\ \\ x \\ \\ \\ \end{bmatrix}\right) = h\left(\begin{matrix} \begin{bmatrix} x_1 \end{bmatrix} \\ \begin{bmatrix} x_2 \end{bmatrix} \\ \vdots \\ \begin{bmatrix} x_p \end{bmatrix} \end{matrix}\right) = \sum_{j=1}^{p} h_j(x_j)$$

where, for every $j \in \{1, \dots, p\}$,
h_j convex μ_j-Lipschitz
differentiable with $\mu_j \in]0, +\infty[$.

Block-coordinate strategy

$\Rightarrow$ At each iteration n, update only a subset of components ($\sim$ Gauss-Seidel).

Block-coordinate strategy

$\Rightarrow$ At each iteration n, update only a subset of components ($\sim$ Gauss-Seidel).

Advantages:

- Reduced complexity
- Less memory requirements per iteration
- More flexibility

$\Rightarrow$ Useful for large-scale optimization
(see e.g. [Richtárik andTakáč, 2014])

Difficulty of block-coordinate fixed point algorithms

$$\mathsf{T}\colon (\mathsf{x}_1, \mathsf{x}_2) \mapsto (-\mathsf{x}_2, \mathsf{x}_1) \text{ is nonexpansive}$$

but

$$\left.\begin{array}{r} \mathsf{T}_1\colon (\mathsf{x}_1, \mathsf{x}_2) \mapsto (\mathsf{x}_1, \mathsf{x}_1) \\ \mathsf{T}_2\colon (\mathsf{x}_1, \mathsf{x}_2) \mapsto (-\mathsf{x}_2, \mathsf{x}_2) \\ \mathsf{T}_1 \circ \mathsf{T}_2 \\ \mathsf{T}_2 \circ \mathsf{T}_1 \end{array}\right\} \text{are not nonexpansive}$$

$\rightsquigarrow$ introduce stochasticity

Parallel proximal primal-dual algorithm

Optimization problem: Minimization of

$$(\forall x \in \mathsf{H}) \qquad \Phi(x) = f(x) + h(x) + \sum_{k=1}^{q} (g_k \,\Box\, l_k)(L_k x)$$

Parallel proximal primal-dual algorithm

Optimization problem: Minimization of

$$(\forall x \in \mathsf{H}) \qquad \Phi(x) = f(x) + h(x) + \sum_{k=1}^{q} (g_k \,\Box\, l_k)(L_k x)$$

where

- $(\forall k \in \{1,\ldots,q\})$ $g_k \in \Gamma_0(\mathsf{G}_k)$, G_k separable real Hilbert space
- $l_k \in \Gamma_0(\mathsf{G}_k)$ ν_k-strongly convex with $\nu_k \in \,]0,+\infty[$
- $(\forall x \in \mathsf{H})$ $L_k x = \sum_{j=1}^{p} L_{k,j} x_j$, where $(\forall j \in \{1,\ldots,p\})$ $L_{k,j} \colon \mathsf{H}_j \to \mathsf{G}_k$ bounded linear operator such that

$$\mathbb{L}_k = \big\{ j \in \{1,\ldots,p\} \bigm| L_{k,j} \neq 0 \big\} \neq \varnothing$$
$$\mathbb{L}_j^* = \big\{ k \in \{1,\ldots,q\} \bigm| L_{k,j} \neq 0 \big\} \neq \varnothing.$$

Block parallel proximal primal-dual problem

Primal problem : Find an element of the set F of solutions to

$$\underset{x_1\in\mathsf{H}_1,\ldots,x_p\in\mathsf{H}_p}{\text{minimize}} \quad \sum_{j=1}^{p}\big(f_j(x_j)+h_j(x_j)\big)+\sum_{k=1}^{q}(g_k\,\Box\,l_k)\Big(\sum_{j=1}^{p}L_{k,j}x_j\Big)$$

Dual problem : Find an element of the set F^* of solutions to

$$\underset{v_1\in\mathsf{G}_1,\ldots,v_q\in\mathsf{G}_q}{\text{minimize}} \quad \sum_{j=1}^{p}(f_j^*\,\Box\,h_j^*)\Big(-\sum_{k=1}^{q}L_{k,j}^*v_k\Big)+\sum_{k=1}^{q}\big(g_k^*(v_k)+l_k^*(v_k)\big)$$

We assume that there exists $(\overline{x}_1,\ldots,\overline{x}_p)\in\mathsf{H}_1\times\cdots\times\mathsf{H}_p$ such that

$$(\forall j\in\{1,\ldots,p\})\quad 0\in\partial f_j(\overline{x}_j)+\nabla h_j(\overline{x}_j)+\sum_{k=1}^{q}L_{k,j}^*\partial(g_k\,\Box\,l_k)\Big(\sum_{j'=1}^{p}L_{k,j'}\overline{x}_{j'}\Big).$$

Random block-coordinate primal-dual algorithm: Type I

for $n = 0, 1, \ldots$

$$
\left\lfloor
\begin{array}{l}
\text{for } k = 1, \ldots, q \\
\left\lfloor
\begin{array}{l}
u_{k,n} = \varepsilon_{p+k,n}\Big(\mathrm{prox}_{g_k^*}^{\mathsf{U}_k^{-1}}\big(v_{k,n} + \mathsf{U}_k(\sum_{j\in\mathbb{L}_k} L_{k,j}x_{j,n} - \nabla l_k^*(v_{k,n}) + d_{k,n})\big) + b_{k,n}\Big) \\
v_{k,n+1} = v_{k,n} + \lambda_n\varepsilon_{p+k,n}(u_{k,n} - v_{k,n}),
\end{array}
\right. \\
\text{for } j = 1, \ldots, p \\
\left\lfloor
\begin{array}{l}
y_{j,n} = \varepsilon_{j,n}\Big(\mathrm{prox}_{f_j}^{\mathsf{W}_j^{-1}}\big(x_{j,n} - \mathsf{W}_j(\sum_{k\in\mathbb{L}_j^*} L_{k,j}^*(2u_{k,n} - v_{k,n}) + \nabla h_j(x_{j,n}) + c_{j,n})\big) + a_{j,n}\Big) \\
x_{j,n+1} = x_{j,n} + \lambda_n\varepsilon_{j,n}(y_{j,n} - x_{j,n})
\end{array}
\right.
\end{array}
\right.
$$

Random block-coordinate primal-dual algorithm: Type I

for $n = 0, 1, \dots$
for $k = 1, \dots, q$

$$u_{k,n} = \varepsilon_{p+k,n}\Big(\operatorname{prox}_{g_k^*}^{\mathsf{U}_k^{-1}}\big(v_{k,n} + \mathsf{U}_k\big(\sum_{j\in\mathbb{L}_k} L_{k,j}x_{j,n} - \nabla l_k^*(v_{k,n}) + d_{k,n}\big)\big) + b_{k,n}\Big)$$

$$v_{k,n+1} = v_{k,n} + \lambda_n \varepsilon_{p+k,n}(u_{k,n} - v_{k,n}),$$

for $j = 1, \dots, p$

$$y_{j,n} = \varepsilon_{j,n}\Big(\operatorname{prox}_{f_j}^{\mathsf{W}_j^{-1}}\big(x_{j,n} - \mathsf{W}_j\big(\sum_{k\in\mathbb{L}_j^*} L_{k,j}^*(2u_{k,n} - v_{k,n}) + \nabla h_j(x_{j,n}) + c_{j,n}\big) + a_{j,n}\Big)$$

$$x_{j,n+1} = x_{j,n} + \lambda_n \varepsilon_{j,n}(y_{j,n} - x_{j,n})$$

where

- $(\varepsilon_n)_{n\in\mathbb{N}}$ identically distributed $\mathbb{D}$-valued random variables with $\mathbb{D} = \{0,1\}^{p+q} \smallsetminus \{\mathbf{0}\}$
  $\rightsquigarrow$ binary variables signaling the blocks to be activated

Random block-coordinate primal-dual algorithm: Type I

for $n = 0, 1, \ldots$
  for $k = 1, \ldots, q$
    $u_{k,n} = \varepsilon_{p+k,n}\Big(\mathrm{prox}^{\mathsf{U}_k^{-1}}_{g_k^*}\big(v_{k,n} + \mathsf{U}_k(\sum_{j \in \mathbb{L}_k} L_{k,j} x_{j,n} - \nabla l_k^*(v_{k,n}) + d_{k,n})\big) + b_{k,n}\Big)$
    $v_{k,n+1} = v_{k,n} + \lambda_n \varepsilon_{p+k,n}(u_{k,n} - v_{k,n}),$
  for $j = 1, \ldots, p$
    $y_{j,n} = \varepsilon_{j,n}\Big(\mathrm{prox}^{\mathsf{W}_j^{-1}}_{f_j}\big(x_{j,n} - \mathsf{W}_j(\sum_{k \in \mathbb{L}_j^*} L_{k,j}^*(2u_{k,n} - v_{k,n}) + \nabla h_j(x_{j,n}) + c_{j,n}\big) + a_{j,n}\Big)$
    $x_{j,n+1} = x_{j,n} + \lambda_n \varepsilon_{j,n}(y_{j,n} - x_{j,n})$

where

- $(\varepsilon_n)_{n\in\mathbb{N}} \rightsquigarrow$ binary variables signaling the blocks to be activated
- x_0, $(a_n)_{n\in\mathbb{N}}$, and $(c_n)_{n\in\mathbb{N}}$ H-valued random variables, v_0, $(b_n)_{n\in\mathbb{N}}$, and $(d_n)_{n\in\mathbb{N}}$ G-valued random variables with $\mathsf{G} = \mathsf{G}_1 \times \cdots \times \mathsf{G}_q$
  $\rightsquigarrow$ $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$, and $(d_n)_{n\in\mathbb{N}}$: error terms

Random block-coordinate primal-dual algorithm: Type I

for $n = 0, 1, \ldots$

for $k = 1, \ldots, q$

$$u_{k,n} = \varepsilon_{p+k,n}\Big(\mathrm{prox}^{\mathsf{U}_k^{-1}}_{g_k^*}\big(v_{k,n} + \mathsf{U}_k\big(\sum_{j\in\mathbb{L}_k} L_{k,j}x_{j,n} - \nabla l_k^*(v_{k,n}) + d_{k,n}\big)\big) + b_{k,n}\Big)$$

$$v_{k,n+1} = v_{k,n} + \lambda_n\varepsilon_{p+k,n}(u_{k,n} - v_{k,n}),$$

for $j = 1, \ldots, p$

$$y_{j,n} = \varepsilon_{j,n}\Big(\mathrm{prox}^{\mathsf{W}_j^{-1}}_{f_j}\big(x_{j,n} - \mathsf{W}_j\big(\sum_{k\in\mathbb{L}_j^*} L_{k,j}^*(2u_{k,n} - v_{k,n}) + \nabla h_j(x_{j,n}) + c_{j,n}\big) + a_{j,n}\Big)$$

$$x_{j,n+1} = x_{j,n} + \lambda_n\varepsilon_{j,n}(y_{j,n} - x_{j,n})$$

where

- $(\varepsilon_n)_{n\in\mathbb{N}}$ $\rightsquigarrow$ binary variables signaling the blocks to be activated
- $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$, and $(d_n)_{n\in\mathbb{N}}$: error terms
- $(\forall j\in\{1,\ldots,p\})$ $\mathsf{W}_j\colon \mathsf{H}_j \to \mathsf{H}_j$ and $(\forall k\in\{1,\ldots,q\})$ $\mathsf{U}_k\colon \mathsf{G}_k \to \mathsf{G}_k$ strongly positive self-adjoint preconditioning linear operators such that

$$1 - \Big(\sum_{j=1}^{p}\sum_{k=1}^{q}\|\mathsf{U}_k^{1/2}L_{k,j}\mathsf{W}_j^{1/2}\|^2\Big)^{1/2} > \frac{1}{2}\max\{(\|\mathsf{W}_j\|\mu_j)_{1\leqslant j\leqslant p}, (\|\mathsf{U}_k\|\nu_k)_{1\leqslant k\leqslant q}\}.$$

Random block-coordinate primal-dual algorithm: Type I

$$
\begin{array}{l}
\text{for } n = 0, 1, \ldots \\
\quad \text{for } k = 1, \ldots, q \\
\quad\quad u_{k,n} = \varepsilon_{p+k,n} \Big(\operatorname{prox}_{g_k^*}^{\mathsf{U}_k^{-1}} \big(v_{k,n} + \mathsf{U}_k \big(\sum_{j \in \mathbb{L}_k} L_{k,j} x_{j,n} - \nabla l_k^*(v_{k,n}) + d_{k,n} \big) \big) + b_{k,n} \Big) \\
\quad\quad v_{k,n+1} = v_{k,n} + \lambda_n \varepsilon_{p+k,n} (u_{k,n} - v_{k,n}), \\
\quad \text{for } j = 1, \ldots, p \\
\quad\quad y_{j,n} = \varepsilon_{j,n} \Big(\operatorname{prox}_{f_j}^{\mathsf{W}_j^{-1}} \big(x_{j,n} - \mathsf{W}_j \big(\sum_{k \in \mathbb{L}_j^*} L_{k,j}^* (2u_{k,n} - v_{k,n}) + \nabla h_j(x_{j,n}) + c_{j,n} \big) + a_{j,n} \Big) \\
\quad\quad x_{j,n+1} = x_{j,n} + \lambda_n \varepsilon_{j,n} (y_{j,n} - x_{j,n})
\end{array}
$$

where

- $(\varepsilon_n)_{n\in\mathbb{N}} \rightsquigarrow$ binary variables signaling the blocks to be activated
- $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$, and $(d_n)_{n\in\mathbb{N}}$: error terms
- $(\forall j \in \{1,\ldots,p\})$ $\mathsf{W}_j : \mathsf{H}_j \to \mathsf{H}_j$ and $(\forall k \in \{1,\ldots,q\})$ $\mathsf{U}_k : \mathsf{G}_k \to \mathsf{G}_k$ strongly positive self-adjoint preconditioning linear operators
- $(\forall n \in \mathbb{N})$ $\lambda_n \in \,]0,1]$ such that $\inf_{n\in\mathbb{N}} \lambda_n > 0$.

Random block-coordinate primal-dual algorithm: Type I

Theorem

Let $(\Omega, \mathcal{F}, \mathsf{P})$ *be the underlying probability space. Set* $(\forall n \in \mathbb{N})$ $\mathcal{X}_n = \sigma(x_{n'}, v_{n'})_{0 \leqslant n' \leqslant n}$. *Assume that*

1. $\sum_{n\in\mathbb{N}} \sqrt{\mathsf{E}(\|a_n\|^2 \,|\, \mathcal{X}_n)} < +\infty$, $\sum_{n\in\mathbb{N}} \sqrt{\mathsf{E}(\|b_n\|^2 \,|\, \mathcal{X}_n)} < +\infty$, $\sum_{n\in\mathbb{N}} \sqrt{\mathsf{E}(\|c_n\|^2 \,|\, \mathcal{X}_n)} < +\infty$, *and* $\sum_{n\in\mathbb{N}} \sqrt{\mathsf{E}(\|d_n\|^2 \,|\, \mathcal{X}_n)} < +\infty$ P-a.s.
2. *The variables* $(\varepsilon_n)_{n\in\mathbb{N}}$ *are identically distributed such that* $(\forall j \in \{1,\ldots,p\})$ $\mathsf{P}[\varepsilon_{j,0} = 1] > 0$.
3. *For every* $n \in \mathbb{N}$, ε_n *and* $\mathcal{X}_n$ *are independent.*
4. *For every* $k \in \{1,\ldots,q\}$ *and* $n \in \mathbb{N}$,
$$\bigcup_{j\in\mathbb{L}_k} \big\{\omega \in \Omega \;\big|\; \varepsilon_{j,n}(\omega) = 1\big\} \subset \big\{\omega \in \Omega \;\big|\; \varepsilon_{p+k,n}(\omega) = 1\big\}.$$

Then, $(x_n)_{n\in\mathbb{N}}$ *converges weakly* P-a.s. *to an* F*-valued random variable, and* $(v_n)_{n\in\mathbb{N}}$ *converges weakly* P-a.s. *to an* F^**-valued random variable.*

Proof: based on properties of quasi-Fejér stochastic sequences [Combettes and Pesquet, 2014].

Random block-coordinate primal-dual algorithm: Type I

Theorem

Let $(\Omega, \mathcal{F}, \mathsf{P})$ *be the underlying probability space.*
Set $(\forall n \in \mathbb{N})\ \mathcal{X}_n = \sigma(x_{n'}, v_{n'})_{0 \leqslant n' \leqslant n}$. *Assume that*

1. $\sum_{n\in\mathbb{N}} \sqrt{\mathsf{E}(\|a_n\|^2 \,|\, \mathcal{X}_n)} < +\infty$, $\sum_{n\in\mathbb{N}} \sqrt{\mathsf{E}(\|b_n\|^2 \,|\, \mathcal{X}_n)} < +\infty$, $\sum_{n\in\mathbb{N}} \sqrt{\mathsf{E}(\|c_n\|^2 \,|\, \mathcal{X}_n)} < +\infty$, *and* $\sum_{n\in\mathbb{N}} \sqrt{\mathsf{E}(\|d_n\|^2 \,|\, \mathcal{X}_n)} < +\infty$ P-a.s.
2. *The variables* $(\varepsilon_n)_{n\in\mathbb{N}}$ *are identically distributed such that* $(\forall j \in \{1, \ldots, p\})\ \mathsf{P}[\varepsilon_{j,0} = 1] > 0$.
3. *For every* $n \in \mathbb{N}$, ε_n *and* $\mathcal{X}_n$ *are independent.*
4. *For every* $k \in \{1, \ldots, q\}$ *and* $n \in \mathbb{N}$,

$$\boxed{\varepsilon_{p+k,n} = \max_{1\leqslant j\leqslant p} \left\{ \varepsilon_{j,n} \;\middle|\; j \in \mathbb{L}_k \right\}}.$$

Then, $(x_n)_{n\in\mathbb{N}}$ *converges weakly* P-a.s. *to an* F*-valued random variable, and* $(v_n)_{n\in\mathbb{N}}$ *converges weakly* P-a.s. *to an* F^**-valued random variable.*

Proof: based on properties of quasi-Fejér stochastic sequences [Combettes and Pesquet, 2014].

Stochastic quasi-Fejér sequences

F: a nonempty closed subset of H

- **Deterministic definition**: A sequence $(x_n)_{n\in\mathbb{N}}$ of H is Fejér monotone w.r.t. F if for every $\mathsf{z} \in$ F,

$$(\forall n \in \mathbb{N})\ \|x_{n+1} - \mathsf{z}\| \leqslant \|x_n - \mathsf{z}\|.$$

Stochastic quasi-Fejér sequences

F: a nonempty closed subset of H

$\phi\colon\ [0,+\infty[\ \to [0,+\infty[$: strictly increasing with $\lim_{t\to+\infty}\phi(t)=+\infty$.

- **Deterministic definition**: A sequence $(x_n)_{n\in\mathbb{N}}$ of H is Fejér monotone w.r.t. F if for every $\mathsf{z}\in$ F,

$$(\forall n\in\mathbb{N})\ \phi(\|x_{n+1}-\mathsf{z}\|)\leqslant\phi(\|x_n-\mathsf{z}\|).$$

Stochastic quasi-Fejér sequences

F: a nonempty closed subset of H

- **Stochastic definition**: A sequence $(x_n)_{n\in\mathbb{N}}$ of H-valued random variables is Fejér monotone w.r.t. F if for every $\mathsf{z} \in \mathsf{F}$, the following is satisfied P-a.s.:

$$(\forall n \in \mathbb{N})\ \mathsf{E}(\phi(\|x_{n+1} - \mathsf{z}\| \,|\, \mathcal{X}_n) \leqslant \phi(\|x_n - \mathsf{z}\|)$$

where $\mathcal{X}_n = \sigma(x_0, \ldots, x_n)$.

Stochastic quasi-Fejér sequences

F: a nonempty closed subset of H

- **Stochastic definition**: A sequence $(x_n)_{n\in\mathbb{N}}$ of H-valued random variables is quasi-Fejér monotone w.r.t. F if for every $\mathsf{z} \in$ F, there exist $(\chi_n(\mathsf{z}))_{n\in\mathbb{N}} \in \ell_+^1(\mathscr{X})$, $(\vartheta_n(\mathsf{z}))_{n\in\mathbb{N}} \in \ell_+(\mathscr{X})$, and $(\eta_n(\mathsf{z}))_{n\in\mathbb{N}} \in \ell_+^1(\mathscr{X})$ such that the following is satisfied P-a.s.:

  $$(\forall n \in \mathbb{N})\ \mathsf{E}(\phi(\|x_{n+1}-\mathsf{z}\|)\,|\,\mathcal{X}_n)+\vartheta_n(\mathsf{z}) \leqslant (1+\chi_n(\mathsf{z}))\phi(\|x_n-\mathsf{z}\|)+\eta_n(\mathsf{z})$$

  where $\mathcal{X}_n = \sigma(x_0,\ldots,x_n)$ and $\mathscr{X} = (\mathcal{X}_n)_{n\in\mathbb{N}}$.

Stochastic quasi-Fejér sequences

F: a nonempty closed subset of H

- **Stochastic definition**: A sequence $(x_n)_{n\in\mathbb{N}}$ of H-valued random variables is quasi-Fejér monotone w.r.t. F if for every $\mathsf{z} \in \mathsf{F}$, there exist $(\chi_n(\mathsf{z}))_{n\in\mathbb{N}} \in \ell_+^1(\mathscr{X})$, $(\vartheta_n(\mathsf{z}))_{n\in\mathbb{N}} \in \ell_+(\mathscr{X})$, and $(\eta_n(\mathsf{z}))_{n\in\mathbb{N}} \in \ell_+^1(\mathscr{X})$ such that the following is satisfied P-a.s.:

$$(\forall n \in \mathbb{N})\ \mathsf{E}(\phi(\|x_{n+1}-\mathsf{z}\|)\,|\,\mathfrak{X}_n)+\vartheta_n(\mathsf{z}) \leqslant (1+\chi_n(\mathsf{z}))\phi(\|x_n-\mathsf{z}\|)+\eta_n(\mathsf{z})$$

where $\mathfrak{X}_n = \sigma(x_0, \ldots, x_n)$ and $\mathscr{X} = (\mathfrak{X}_n)_{n\in\mathbb{N}}$.

Theorem

If $(x_n)_{n\in\mathbb{N}}$ is quasi-Fejér monotone w.r.t. F, *then*

1. $(\forall \mathsf{z} \in \mathsf{F})\ \left[\ \sum_{n\in\mathbb{N}} \vartheta_n(\mathsf{z}) < +\infty\ \mathsf{P}\text{-a.s.}\right]$
2. $[\mathfrak{W}(x_n)_{n\in\mathbb{N}} \subset \mathsf{F}\ \ \mathsf{P}\text{-a.s.}] \Rightarrow [(x_n)_{n\in\mathbb{N}}$ *converges weakly* P-a.s. *to an* F*-valued random variable*].

$\mathfrak{W}(x_n)_{n\in\mathbb{N}}$: set of weak sequential cluster points of $(x_n)_{n\in\mathbb{N}}$.

An abstract iterative scheme

Theorem

Let F *be a nonempty closed subset of* H*. Suppose that* $(x_n)_{n\in\mathbb{N}}$, $(t_n)_{n\in\mathbb{N}}$, *and* $(e_n)_{n\in\mathbb{N}}$ *are sequences of* H*-valued random variables such that:*

1. $(\forall n \in \mathbb{N})\ x_{n+1} = x_n + \lambda_n(t_n + e_n - x_n), \quad \lambda_n \in\,]0,1]$.
2. $\sum_{n\in\mathbb{N}} \lambda_n \sqrt{\mathsf{E}(\|e_n\|^2 \,|\, \mathfrak{X}_n)} < +\infty$ P-a.s.
3. *For every* $\mathsf{z} \in \mathsf{F}$*, there exist* $(\theta_n(\mathsf{z}))_{n\in\mathbb{N}} \in \ell_+(\mathscr{X})$, $(\mu_n(\mathsf{z}))_{n\in\mathbb{N}} \in \ell_+^\infty(\mathscr{X})$*, and* $(\nu_n(\mathsf{z}))_{n\in\mathbb{N}} \in \ell_+^\infty(\mathscr{X})$ *such that* $(\lambda_n\mu_n(\mathsf{z}))_{n\in\mathbb{N}} \in \ell_+^1(\mathscr{X})$, $(\lambda_n\nu_n(\mathsf{z}))_{n\in\mathbb{N}} \in \ell_+^{1/2}(\mathscr{X})$*, and the following is satisfied* P-a.s.*:*

$$(\forall n \in \mathbb{N}) \quad \mathsf{E}(\|t_n - \mathsf{z}\|^2 \,|\, \mathfrak{X}_n) + \theta_n(\mathsf{z}) \leqslant (1 + \mu_n(\mathsf{z}))\|x_n - \mathsf{z}\|^2 + \nu_n(\mathsf{z}).$$

Then $(\forall \mathsf{z} \in \mathsf{F}) \quad \left[\ \sum_{n\in\mathbb{N}} \lambda_n\theta_n(\mathsf{z}) < +\infty \text{ P-a.s.}\ \right]$ *and* $[\mathfrak{W}(x_n)_{n\in\mathbb{N}} \subset \mathsf{F}\ \text{ P-a.s.}] \Rightarrow (x_n)_{n\in\mathbb{N}}$ *converges weakly* P-a.s. *to an* F*-valued random variable.*

Random block-coordinate primal-dual algorithm: Type II

⇝ When $(\forall j \in \{1,\ldots,p\})\ f_j = 0$

for $n = 0, 1, \ldots$
- for $j = 1, \ldots, p$
  - $\eta_{j,n} = \max\left\{\varepsilon_{p+k,n} \mid k \in \mathbb{L}_j^*\right\}$
  - $s_{j,n} = \eta_{j,n}\Big(x_{j,n} - \mathsf{W}_j\big(\nabla h_j(x_{j,n}) + a_{j,n}\big)\Big)$
  - $y_{j,n} = \eta_{j,n}\big(s_{j,n} - \mathsf{W}_j \sum_{k\in\mathbb{L}_j^*} L_{k,j}^* v_{k,n}\big)$
- for $k = 1, \ldots, q$
  - $u_{k,n} = \varepsilon_{p+k,n}\Big(\mathrm{prox}_{g_k^*}^{\mathsf{U}_k^{-1}}\Big(v_{k,n} + \mathsf{U}_k \sum_{j\in\mathbb{L}_k} L_{k,j} y_{j,n} - \mathsf{U}_k\big(\nabla l_k^*(v_{k,n}) + c_{k,n}\big)\Big) + b_{k,n}\Big)$
  - $v_{k,n+1} = v_{k,n} + \lambda_n \varepsilon_{p+k,n}(u_{k,n} - v_{k,n})$
- for $j = 1, \ldots, p$
  - $p_{j,n} = \varepsilon_{j,n}\Big(s_{j,n} - \mathsf{W}_j \sum_{k\in\mathbb{L}_j^*} L_{k,j}^* u_{k,n}\Big)$
  - $x_{j,n+1} = x_{j,n} + \lambda_n \varepsilon_{j,n}(p_{j,n} - x_{j,n}).$

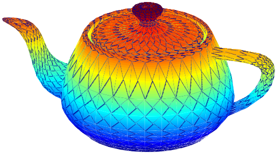
- Less restrictive conditions on $(\mathsf{W}_j)_{1\leqslant j\leqslant p}$ and $(\mathsf{U}_k)_{1\leqslant k\leqslant q}$:

$$\min\left\{(\|\mathsf{W}_j\|^{-1}\mu_j^{-1})_{1\leqslant j\leqslant p},\ \Big(1 - \sum_{j=1}^{p}\sum_{k=1}^{q} \|\mathsf{U}_k^{1/2} L_{k,j} \mathsf{W}_j^{1/2}\|^2\Big)(\|\mathsf{U}_k\|^{-1}\nu_k^{-1})_{1\leqslant k\leqslant q}\right\} > \frac{1}{2}.$$

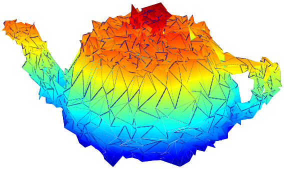
- Similar convergence results as for type I.

Application to 3D mesh denoising

Mesh denoising problem



Initial mesh $\overline{x}$



Observed mesh z

Undirected nonreflexive graph

OBJECTIVE:
Estimate $\overline{x} = (\overline{x}^{(i)})_{1 \leqslant i \leqslant M}$ from noisy observations $z = (z^{(i)})_{1 \leqslant i \leqslant M}$ where, for every $i \in \{1, \ldots, M\}$, $\overline{x}^{(i)} \in \mathbb{R}^3$ is the vector of 3D coordinates of the i-th vertex of a mesh

$\rightsquigarrow$ $\mathsf{H} = (\mathbb{R}^3)^M$

Mesh denoising problem

OBJECTIVE:

Estimate $\overline{x} = (\overline{x}^{(i)})_{1\leqslant i\leqslant M}$ from noisy observations $z = (z^{(i)})_{1\leqslant i\leqslant M}$ where, for every $i \in \{1,\ldots,M\}$, $\overline{x}^{(i)} \in \mathbb{R}^3$ is the vector of 3D coordinates of the i-th vertex of a mesh

COST FUNCTION:

$$\Phi(x) = \sum_{j=1}^{M} \psi_j(x^{(j)} - z^{(j)}) + \iota_{C_j}(x^{(j)}) + \eta_j \|(x^{(j)} - x^{(i)})_{i\in\mathcal{N}_j}\|_{1,2}$$

where, for every $j \in \{1,\ldots,M\}$,

- $\psi_j : \mathbb{R}^3 \to \mathbb{R}$: $\ell_2 - \ell_1$ Huber function
  - robust data fidelity measure
- C_j: nonempty convex subset of $\mathbb{R}^3$
  - box constraint
- $\mathcal{N}_j$: neighborhood of j-th vertex
- $(\eta_j)_{1\leqslant j\leqslant M}$: nonnegative regularization constants.

Mesh denoising problem

OBJECTIVE:

Estimate $\overline{x} = (\overline{x}^{(i)})_{1 \leqslant i \leqslant M}$ from noisy observations $z = (z^{(i)})_{1 \leqslant i \leqslant M}$ where, for every $i \in \{1, \dots, M\}$, $\overline{x}^{(i)} \in \mathbb{R}^3$ is the vector of 3D coordinates of the i-th vertex of a mesh

COST FUNCTION:

$$\Phi(x) = \sum_{j=1}^{M} \psi_j(x^{(j)} - z^{(j)}) + \iota_{C_j}(x^{(j)}) + \eta_j \|(x^{(j)} - x^{(i)})_{i \in \mathcal{N}_j}\|_{1,2}$$

IMPLEMENTATION DETAILS:

a block $\equiv$ a vertex $\quad \Rightarrow \quad p = M$

Algorithm type I:

$(\forall j \in \{1, \dots, M\})$

- $h_j = \psi_j(\cdot - z^{(j)})$
- $f_j = \iota_{C_j}$

$q = M$

$(\forall k \in \{1, \dots, M\})(\forall x \in \mathsf{H})$

- $g_k(L_k x) = \|(x^{(k)} - x^{(i)})_{i \in \mathcal{N}_k}\|_{1,2}$
- $l_k = \iota_{\{0\}}$

Algorithm type II:

$(\forall j \in \{1, \dots, M\})$

- $h_j = \psi_j(\cdot - z^{(j)})$

$q = 2M$

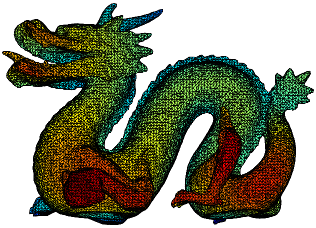
$(\forall k \in \{1, \dots, M\})(\forall x \in \mathsf{H})$

- $g_k(L_k x) = \|(x^{(k)} - x^{(i)})_{i \in \mathcal{N}_k}\|_{1,2}$
- $g_{M+k}(L_{M+k} x) = \iota_{C_k}(x^{(k)})$
- $l_k = \iota_{\{0\}}$

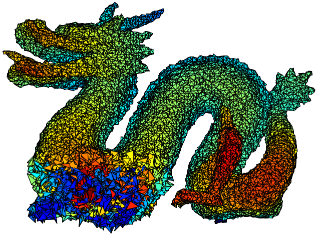
Simulation results (algorithm type I)

- $\{1,\ldots,M\} = \mathcal{V}_1 \cup \mathcal{V}_2$ with $|\mathcal{V}_1| > |\mathcal{V}_2|$.
- additive independent noise with distribution
  $\mathcal{V}_1 \rightsquigarrow \mathcal{N}(0,\sigma_1^2)$,
  $\mathcal{V}_2 \rightsquigarrow \pi\,\mathcal{N}(0,\sigma_2^2) + (1-\pi)\,\mathcal{N}(0,(\sigma_2')^2)$, $\pi \in (0,1)$.
- probability of variable activation

$$(\forall j \in \{1,\ldots,M\})(\forall n \in \mathbb{N}) \quad \mathsf{P}(\varepsilon_{j,n} = 1) = \begin{cases} \mathsf{p} & \text{if } j \in \mathcal{V}_1 \\ 1 & \text{otherwise.} \end{cases}$$

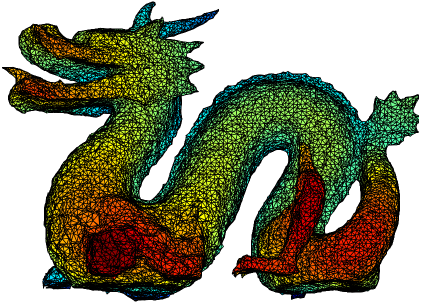


Original mesh,
$M = 22998$, $|\mathcal{V}_1| = 18492$ and $|\mathcal{V}_2| = 4506$.



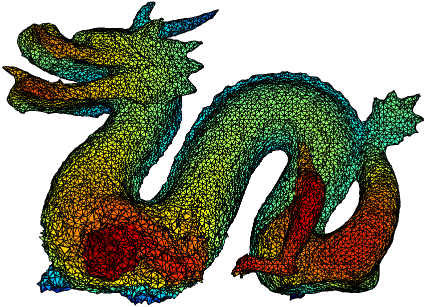
Noisy mesh,
MSE = 1.08×10^{-6}.

Simulation results (algorithm type I)



Proposed reconstruction,

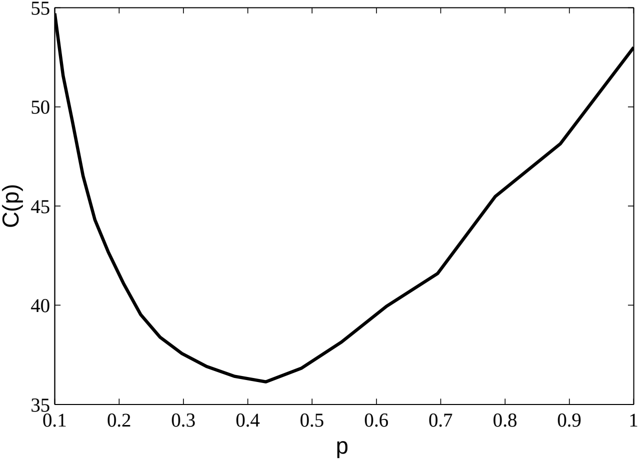
MSE = 2.19×10^{-7}.



Laplacian smoothing,

MSE = 2.95×10^{-7}.

Complexity (algorithm type I)

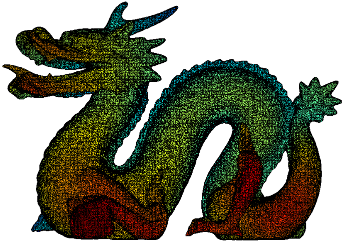


Simulation results (algorithm type II)

- positions of the original mesh are corrupted through an i.i.d zero-mean Gaussian mixture noise model.
- a limited number r of variables can be handled at each iteration, where

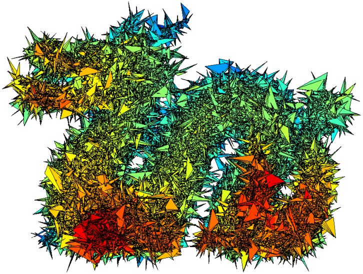
$$\sum_{j=1}^{p} \varepsilon_{j,n} = r \leqslant p.$$

- mesh decomposed into p/r non-overlapping sets.



Original mesh,

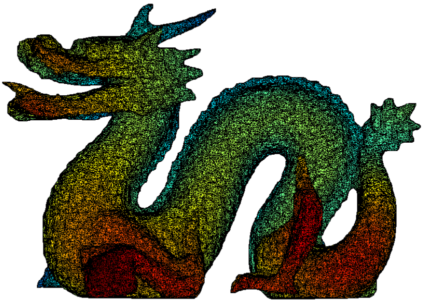
$M = 100250.$



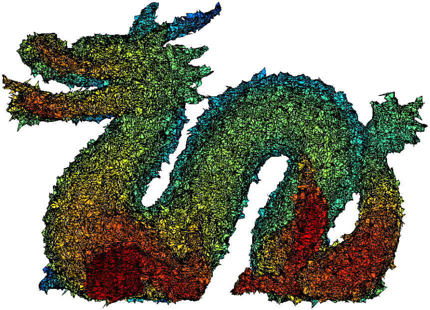
Noisy mesh,

MSE = 2.89×10^{-6}.

Simulation results (algorithm type II)

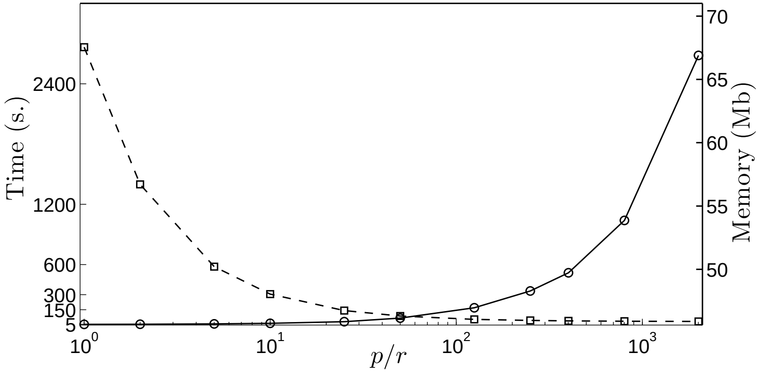


Proposed reconstruction,
MSE = 8.09×10^{-8}.



Laplacian smoothing,
MSE = 5.23×10^{-7}.

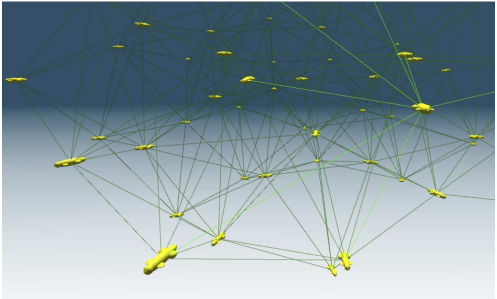
Complexity (algorithm type II)



- dashed line: required memory
- continuous line: reconstruction time

Available extensions

Asynchronous distributed algorithms
(stochastic, primal-dual, proximal, defined on a hypergraph)
[Pesquet and Repetti - 2014]



Conclusion

- No linear operator inversion.
- Existing deterministic parallel proximal primal-dual algorithms recovered when $p = 1$ and $\varepsilon_0 = (1, \dots, 1)$.
- Flexibility in the random activation of primal/dual components.
  $\rightsquigarrow$ *arbitrary sampling*
- Possibility to address other graph processing problems than denoising
  [Couprie et al.,2013]

Some references

P. L. Combettes and J.-C. Pesquet
Proximal splitting methods in signal processing
in *Fixed-Point Algorithms for Inverse Problems in Science and Engineering*, H. H. Bauschke, R. Burachik, P. L. Combettes, V. Elser, D. R. Luke, and H. Wolkowicz editors. Springer-Verlag, New York, pp. 185-212, 2011.

C. Couprie, L. Grady, L. Najman, J.-C. Pesquet, and H. Talbot
Dual constrained TV-based regularization on graphs
SIAM Journal on Imaging Sciences, vol. 6, no 3, pp. 1246-1273, 2013.

P. L. Combettes, L. Condat, J.-C. Pesquet, and B. C. Vũ
A forward-backward view of some primal-dual optimization methods in image recovery
IEEE International Conference on Image Processing (ICIP 2014), 5 p., Paris, France, Oct. 27-30, 2014.

P. Combettes and J.-C Pesquet
Stochastic quasi-Fejér block-coordinate fixed point iterations with random sweeping
to appear in *SIAM Journal on Optimization*, 2015.

N. Komodakis and J.-C. Pesquet
Playing with duality: An overview of recent primal-dual approaches for solving large-scale optimization problems
to appear in *IEEE Signal Processing Magazine*, 2015.

J.-C. Pesquet and A. Repetti
A class of randomized primal-dual algorithms for distributed optimization
to appear in *Journal of Nonlinear and Convex Analysis*, 2015.

A. Repetti, E. Chouzenoux and J.-C. Pesquet
A random block-coordinate primal-dual proximal algorithm with application to 3D mesh denoising
IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP 2015).