Combinatorics on words: Introduction

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Aix-Marseille Université, September 2020

Alphabets and words

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ε is the empty word of length 0.

Classic theorem: Fine and Wilf

Theorem

Let w be a finite word with periods p and q, of length

$$|w| \geq p + q - (p, q).$$

Then w is also periodic with period (p, q).

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$$abaababaaba$$

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There exists a word over the ternary alphabet which does not contain two consecutive equal factors.

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No *abcabc* and in general, no XX for any finite X.

Connections

I. Algebra

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We can also consider other semigroups and groups.

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Theorem (Novikov, Adian, 1968)

For every odd number n with $n > 4381$, there exist infinite, finitely generated groups of exponent n.

Exponent of a group G is the least n such that $g^n = 1$ for all $g \in G$.

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Theorem (Novikov, Adian, 1968)

For every odd number n with $n > 4381$, there exist infinite, finitely generated groups of exponent n.

Exponent of a group G is the least n such that $g^n = 1$ for all $g \in G$.

This is the solution of the *bounded Burnside problem for groups*, and the proof uses the existence of a square-free word over a finite alphabet.

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II.a Symbolic dynamics

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Code of a trajectory of a point.

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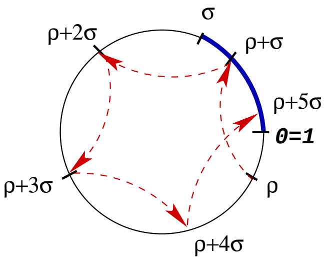

$w = 0{\color{blue}1}000{\color{blue}1}\cdots$

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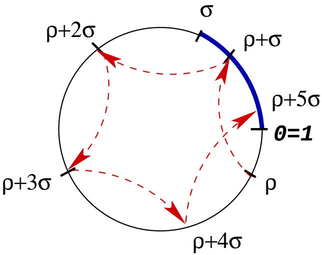

$w = 010001\cdots$

Rotation words are periodic or *Sturmian*.

Connections

II.b Discrete dynamics: shifts spaces

The shift operator σ:

$$\sigma(a[0]a[1]a[2]\cdots a[n]\cdots) = a[1]a[2]a[3]\cdots a[n+1]\cdots$$

Shift space = a closed set of infinite words invariant under σ

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Shift space = a closed set of infinite words invariant under σ

Many properties of a *uniformly recurrent* infinite word depend only on its subshift.

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III. Discrete geometry

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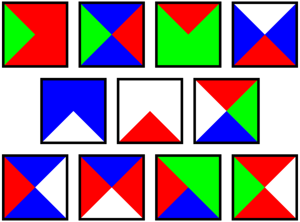
Example: Wang tiles



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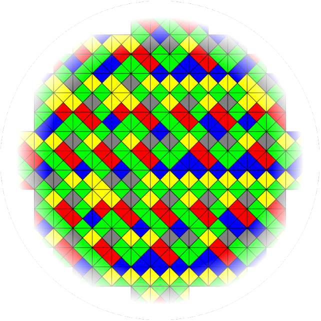

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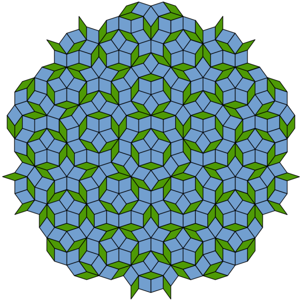
Another example: Penrose tilings



Connections

IV. Number theory
Decimal (or k-ary) expansions of irrational numbers are infinite words.
What are their properties?

$$\sqrt{2} = 1.41421356237309504880168872420969807856967187 5 \cdots$$

Does 7 occur in this word an infinite number of times?

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$$\sqrt{2} = 1.4142135623730950488016887242096980785696718750\cdots$$

Does 7 occur in this word an infinite number of times?

NOBODY KNOWS

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Conjecture

All k-ary expansions of irrational algebraic numbers are normal, meaning that every finite pattern of length n occurs in this expansion with the expected limiting frequency k^n.

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This conjecture is not proven for *any* irrational algebraic number.

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IV. Number theory

Theorem (Adamczewski and Bugeaud, 2007)

The complexity function $p(n)$, defined as the number of different patterns of length n, of the k-ary expansion of every irrational algebraic number satisfies

$$\liminf_{n\to\infty} \frac{p(n)}{n} = \infty.$$

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The proof uses combinatorics on words.

Other connections

- Formal languages

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- Formal languages
- Algorithms on strings

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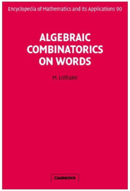
- Formal languages
- Algorithms on strings
- Applied algorithms: bioinformatics etc.

Main sources

M. Lothaire, *Algebraic Combinatorics on Words*. Cambridge Univ. Press, 2002.



available online

Main sources

Jean-Paul Allouche, Jeffrey Shallit, *Automatic Sequences — Theory, Applications, Generalizations*. Cambridge Univ. Press, 2003.