Combinatorics on words: Automatic words by examples

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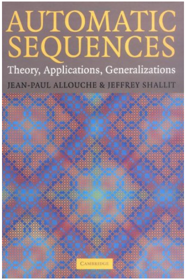
The main source

Jean-Paul Allouche, Jeffrey Shallit, *Automatic Sequences — Theory, Applications, Generalizations*. Cambridge Univ. Press, 2003.



Thue-Morse word

$$\mu : \begin{cases} 0 \to 01, \\ 1 \to 10. \end{cases}$$

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Thue-Morse automaton

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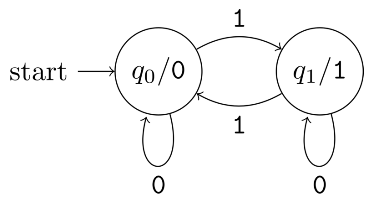


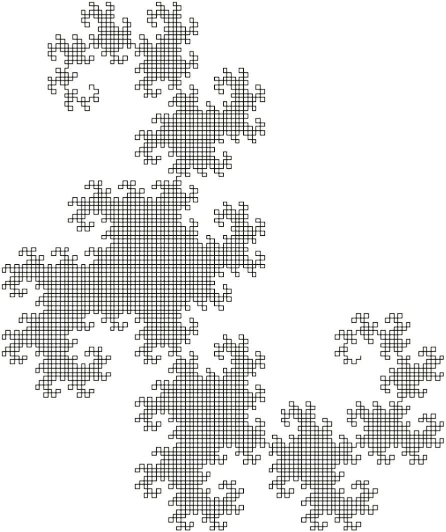
The paperfolding word

$$\mathbf{w} = w[1]w[2]w[3]\cdots \in \{1,-1\}^\omega$$

$$P = 1 \diamond -1 \diamond$$

$w(0)$	=	◇	◇	◇	◇	◇	◇	◇	◇	◇	◇	◇	◇	◇	◇	◇
$w(1)$	=	1	◇	-1	◇	1	◇	-1	◇	1	◇	-1	◇	1	◇	-1
$w(2)$	=	1	1	-1	◇	1	-1	-1	◇	1	1	-1	◇	1	-1	-1
$w(3)$	=	1	1	-1	1	1	-1	-1	◇	1	1	-1	-1	1	-1	-1
$\mathbf{w}$	=	1	1	-1	1	1	-1	-1	1	1	1	-1	-1	1	-1	-1

Dragon curve



Paperfolding word over four letters

$\mathbf{w}$=1 1 -1 1 1 -1 -1 1 1 1 -1 -1 1 -1 -1 $\cdots$

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$a = 1$ at an even position;
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$$\varphi : \begin{cases} a \to ab \\ b \to cb \\ c \to ad \\ d \to cd \end{cases}$$

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$\mathbf{w} = \psi(\varphi^\omega(a))$, where $\psi : a, b \to 1; c, d \to -1$.

Cobham theorem

Theorem (Cobham,1972)

An infinite word $\mathbf{w}$ *is* k*-automatic if and only if* $\mathbf{w} = \psi(\varphi^\omega(a))$ *for some* a*, where* φ *is a* k*-uniform morphism and* ψ *is a coding.*