Combinatorics on words: Properties of automatic words

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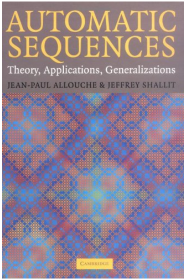
The main source

Jean-Paul Allouche, Jeffrey Shallit, *Automatic Sequences — Theory, Applications, Generalizations*. Cambridge Univ. Press, 2003.



Formal definition

$$\Sigma_k = \{0, \ldots, k-1\}.$$

Definition

An infinite word $\mathbf{u} = u[0]u[1] \cdots u[n] \cdots$ over an alphabet Δ is *k-automatic* for a $k > 1$, if there exists a DFAO $A = (Q, \Sigma_k, \delta, q_0, \Delta, \lambda)$ such that $u[n] = \lambda(\delta(q_0, a))$ for all $n \geq 0$ and for all k-ary representations $a \in \Sigma_k^*$ of n.

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So, the input of A is a k-ary representation of n and the output is $u[n]$.

Cobham theorem

Theorem (Cobham,1972)

An infinite word $\mathbf{w}$ *is* k*-automatic if and only if* $\mathbf{w} = \psi(\varphi^\omega(a))$ *for some* a*, where* φ *is a* k*-uniform morphism and* ψ *is a coding.*

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- A word $\mathbf{u}$ over an alphabel Δ is k-automatic iff for all $a \in \Delta$, the language $F_a = \{(n)_k | u[n] = a\}$ is regular.
- An infinite word obtained from a k-automatic word by changing a finite number of symbols is k-automatic.

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Lemma

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PROOF. It is sufficient to consider a pure period v (that is, $u = \varepsilon$).

$$Q = \{0, 1, \ldots, t-1\};$$

$$\delta(q, b) = (kq + b) \pmod{t};$$

$$\tau(q) = w[q].$$

Another Cobham theorem

Integers k and l are called *multiplicatively dependent* if $k^r = l^s$ for some integer r, s.

Theorem (Cobham, 1969)

Let k and l be multiplicatively independent integers. Then an infinite word which is k- and l-automatic is ultimately periodic.

k-kernel

The *k-kernel* of an infinite word $\mathbf{w} = w[0] \cdots w[n] \cdots$ is the set of infinite words of the form

$$\mathbf{w}_m^i = w[i]w[i + k^m] \cdots w[i + nk^m] \cdots ,$$

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Theorem

A word is k-automatic if and only if its k-kernel is finite.

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This is one of tools to prove that an infinite word is *not* automatic.

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- Find a non-periodic sequence $\{w_{[uv^i]_k}\}_{i=1}^{\infty}$;
- Find a symbol of irrational frequency;
- Sometimes it is not clear!

Decidability and enumeration

Theorem (Charlier, Rampersad, Shallit, 2011)

If we can express a property of a k-automatic sequence $\mathbf{w}$ using quantifiers, logical operations, integer variables, the operations of addition, subtraction, indexing into $\mathbf{w}$, and comparison of integers or elements of $\mathbf{w}$, then this property is decidable.

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Many sequences related to k-automatic words are *k-regular*.

Examples: number of factors of length n; number of palindromes of a given length; indices where these palindromes start, etc.

Walnut software

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So, the most interesting questions about k-automatic words are now those which cannot be solved by it!