Sturmian words: equivalent definitions

Anna FRID

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A word $\mathbf{u}$ of complexity $p_{\mathbf{u}}(n) = n+1$ is called *Sturmian*.

Fibonacci word

Example (Fibonacci morphism)

$$\varphi(0) = 01, \varphi(1) = 0$$

$$0 \rightarrow 01 \rightarrow 01\ 0 \rightarrow 010\ 01 \rightarrow 01001\ 010 \rightarrow 01001010\ 01001 \rightarrow \cdots$$

Its fixed point is the *Fibonacci word*

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Lemma

The Fibonacci word is Sturmian.

Balanced words

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$$\delta(x, y) = ||x|_1 - |y|_1| \leq 1.$$

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- In a balanced set of factors of length n, there are at most $n+1$ elements;
- A set of factors F is not balanced $\iff$ there exists a strong bispecial $w|\ 0w0, 1w1 \in F$.

One- and two-sided words

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Attention:

$$\cdots 00000010000000 \cdots$$

is not considered to be Sturmian, even though its complexity is $n+1$. It is two-sided and "half-periodic".

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A balanced word is periodic if and only if its slope is rational.

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Example

The slope of the Fibonacci word $0\ 1\ 0\ 01\ 010\ 01001\ 01001010 \cdots$ is

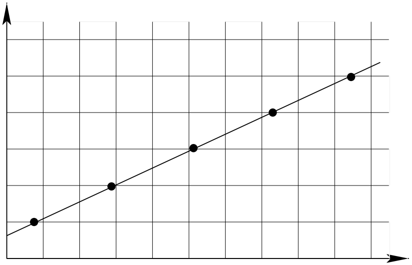
$$\lim_{n\to\infty} \frac{|\varphi^n(0)|_1}{|\varphi^n(0)|} = \lim_{n\to\infty} \frac{F_{n-2}}{F_n} = \frac{1}{\tau^2},$$

where $\tau = (1+\sqrt{5})/2$.

$$\frac{1}{\tau^2} = 0,38 \cdots .$$

Mechanical words

$$y = \sigma x + \rho, \quad 0 \leq \sigma, \rho < 1.$$

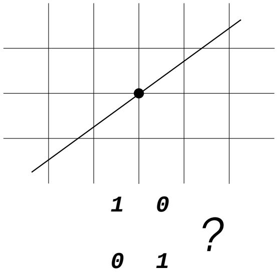


1 0 1 0 0 1 0 1 0 1

$$\mathbf{w} = w[0]w[1]\cdots$$

$$w[n] = \lfloor (n+1)\sigma + \rho \rfloor - \lfloor n\sigma + \rho \rfloor.$$

Important choice



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Mechanical words

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An infinite word $\mathbf{w} = w[0]w[1]\cdots$ over $\{0,1\}$ is *mechanical*, if for all $n \geq 0$ we have

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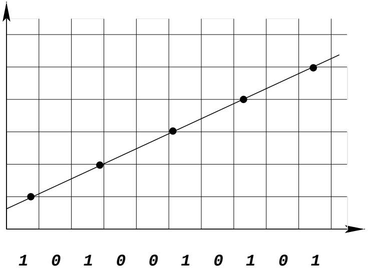

Three equivalent definitions

Theorem

For a right infinite word $\mathbf{x}$ *over* $\{0,1\}$*, the following conditions are equivalent:*

- $p_{\mathbf{x}}(n) = n+1 \quad \forall n$;
- x *is a non-periodic balanced word;*
- x *is a mechanical word with an irrational slope* σ.

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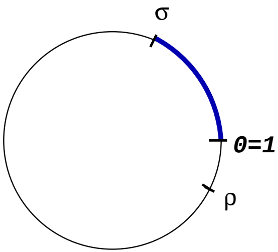
- x *is a mechanical word with an irrational slope* σ.

If any of the conditions holds, the word is Sturmian.

Mechanical words and rotations



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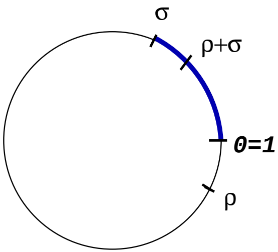

$\mathbf{w} = 1 \cdots$

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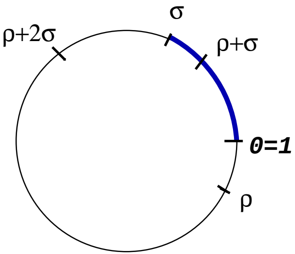
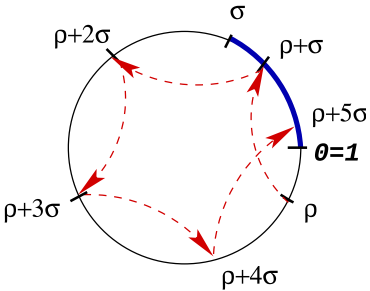


$\mathbf{w} = 10\cdots$

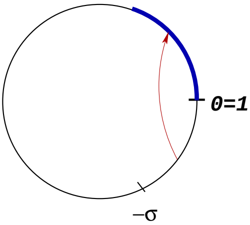
Mechanical words and rotations



$\mathbf{w} = 10001\cdots$

Complexity of rotation words

$w[0] = 1 \iff 1 - \sigma < \rho < 1$

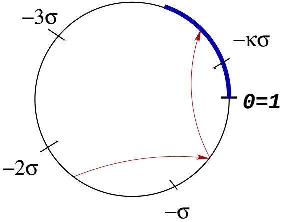


The first symbol is determined by one of two intervals where ρ is located

$p_{\mathbf{w}}(1) = 2$

Complexity of rotation words

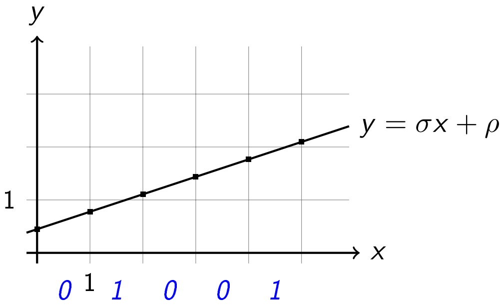
$w[k] = 1 \iff -k\sigma < \rho < -(k+1)\sigma$



The prefix of length k is determined by one of $k+1$ intervals where ρ is located

$p_w(k) = k + 1.$

Mechanical vs. billiard definition



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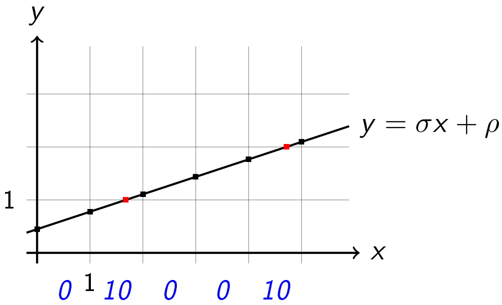
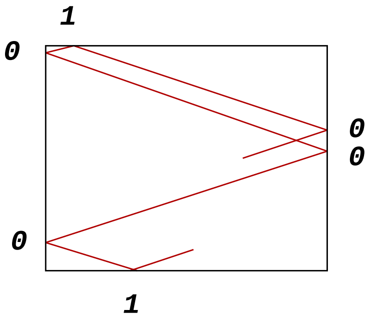
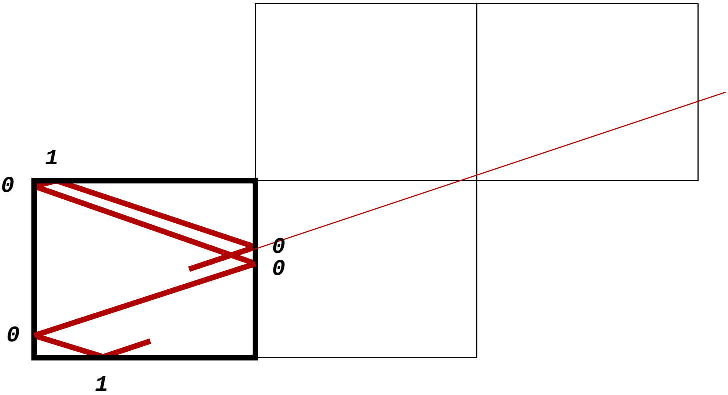


Billiards



$$010001\cdots$$

Billiard words are Sturmian



$$010001\cdots$$

Properties of Sturmian words

Lemma

A Sturmian word is never k-automatic.

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PROOF. The frequency of 1 in a Sturmian word is irrational (and equal to the slope). In a k-automatic word, this frequency is rational.

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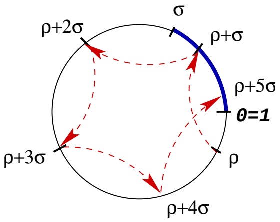

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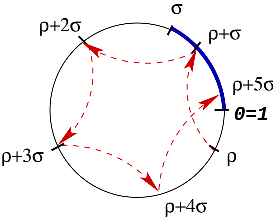

So, for many arguments we may take $\rho = \sigma$. Such Sturmian words are *characteristic*.

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Let

$$\sigma = \cfrac{1}{m_1 + 1 + \cfrac{1}{m_2 + \cfrac{1}{m_3 + \cfrac{1}{m_4 + \cdots}}}} = [0, m_1 + 1, m_2, m_3, \cdots].$$

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Then $c_\sigma = \lim_{n\to\infty} s_n$, where

$$s_{-1} = 1, s_0 = 0, \qquad s_n = s_{n-1}^{m_n} s_{n-2}.$$

Example: the Fibonacci word

The slope of the Fibonacci word is

$$\frac{1}{\tau^2} = [0, 2, 1, 1, 1, 1, \cdots], \text{ where } \tau = \frac{1+\sqrt{5}}{2}.$$

$$\begin{aligned} s_{-1} &= 1 \\ s_0 &= 0 \\ s_1 &= 01 \\ s_2 &= 01\ 0 \\ s_3 &= 010\ 01 \\ s_4 &= 01001\ 010 \end{aligned}$$

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So, it is indeed the Fibonacci word.