Total number of Sturmian factors

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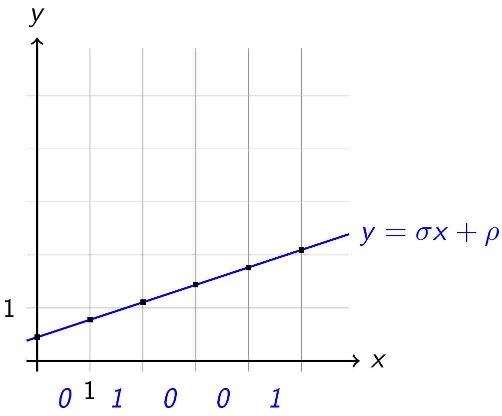
Aix-Marseille Université, September 2020

Construction of Sturmian words

$0 \leq \sigma, \rho < 1$

$\sigma$is irrational



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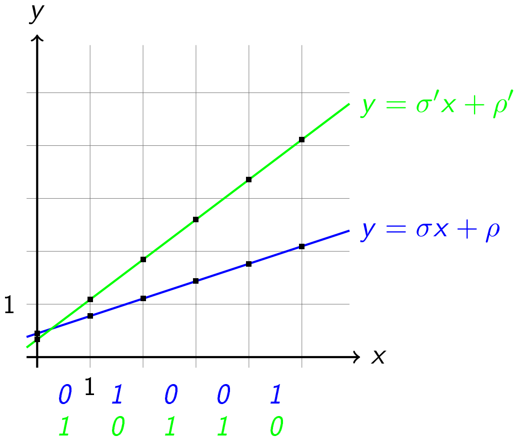

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https://arxiv.org/abs/1901.01952

Mechanical definition

Definition

An infinite word $\mathbf{w} = w[0]w[1]\cdots$ over $\{0,1\}$ is Sturmian if and only if for all $n \geq 0$ we have

$$w[n] = \lfloor (n+1)\sigma + \rho \rfloor - \lfloor n\sigma + \rho \rfloor$$

or

$$w[n] = \lceil (n+1)\sigma + \rho \rceil - \lceil n\sigma + \rho \rceil$$

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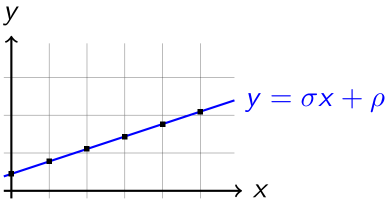

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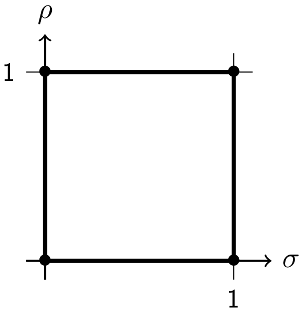
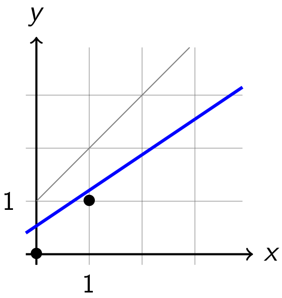
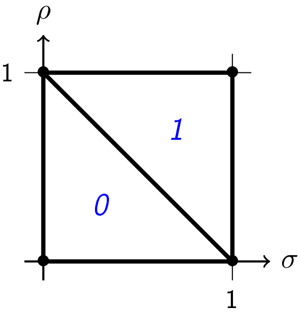


Arrangement of order 1

$\mathbf{w}[0] = \lfloor \sigma + \rho \rfloor - \lfloor \rho \rfloor = 1$ if and only if $\sigma + \rho > 1$.

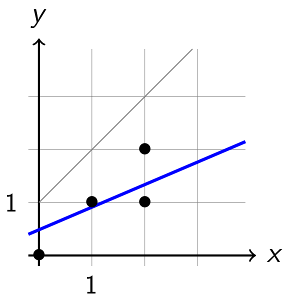


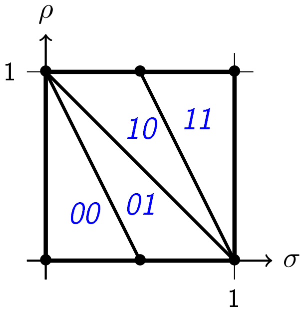


Arrangement of order 2

$\mathbf{w}[1] = \lfloor 2\sigma + \rho \rfloor - \lfloor \sigma + \rho \rfloor = 1$ if and only if

$$\begin{cases} 0 < \sigma + \rho < 1, 1 < 2\sigma + \rho < 2, & \text{or} \\ 1 < \sigma + \rho < 2, 2 < 2\sigma + \rho < 3. \end{cases}$$





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$\mathbf{w}[1] = \lfloor 2\sigma + \rho \rfloor - \lfloor \sigma + \rho \rfloor = 1$ if and only if

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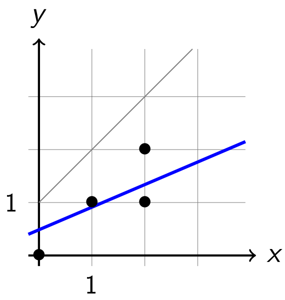
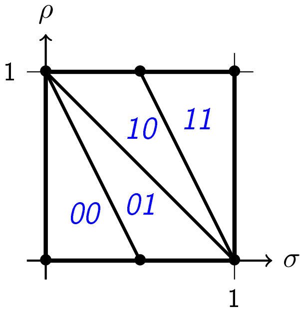
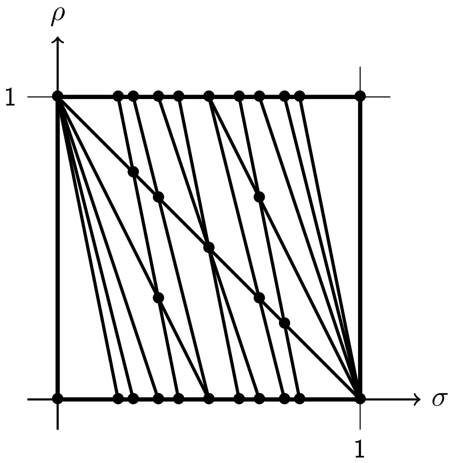




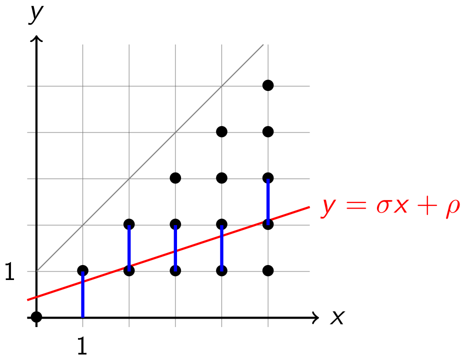
Arrangement of a high order

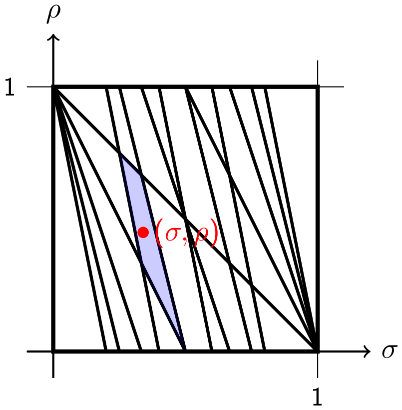
Lines in the square: $k\sigma + \rho = l$, where $0 < l \leq k \leq n$.



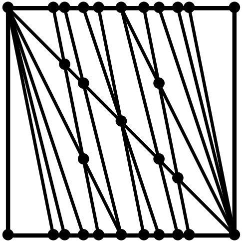
Duality

lines $\longleftrightarrow$ points, points $\longleftrightarrow$ lines
faces $\longleftrightarrow$ Sturmian words





Euler's formula for planar graphs



$$v - e + f = 1$$

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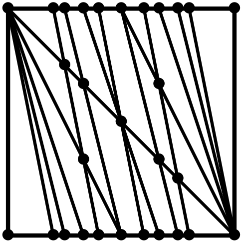

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$$2e = \sum_{t \in V} deg(t)$$

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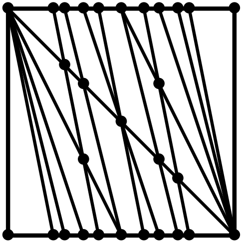
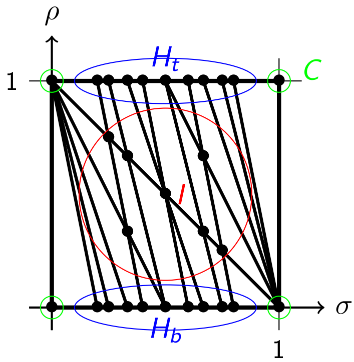


$$v - e + f = 1$$

$$2e = \sum_{t \in V} deg(t)$$

$$p_s(n) = f = 1 + e - v = 1 + \sum_{t \in V} \left(\frac{deg(t)}{2} - 1 \right).$$

Four groups of vertices



$$p_s(n) = 1 + \sum_{t \in C} \left(\frac{deg(t)}{2} - 1 \right) + \sum_{t \in H_t \cup H_b} \left(\frac{deg(t)}{2} - 1 \right) + \sum_{t \in I} \left(\frac{deg(t)}{2} - 1 \right)$$

$$= 1 + n + 2 \sum_{t \in H_b} \left(\frac{deg(t)}{2} - 1 \right) + \sum_{t \in I} \left(\frac{deg(t)}{2} - 1 \right).$$

Formula with the number of lines

$$p_s(n) = 1 + n + 2\sum_{t\in H_b}\left(\frac{deg(t)}{2} - 1\right) + \sum_{t\in I}\left(\frac{deg(t)}{2} - 1\right)$$
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But where do lines intersect?

Intersections

Suppose that $0 \leq k' < k \leq n$.

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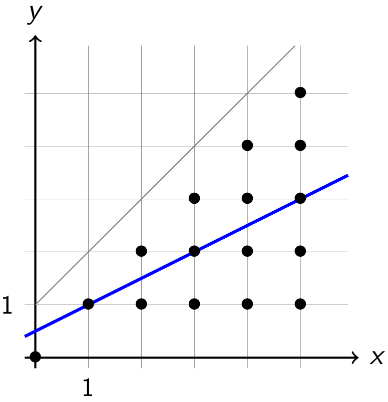
All intersection points are of the form

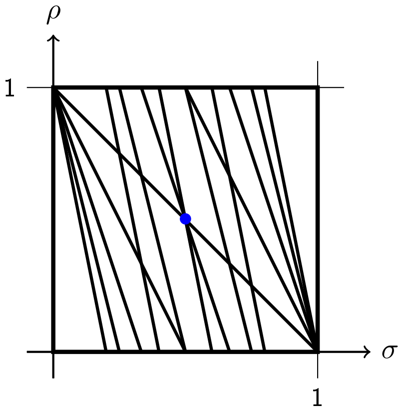
$$\left(\frac{L}{K}, \frac{M}{K} \right),$$

where $1 < K \leq n$, $1 \leq L < K$, $0 \leq M < K$.

How many lines at one point?

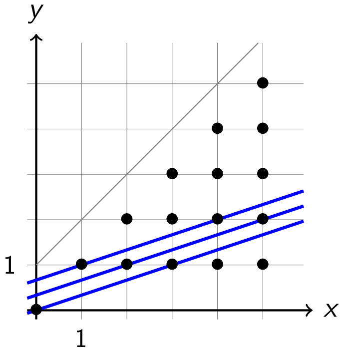
As many as integer points on the dual line.

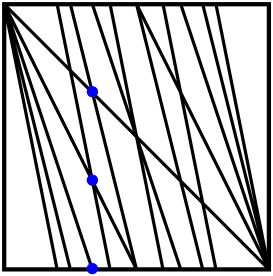




How many lines at points with the same abscissa?

For $\frac{L}{K}$: $n+1$ lines at K points





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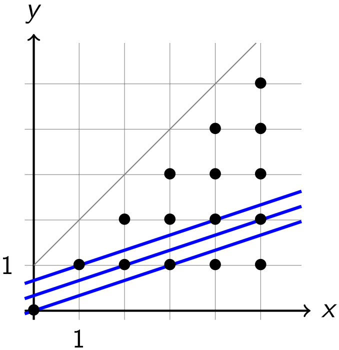

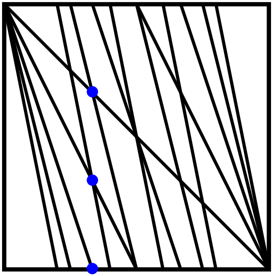

So, we can sum over abscissas L/K, where $2 \leq K \leq n$, $0 \leq L < K$; that is, over the Farey series.

The final formula

We sum over the abscissas $\frac{L}{K}$; the contribution of each of them to the sum is $n+1-K$.

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$$\begin{aligned} p_s(n) &= 1+n+\sum_{t\in H_b\cup I}(c(t)-1) \\ &= 1+n+\sum_{K=2}^{n}\varphi(K)(n+1-K) \\ &= 1+\sum_{K=1}^{n}\varphi(K)(n+1-K), \end{aligned}$$

where $\varphi(K)$ is the Euler's totient function $=$
$\#$ (positive integers $\leq K$ relatively prime to K)

Asymptotics

$$p_s(n) = 1 + \sum_{K=1}^{n} \varphi(K)(n+1-K) = \frac{n^3}{\pi^2} + O(n^2 \log n).$$

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- Always publish important results in English. Make them accessible for everyone.