The Ellis semigroup of bijective substitution shifts

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Semigroup basics

A semigroup is a set S with an associative binary operation, which we denote multiplicatively. Some of the semigroups in this talk have an identity element, but some do not. However they will never have a 0 element.

An idempotent $p \in S$ is an element satisfying $pp = p$.

A (left, right, or bilateral) ideal of S is a nonempty subset $I \subseteq S$ satisfying $SI \subseteq I$, $IS \subseteq I$, or $SI \cup IS \subseteq I$ respectively.

(Left, right, or bilateral) ideals are ordered by inclusion. A *minimal* (left, right, or bilateral) ideal is a minimal element w.r.t. this order.

The idempotent p is *minimal* if it belongs to a minimal ideal.

The *kernel* $\mathcal{M}(S)$ of S is

$$\mathcal{M}(S) := \bigcap_{I : I \text{ ideal}} I$$

A semigroup S is called *simple* if $S = \mathcal{M}(S)$.

Completely simple semigroups; Rees's theorem

A *completely simple* semigroup is a simple semigroup which has minimal idempotents.

Let G be a group, let I and Λ be non-empty sets, and let $A = (a_{\lambda i})_{\lambda \in \Lambda, i \in I}$ be a $\Lambda \times I$ matrix with entries from G. Then the *matrix semigroup* $M[G; I, \Lambda; A]$ is the set $I \times G \times \Lambda$ together with the multiplication

$$(i, g, \lambda)(j, h, \mu) = (i, g a_{\lambda j} h, \mu).$$

An idempotent is of the form

$$(i, a_{\lambda i}^{-1}, \lambda),$$

and $\{i\} \times G \times \{\lambda\}$ is a subsemigroup of $M[G; I, \Lambda; A]$ which is a group whose identity element is $(i, a_{\lambda i}^{-1}, \lambda)$. In particular, S is a disjoint union of groups which are all isomorphic.

Theorem (Rees-Suskevitch)

A semigroup is completely simple if and only if it is isomorphic to a matrix semigroup $M[G; I, \Lambda; A]$ for some group G.

Dynamics setting

(X,T) is a topological dynamical system, where

- X is a compact metric space (here a Cantor space, $X \subset \mathcal{A}^{\mathbb{Z}}$)
- $T : X \to X$ is a homeomorphism (here the shift map σ), so that T defines a $\mathbb{Z}$-action on X.

Definition

The *Ellis semigroup* $E(X)$ of a topological dynamical system (X,T) is the compactification of the $\mathbb{Z}$-action in the topology of pointwise convergence on X^X.

In other words $f : X \to X$ belongs to $E(X)$ iff $f = \lim_k T^{n_k}$ for some net n_k, with the topology of pointwise convergence.

Facts about the Ellis semigroup $E(X)$

- The Bourgain-Fremlin-Talagrand dichotomy:
  Either
  $|E(X)| \leq 2^{\aleph_0}$ in which case (X,T) is called *tame*,
  or
  $|E(X)| = 2^{2^{\aleph_0}}$. This talk is about nontame systems.
- $E(X)$ is a right topological compact semigroup, so by Ellis-Nakamura/Ruppert theorems,
  $E(X)$ admits a kernel $\mathcal{M}(X)$ which contains all minimal idempotents, so that $\mathcal{M}(X) \cong$ a matrix semigroup.
- Two points x and y are proximal in X if there is (n_k) with $d(T^{n_k}x, T^{n_k}y) \to 0$.
  - x and y are proximal if and only if there exists a minimal idempotent p such that $p(x) = p(y)$.
  - If T acts minimally and x, y are proximal then there is a minimal idempotent q such that $y = q(x)$ (and so $y = q(y)$).

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Bijective substitution shifts

A *substitution of length ℓ* is a morphism $\theta : \mathcal{A} \to \mathcal{A}^\ell$.
There are ℓ maps $\theta_i : \mathcal{A} \to \mathcal{A}$, $0 \leq i \leq \ell - 1$, such that

$$\theta(a) = \theta_0(a) \cdots \theta_{\ell-1}(a)$$

for each $a \in \mathcal{A}$.
θ is *bijective* if each of the maps θ_i is a bijection.
A finite word is *allowed* for θ if it appears somewhere in $\theta^k(a)$ for some $a \in \mathcal{A}$ and some $k \in \mathbb{N}$.
The *substitution shift* (X_θ, σ) is the dynamical system where the space X_θ consists of all bi-infinite sequences all of whose subwords are allowed for θ, and σ is the left shift map.
We equip X_θ with the subspace topology of the product topology on $\mathcal{A}^\mathbb{Z}$, making σ a continuous $\mathbb{Z}$-action.

Theorem (Barge-Kellendonk, 2019)

Let θ be a nontrivial primitive bijective substitution of length ℓ which defines the substitution shift (X_θ, σ). Then

$$E(X_\theta) = \mathcal{M}(X_\theta) \cup \mathbb{Z}.$$

Now the kernel $\mathcal{M}(X_\theta)$ is completely simple. What is its Rees semigroup representation?

Theorem (Kellendonk-Y, 2020)

Let (X_θ, σ) be a nontrivial primitive length-ℓ bijective substitution shift. There exist finite groups G_θ, Γ_θ and a finite set I_θ such that, algebraically,

- *if θ has trivial generalised height then $\mathcal{M}(X_\theta) \cong M[\mathcal{G}; I_\theta, \{\pm\}; A]$ where $\mathcal{G} = G_\theta^{\mathbb{Z}_\ell} \rtimes \mathbb{Z}_\ell$.*
- *if θ has generalised height $= h$, and G_θ contains an element of order h, then $\mathcal{M}(X_\theta) \cong M[\mathcal{G}; I_\theta, \{\pm\}; A]$ where $\mathcal{G} = (\overline{\Gamma}_\theta^{\mathbb{Z}_\ell} \rtimes \mathbb{Z}/h\mathbb{Z}) \rtimes \mathbb{Z}_\ell$.*

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Group factors of (X_θ, σ)

If θ has length ℓ, then recognizability implies that for each n and each $x \in X_\theta$, there is a unique $y \in X_\theta$ and $0 \le k < \ell^n$ such that

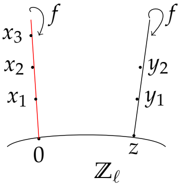
$$x = \sigma^k(\theta^n(y)).$$

Thus there is a factor map $\pi : (X_\theta, \sigma) \to (\mathbb{Z}_\ell, +1)$.
Note that π sends θ-fixed points to 0.
Using π we get a short exact sequence

$$E^{fib} \hookrightarrow E(X_\theta) \overset{\tilde{\pi}}{\twoheadrightarrow} E(\mathbb{Z}_\ell) \cong \mathbb{Z}_\ell$$

where $\tilde{\pi}(fg) = \tilde{\pi}(f) + \tilde{\pi}(g)$ and $E^{fib} = \ker \tilde{\pi}$.

$f \in E^{fib}$ fixes fibres



If e is an idempotent in $E(X_\theta)$,
$\tilde{\pi}(e) = \tilde{\pi}(e^2) = \tilde{\pi}(e) + \tilde{\pi}(e)$
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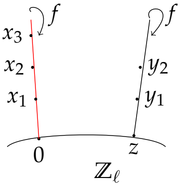

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How to find idempotents

Recall: x and y are proximal in X_θ if $\exists\ (n_k)$, $d(\sigma^{n_k}x, \sigma^{n_k}y) \to 0$.

Example

Consider the Thue-Morse substitution

$$\begin{array}{rcl} a & \mapsto & abba \\ b & \mapsto & baab \end{array}$$

There are four fixed points: $a \cdot a$, $a \cdot b$, $b \cdot a$, $b \cdot b$. We have

$a \cdot a$ and $b \cdot a$ are (right) proximal,

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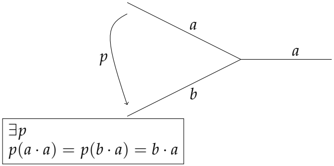
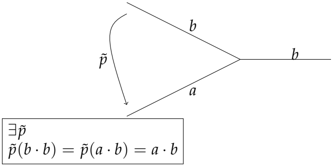
Conspiracies between potential idempotents

We are guaranteed idempotents p and $\tilde{p}$ with:





But it turns out that $p = \tilde{p}$:

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So if $\lim_{n_k} \sigma^{n_k} = p$ is such that

$$p(a \cdot a) = p(b \cdot a) = b \cdot a, \text{ then } p(a \cdot b) = p(b \cdot b) = a \cdot b.$$

But how does p behave elsewhere?

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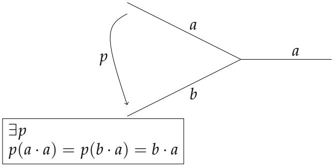

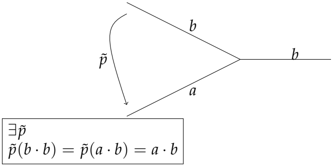

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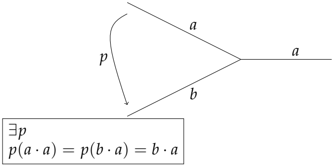

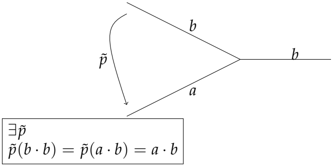

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The dearth of proximality for bijective substitutions

We have found p such that $p^2(x) = p(x)$ for fixed points.

Lemma

For bijective substitutions, the only way a pair of points can be proximal is if they are both in the shift orbit of $\{a \cdot a, a \cdot b, b \cdot a, b \cdot b\}$.

Recall:

- x and y are proximal if and only if there exists a minimal idempotent p such that $p(x) = p(y)$.

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Thue-Morse example, completed

There are 4 idempotents, with $p, \tilde{p}$ coming from forward proximality and $q, \tilde{q}$ from backward proximality:

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Define E_0^{fib} to be the restriction of E^{fib} to the fixed points.

Theorem (Kellendonk-Y,2019, specific to Thue-Morse)

The idempotents generate $E_0^{fib}\setminus \mathrm{Id}$.
Also, algebraically

- $E_0^{fib}\setminus \mathrm{Id} \cong M[G_\theta; S_2, \{\pm\}; A]$ *where* $G_\theta = S_2$.
- $E^{fib}\setminus \mathrm{Id} \cong M[\mathcal{G}^{fib}; S_2, \{\pm\}; A]$ *where* $\mathcal{G}^{fib} \cong G_\theta^{\mathbb{Z}_2/\mathbb{Z}}$;
- $E(X_\theta)\setminus\mathbb{Z} \cong M[\mathcal{G}; S_2, \{\pm\}; A]$ *where* $\mathcal{G} \cong \mathcal{G}^{fib} \rtimes \mathbb{Z}_2$.

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The structure group G_θ and the little structure group Γ_θ

Given a bijective substitution $\theta = \theta_0 \dots \theta_{\ell-1}$, we define

- the *structure group* G_θ of θ to be the group generated by all the bijections $(\theta^n)_i$, $n \in \mathbb{N}$, $i = 0, \cdots, \ell^n - 1$,
- its *R-set* by

$$I_\theta := \{(\theta^n)_i(\theta^n)_{i-1}^{-1} \in G_\theta : n \in \mathbb{N}, i = 1, \cdots, \ell^n - 1\}, \text{ and}$$

- the *little structure group* Γ_θ to be the group generated by

$$\{gh^{-1} : g, h \in I_\theta.\}$$

Example (Thue-Morse)

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \; G_\theta = I_\theta = \overline{\Gamma}_\theta = S_2.$$

Theorem (Kellendonk-Y,2020)

Let θ be a bijective primitive substitution of length ℓ. If $G_\theta = \overline{\Gamma}_\theta$, then the idempotents generate $E_0^{fib} \setminus \mathrm{Id}$, and

$$E(X_\theta)\backslash\mathbb{Z} = \mathcal{M}(X_\theta) \cong M[G_\theta^{\mathbb{Z}_\ell/\mathbb{Z}} \rtimes \mathbb{Z}_\ell; I_\theta, \{\pm\}; A].$$

Do the idempotents always generate $E_0^{fib} \setminus \mathrm{Id}$?

Example

Consider the substitution

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} b \\ a \\ c \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} c \\ b \\ a \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} a \\ c \\ b \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Columns of this substitution are either the identity, or transpositions of S_3. So $G_\theta = S_3$,

$$I_\theta = \left\{ \begin{pmatrix} b \\ a \\ c \end{pmatrix}, \begin{pmatrix} c \\ b \\ a \end{pmatrix}, \begin{pmatrix} a \\ c \\ b \end{pmatrix} \right\},$$ and $\Gamma_\theta = \langle \{ gh^{-1} : g, h \in I_\theta \} \rangle = A_3$

Here $\overline{\Gamma}_\theta = A_3 \neq S_3 = G_\theta$, so our previous theorem does not apply, and in fact the idempotents do not generate E_0^{fib}.

Generalised height

Lemma

$G_\theta/\overline{\Gamma}_\theta$ is a finite cyclic group.

The generalised height h of a primitive aperiodic bijective substitution is the order of $G_\theta/\overline{\Gamma}_\theta$.

Remark

Recall the definition of (classical) height h_{cl} of a primitive substitution of length ℓ : if $u = u_0u_1\ldots$ is a fixed point,

$$h_{cl} = gcd(\ell, \{n : u_n = u_0\}).$$

The generalised height is at least the classical height, but can be larger.

In general...

Theorem (Kellendonk-Y, 2020)

For a bijective substitution θ of length ℓ and generalised height h,

- $E_0^{fib} \backslash \mathrm{Id} \cong M[G_\theta; I_\theta, \{\pm\}; A]$,
- $E^{fib} \backslash \mathrm{Id} \cong M[\mathcal{G}^{fib}; I_\theta, \{\pm\}; A]$ *where if* G_θ *contains an element of order* h *then* $\mathcal{G}^{fib} \cong \overline{\Gamma}_\theta^{\mathbb{Z}_\ell/\mathbb{Z}} \rtimes \mathbb{Z}/h\mathbb{Z}$, *and*
- $E(X_\theta) \backslash \mathbb{Z} \cong M[\mathcal{G}; S_2, \{\pm\}; A]$ *where* $\mathcal{G}$ *is given by*

$$\mathcal{G}^{fib} \hookrightarrow \mathcal{G} \twoheadrightarrow \mathbb{Z}_\ell$$

In particular $\mathcal{G} \cong \mathcal{G}^{fib} \rtimes \mathbb{Z}_\ell$ *if the generalised height equals the classical height.*

Towards topological description of E^{fib}

Theorem (Kellendonk,Y, 2019)

If $\overline{\Gamma}_\theta = G_\theta$*, then there is a topological isomorphism*

$$E^{fib} \cong (M[G_\theta; I_\theta, \{\pm\}; A] \cup \{\mathrm{Id}\}) \;\times \prod_{\substack{[z]\in\mathbb{Z}_\ell/\mathbb{Z}\\ [z]\neq[0]}} G_\theta.$$

Otherwise

$$E^{fib} \cong (M[G_\theta; I_\theta, \{\pm\}; A] \cup \{\mathrm{Id}\}) \;\times \prod_{\substack{[z]\in\mathbb{Z}_\ell/\mathbb{Z}\\ [z]\neq[0]}} \overline{\Gamma}_\theta.$$

What next?

If θ is not bijective,

- it is not necessarily true that $E(X_\theta) = \mathcal{M}(X_\theta) \cup \mathbb{Z}$,
- there can be uncountably many proximal pairs,
- and many conspiracies, both between idempotents, and across fibres

How do we deal with this?