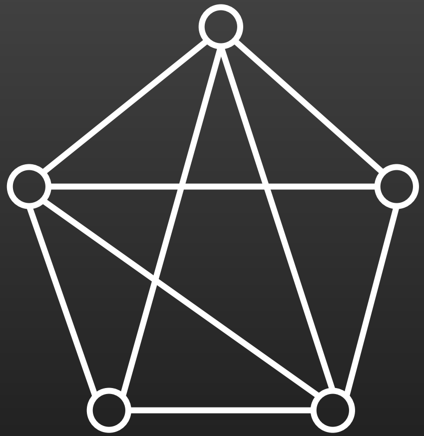
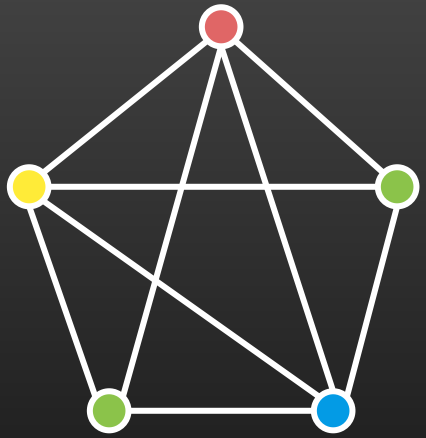
Graph Coloring and Combinatorics on Words

Jarosław Grytczuk

Warsaw University of Technology

Graph coloring problem





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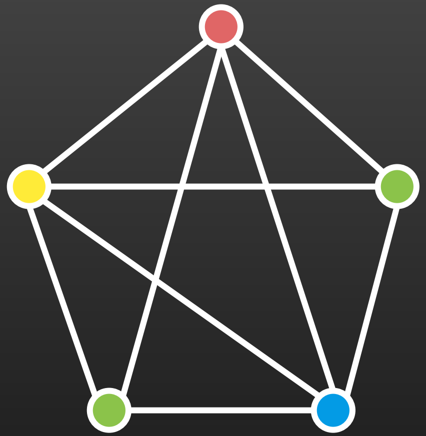

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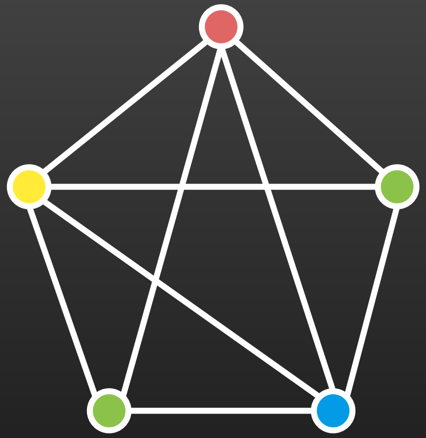

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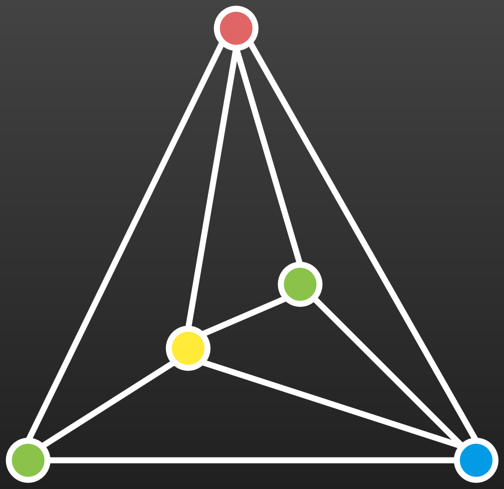

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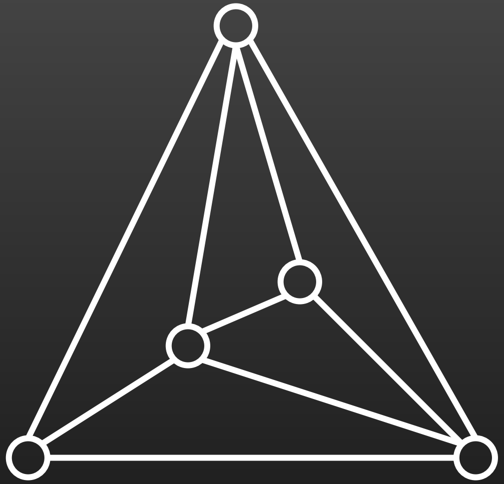

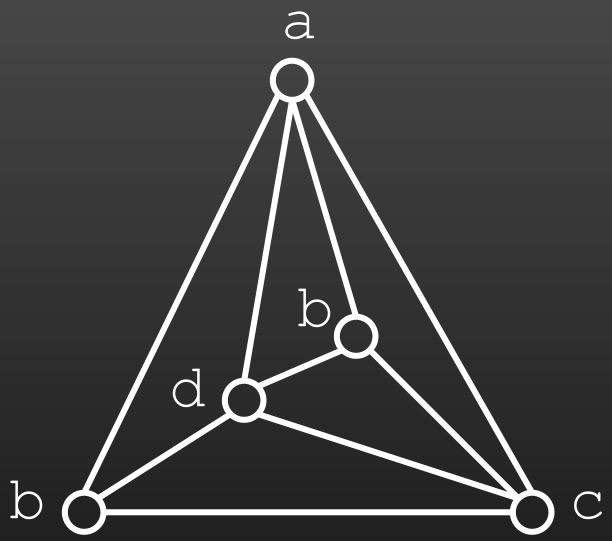
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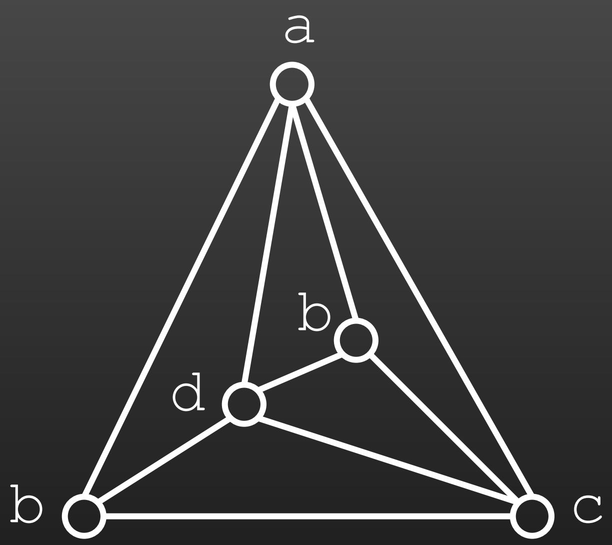
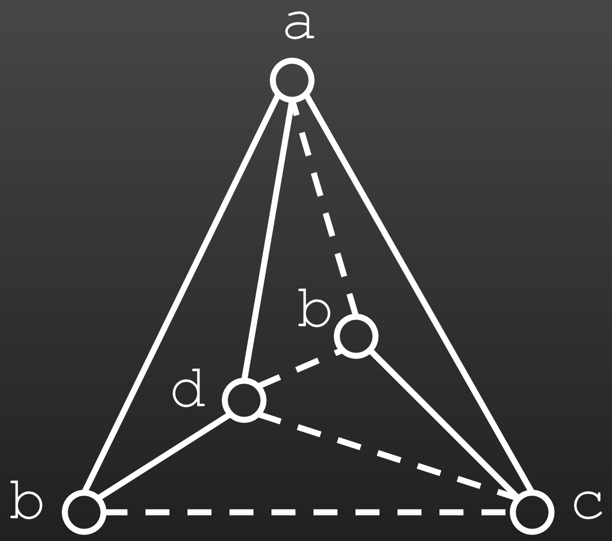


A *coloring* of a graph G is any function from the set of vertices to some alphabet $\mathscr{A}$.



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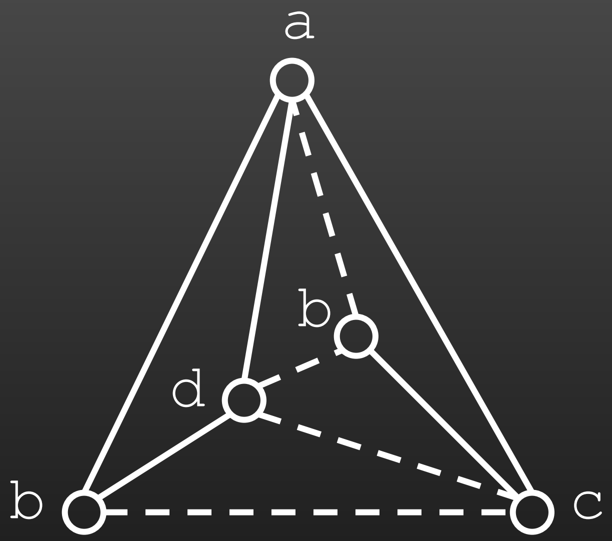

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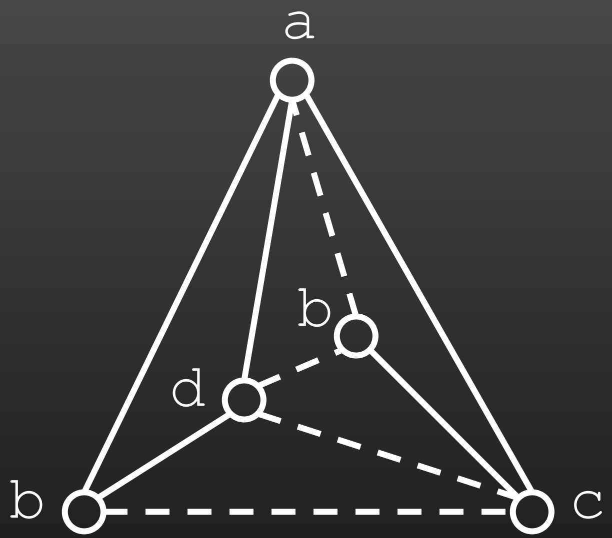

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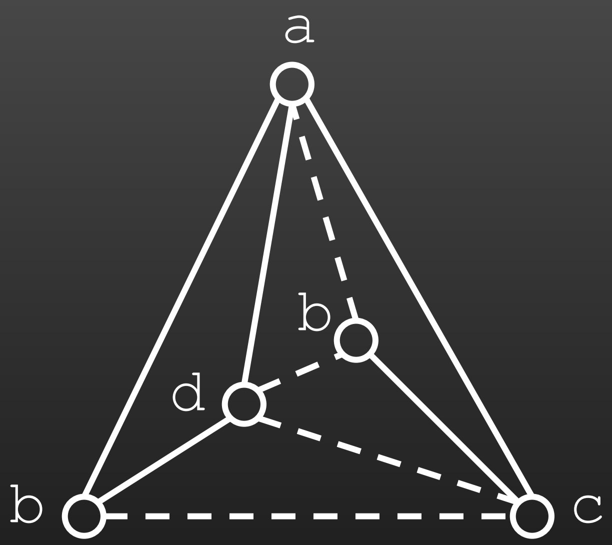

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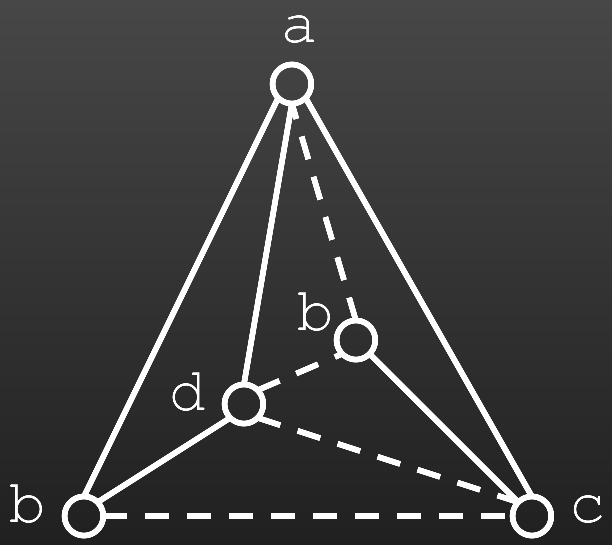

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The Four Color Theorem (Appel, Haken, 1977). *Every planar graph G can be colored using a 4-letter alphabet $\mathscr{A}$ so that the language $L(G)$ does not contain words of the form xx, with $x \in \mathscr{A}$.*

Square-free coloring of graphs

A *square* is a word of the form XX, where X is any nonempty word.

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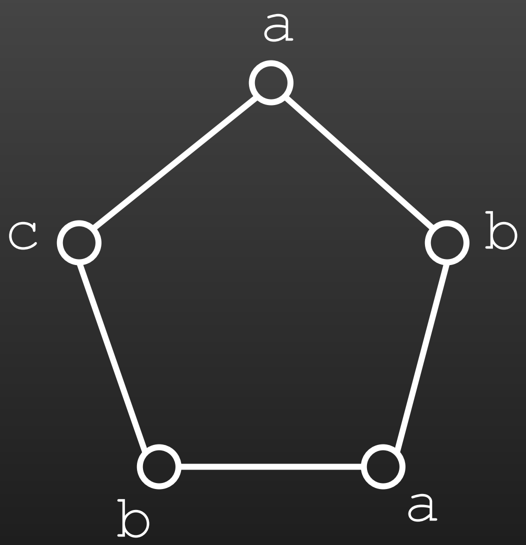

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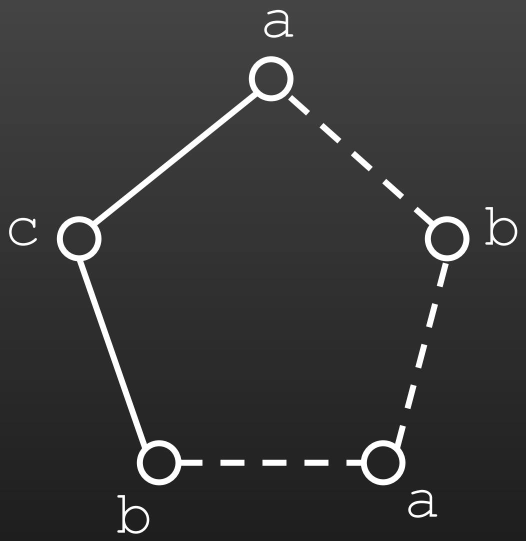

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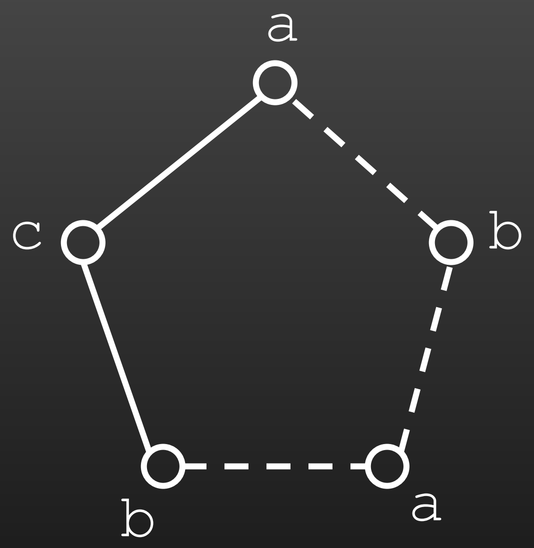

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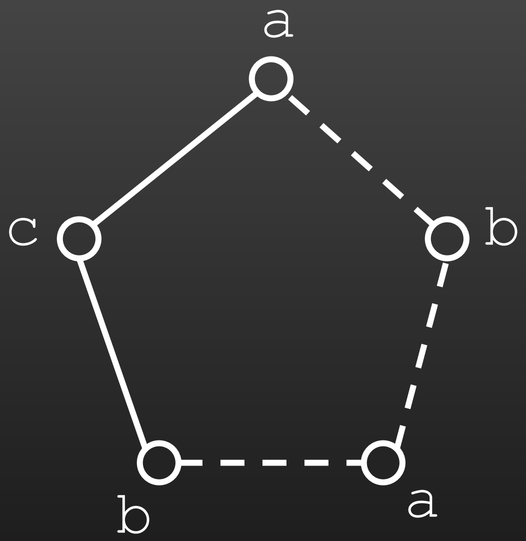

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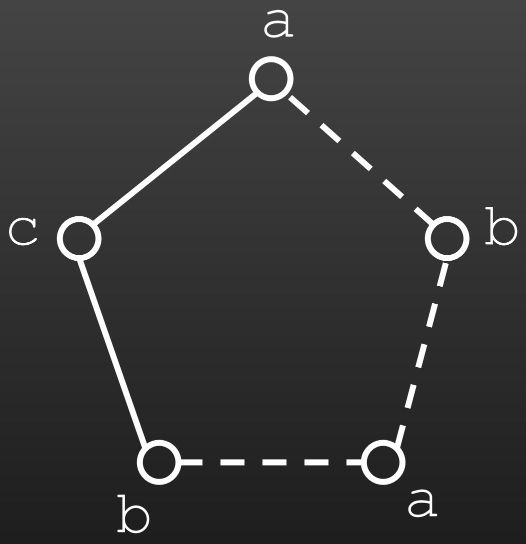

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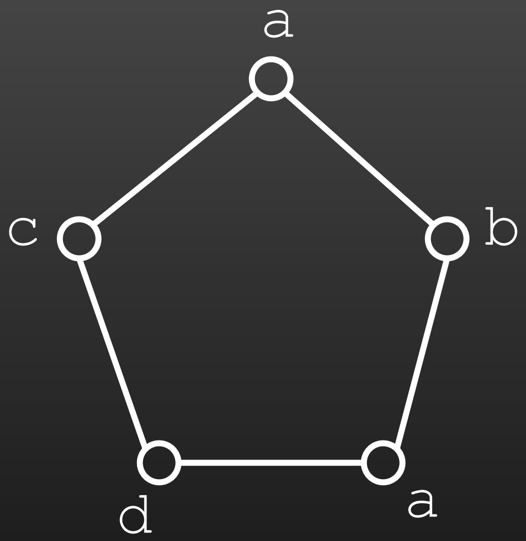

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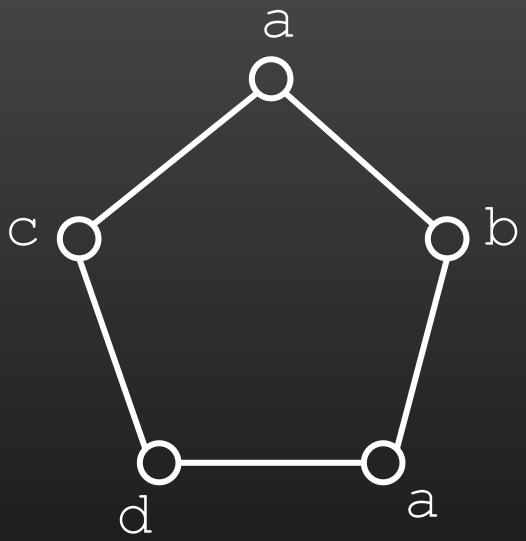

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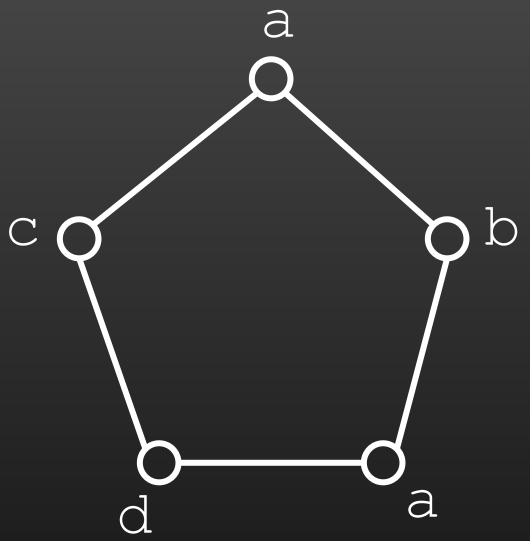

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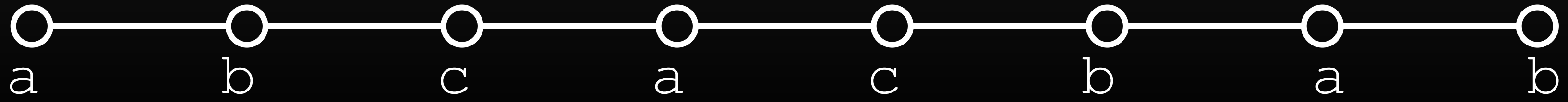

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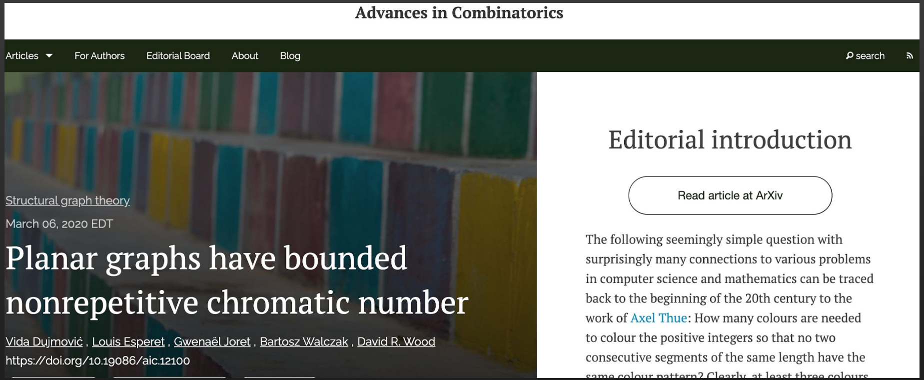

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Vida Dujmović , Louis Esperet , Gwenaël Joret , Bartosz Walczak , David R. Wood

https://doi.org/10.19086/aic.12100

Editorial introduction

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The following seemingly simple question with surprisingly many connections to various problems in computer science and mathematics can be traced back to the beginning of the 20th century to the work of Axel Thue: How many colours are needed to colour the positive integers so that no two consecutive segments of the same length have the same colour pattern? Clearly, at least three colours

Nonrepetitive Graph Colouring

David R. Wood[†]

September 7, 2020

Abstract

A vertex colouring of a graph G is *nonrepetitive* if G contains no path for which the first half of the path is assigned the same sequence of colours as the second half. Thue's famous theorem says that every path is nonrepetitively 3-colourable. This paper surveys results about nonrepetitive colourings of graphs. The goal is to give a unified and comprehensive presentation of the major results and proof methods, as well as to highlight numerous open problems.

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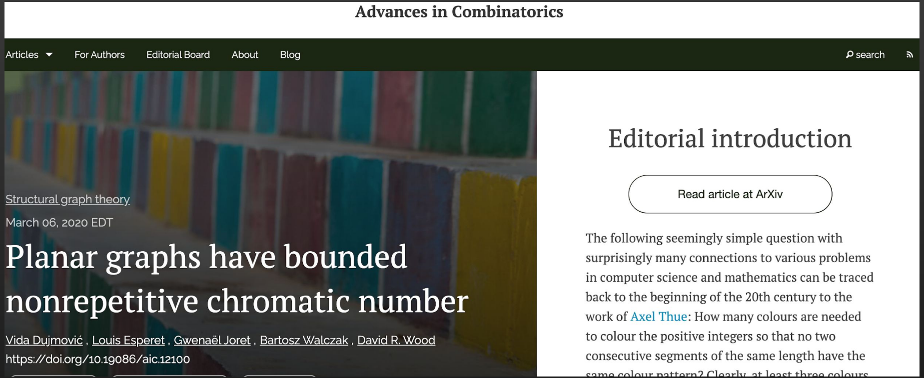

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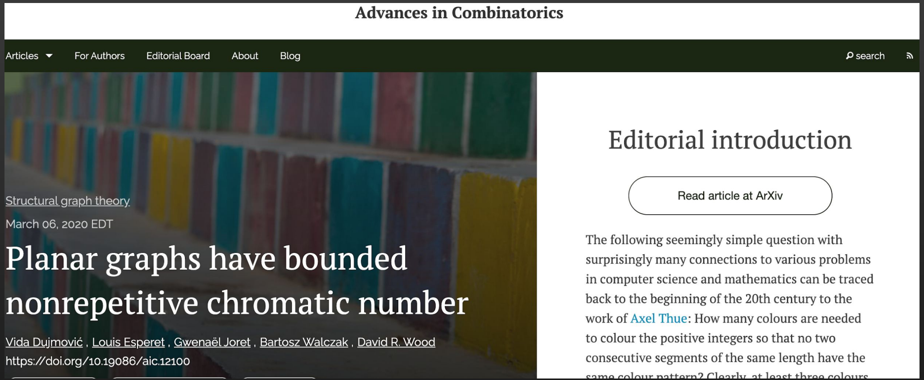
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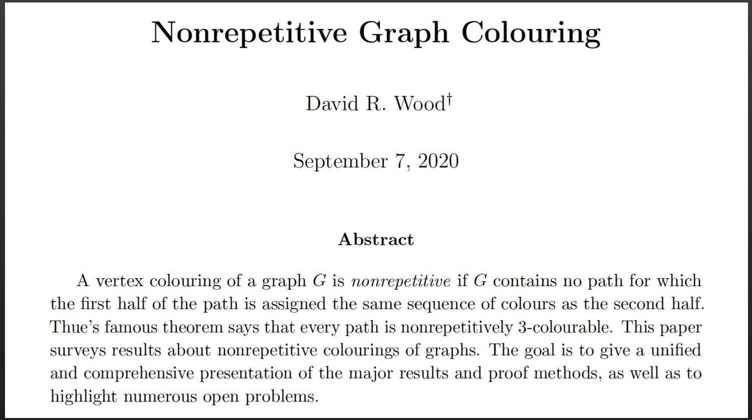
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Theorem (Ochem, 2011). *There exists a planar graph G with $\pi(G) \geqslant 11$.*

Problem. *What is the least number k such that $\pi(G) \leqslant k$, for every planar graph G?*

New four-color conjectures

Conjecture (Grytczuk, 2019). *There is a number k such that every planar graph G has a 4-coloring such that the language $L(G)$ does not contain squares of length greater than k.*

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Thue games

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Theorem (Grytczuk, Szafruga, Zmarz, 2013). *Ann wins the Thue game with* 4 *letters, while Ben can play arbitrarily long with* 12 *letters.*

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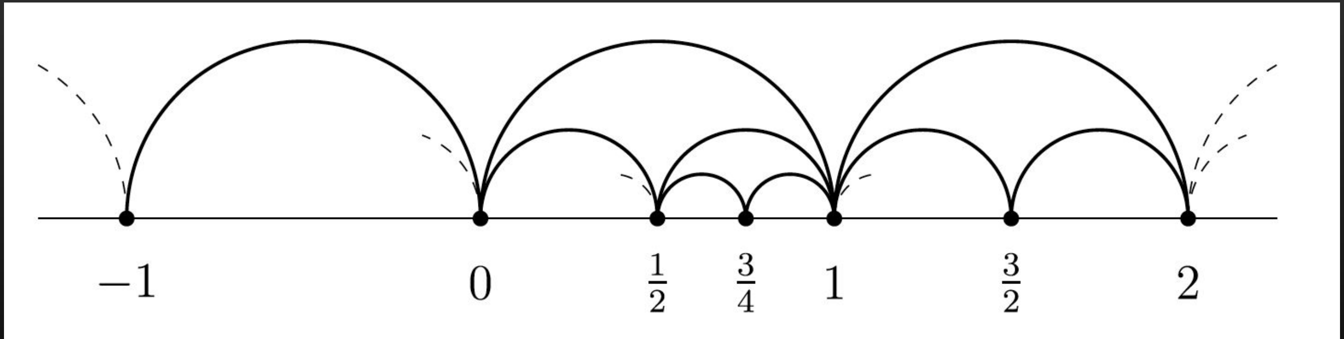

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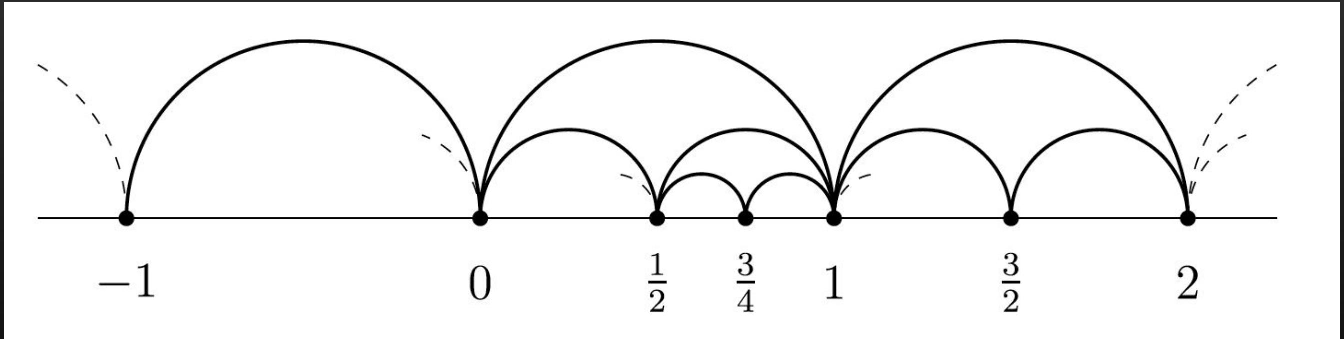

Theorem (Kündgen and Pelsmajer, Barát and Varjú, 2008). *Every* outerplanar *graph G satisfies $\pi(G) \leqslant 12$.*

Extremal words

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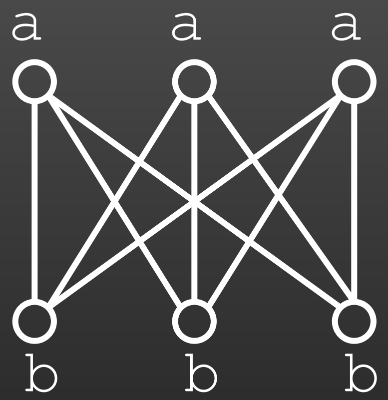
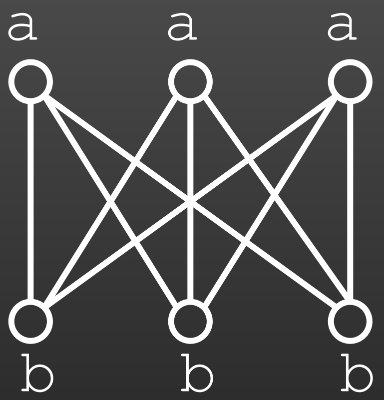
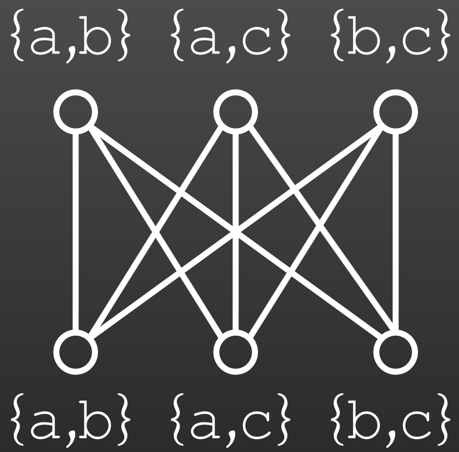
List coloring problems


a
a
a
b
b
b



$$\chi(G) = 2$$



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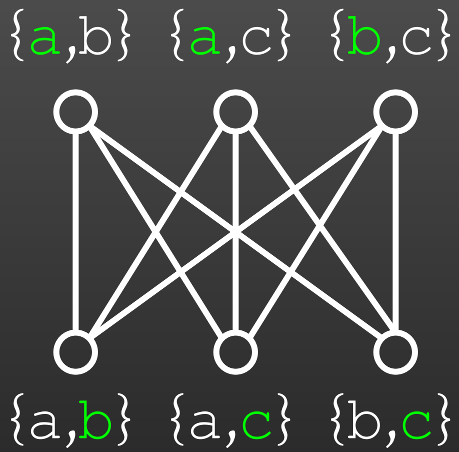

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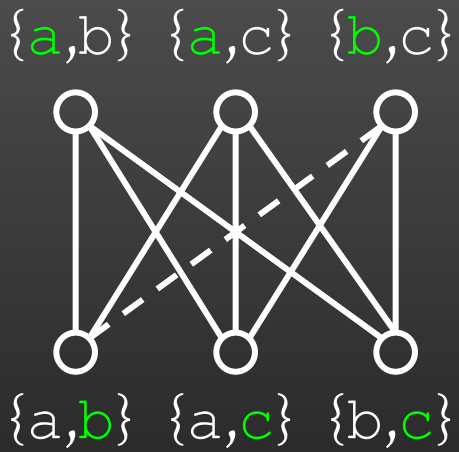

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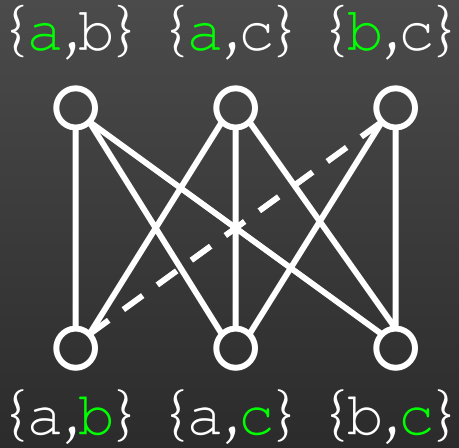
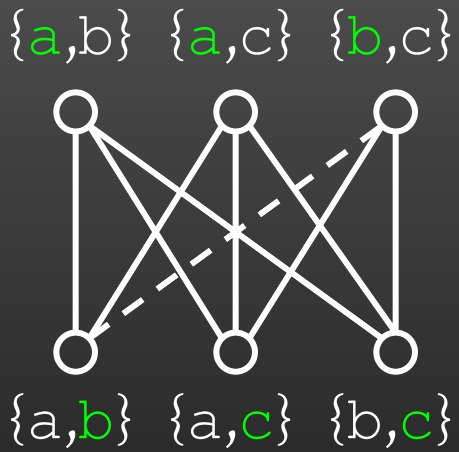


$\chi(G) = 2 \quad \chi_\ell(G) = 3$



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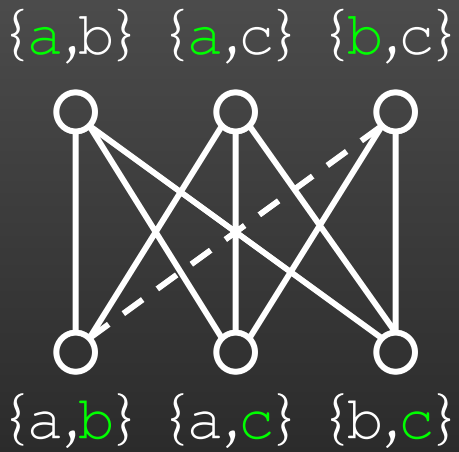

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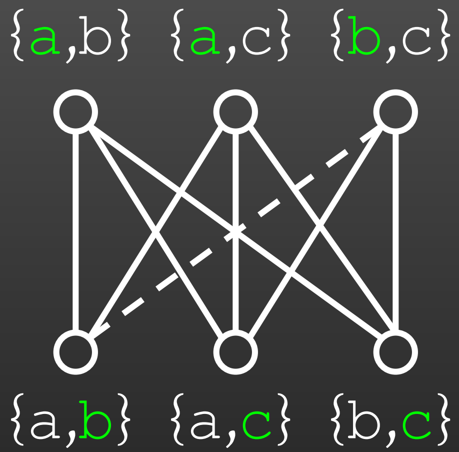

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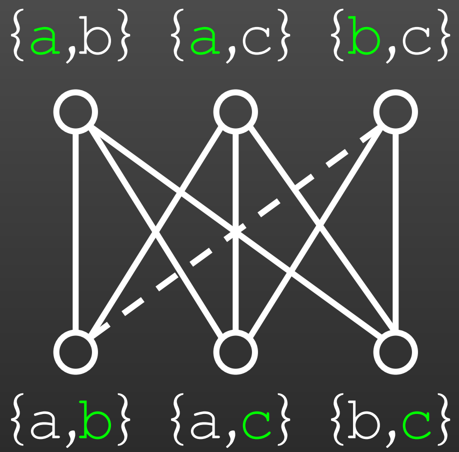

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Theorem (Grytczuk, Zhu, 2020). *Every planar graph G contains a* matching *M such that $\chi_\ell(G - M) \leqslant 4$.*

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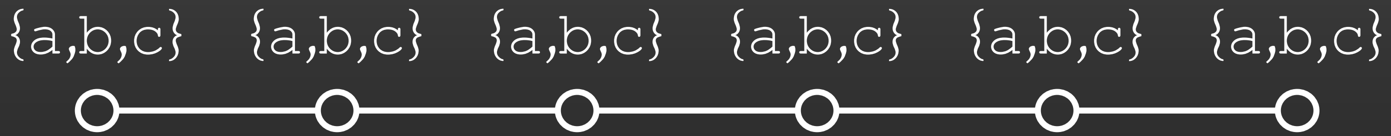

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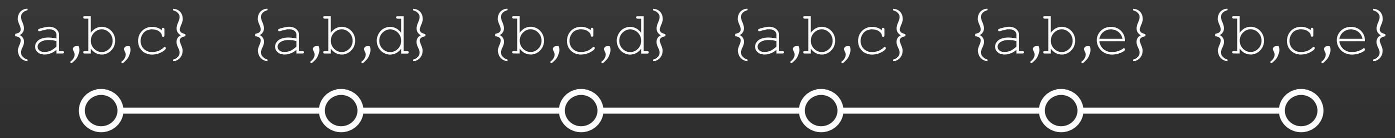

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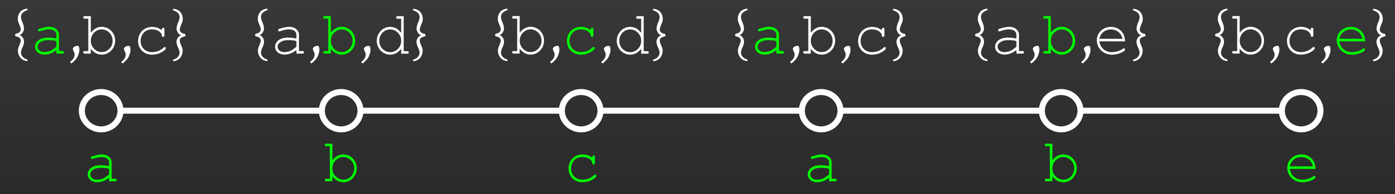

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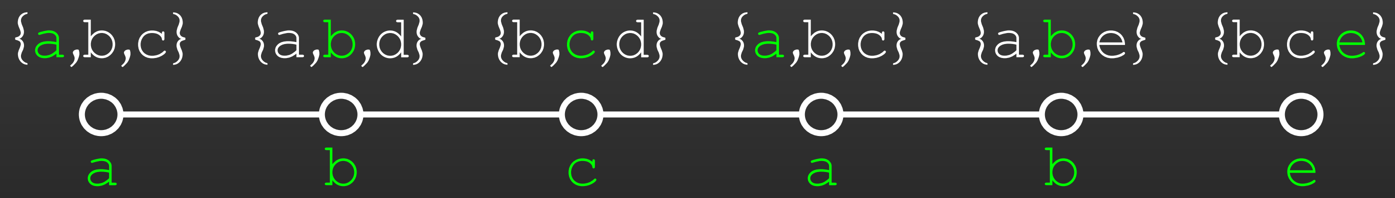

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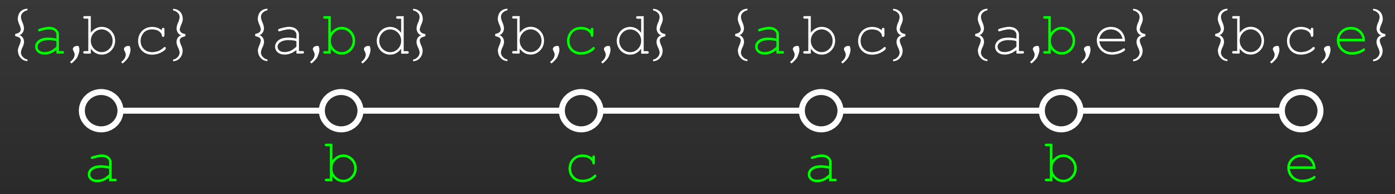

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Cartesian words

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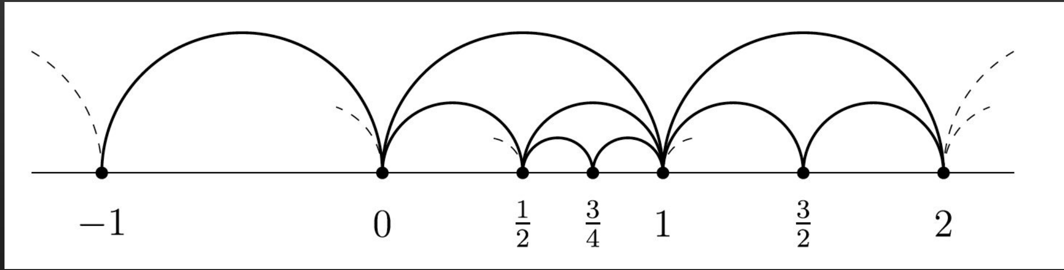

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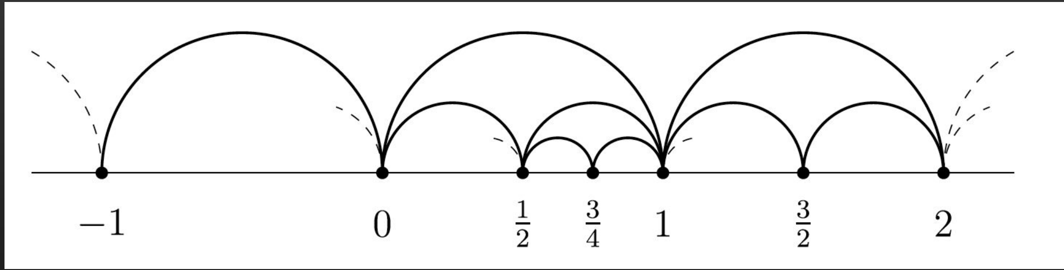

Conjecture (C). *For every n and every sequence $0 \leqslant i(0) < i(1) < \dots < i(m) \leqslant n$, there is a Cartesian word $C = c_0c_1\dots c_n$ such that the subword $c_{i(0)}c_{i(1)}\dots c_{i(m)}$ is also Cartesian.*

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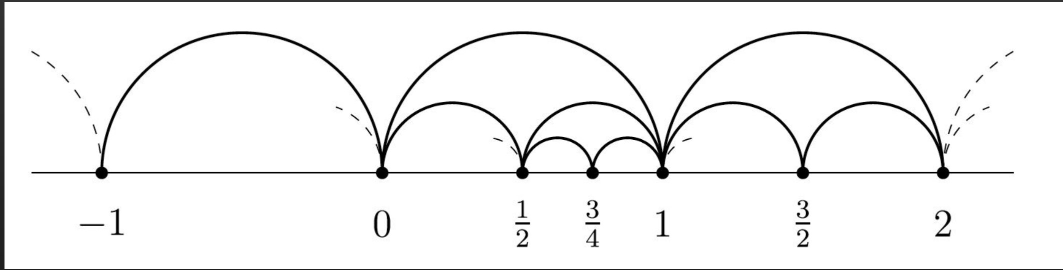

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Theorem (Descartes and Descartes, 1968). *Conjecture (C) is equivalent to the Four Color Theorem.*

Thank You!