A new Approach to Nonrepetitive Colorings of Graphs of Bounded Degree

Matthieu Rosenfeld

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We show instead a stronger result.

Let C_n be the set of square-free words of length n over $\{1, 2, 3, 4\}$.

Lemma

For any integer n,

$$|C_{n+1}| \geq 2|C_n| .$$

The proof by induction that $|C_{n+1}| \geq 2|C_n|$ for all n

Suppose that for all $i < n$, $|C_{i+1}| \geq 2|C_i|$.

Then for all $i \in \{0, \ldots, n\}$,

$$|C_{n-i}| \leq \frac{|C_n|}{2^i} \, .$$

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Let $F = \{ua : u \in C_n, a \in \{1, 2, 3, 4\}\} \setminus C_{n+1}$, then

$$|C_{n+1}| = 4|C_n| - |F| .$$

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For all i, let F_i be the set of words from F that end with a square of period i. Then $|F| \leq \sum_{i \geq 1} |F_i|$ and

$$|C_{n+1}| \geq 4|C_n| - \sum_{i \geq 1} |F_i|$$

Bounding $|F_i|$

Let $w \in F_i$. The last i letters of w are uniquely determined by the prefix v of length $n + 1 - i$. Since $v \in C_{n+1-i}$,

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As desired,

$$|C_{n+1}| \geq 2|C_n| \quad \square$$

An old result

The starting point of combinatorics on words.

Theorem (Thue,1906)

There exists arbitrarily long square-free words over $\{0, 1, 2\}$.

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Many generalizations or variations were studied:

- Cubes, 4th powers, **fractional powers**,
- patterns, formulas (ABABA),
- k-abelian powers, k-binomial powers, additive powers, antipowers,
- **nonrepetitive colorings of graphs** (or other objects).
- ...

Fractional repetitions

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Theorem (Dejean's Conjecture)

For any $k > 5$, there exists a $\frac{k}{k-1}^+$-free word over k letters.

Growth rate over large alphabets

Let L be a language and L_n be the set of words of length n of L. The *growth* of L is the quantity

$$g(L) = \lim_{n\to\infty} |L_n|^{1/n} .$$

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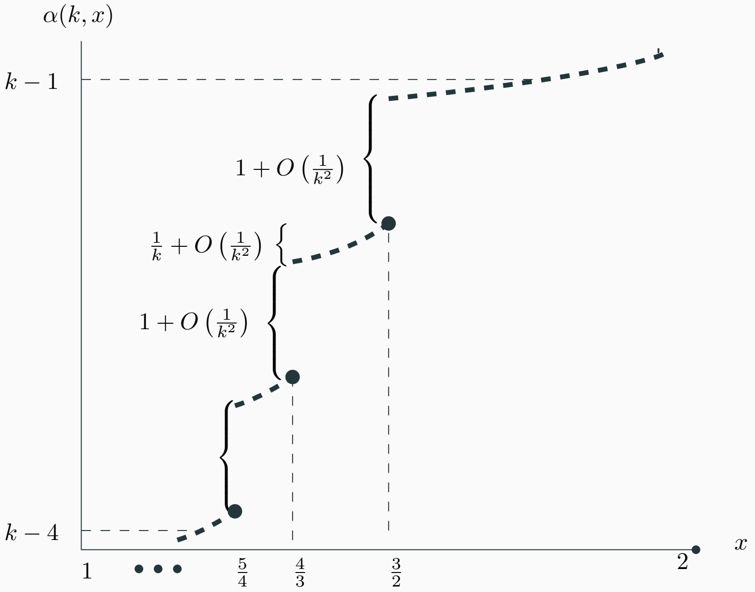
Conjecture (Shur)

For any fixed integer $n \geq 3$ and arbitrarily large integer k the following holds

$$\alpha\left(k, \frac{n}{n-1}\right) = k + 1 - n - \frac{n-1}{k} + O\left(\frac{1}{k^2}\right)$$

$$\alpha\left(k, \frac{n}{n-1}^{+}\right) = k + 2 - n - \frac{n-1}{k} + O\left(\frac{1}{k^2}\right)$$

The gaps



$\alpha(k,x)$
$k-1$
$1+O\left(\frac{1}{k^2}\right)$
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$k-4$
x
1
$\frac{5}{4}$
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2

The Lemma

Lemma (R.)

Let k and n be two integers with $k > n > 1$. For all i, let C_i be the set of $\frac{n}{n-1}$-free words of length i over the alphabet $\{1, 2, \ldots, k\}$. If $\gamma > 1$ is a such that $k - (n - 1)\frac{\gamma}{\gamma-1} \geq \gamma$, then for any integer i,

$$|C_{i+1}| \geq \gamma |C_i| \,.$$

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With the right value of γ, it implies:

Corollary (R.)

For any fixed integer $n \geq 3$ and arbitrarily large integer k the following holds

$$\alpha\left(k, \frac{n}{n-1}\right) \geq k + 1 - n - \frac{n-1}{k} + O\left(\frac{1}{k^2}\right)$$

The proof - Part I: Induction and definition of F_p

Lemma (R.)

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The proof - Part II: Bounding $\mathrm{F_p}$

Let w be a word of $\mathrm{F_p}$. $\mathrm{w = uxy}$ with $|\mathrm{x}| = \mathrm{p}$ and y prefix of x with

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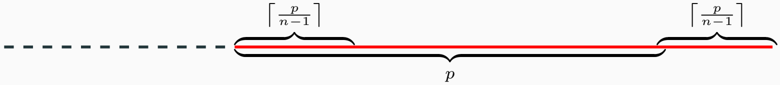

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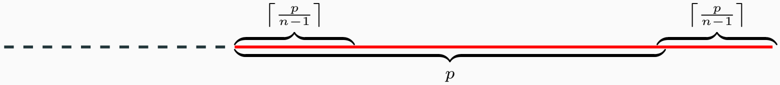

The last $\left\lceil \frac{p}{n-1} \right\rceil$ letters are uniquely determined by the prefix:

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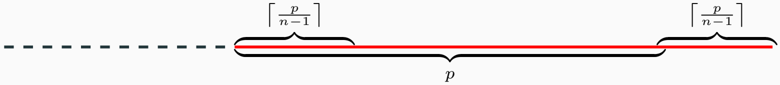

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Recall:

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By assumption $k - (n-1)\frac{\gamma}{\gamma-1} \geq \gamma$. This implies

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The conjecture

Theorem (We showed)

For any fixed integer $n \geq 3$ and arbitrarily large integer k the following holds

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Conjecture (What is left to do)

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The idea of the proof

Let Π_n be the number of **nice** colorings of some objects of size n using c colors (an infinite word is a coloring of the integers).

Suppose that there are $(a_i)_{i\geq 1}$ such that

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Find c and β such that $\beta \geq c - \sum_{i\geq 1} a_i \beta^{1-i}$ $\square$.

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One can choose

$$c = \min_{\beta} \beta + \sum_{i\geq 1} a_i\beta^{1-i}$$

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- ? The rest of this talk.

Nonrepetitive colorings of graphs

A coloring of a graph is *nonrepetitive* if the sequence of colors of any path is square-free.

$\pi(G)$ is the smallest number of colors in any nonrepetitive coloring of the vertices of G.

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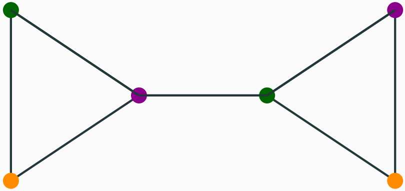
Theorem (Thue, 1906)

For any path P, $\pi(P) \leq 3$.

Remark

For all G, $\chi(G) \leq \pi(G)$.



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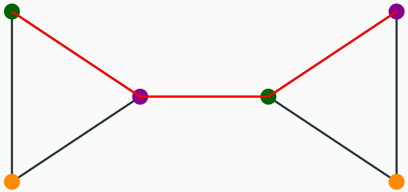

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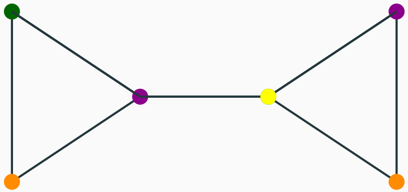

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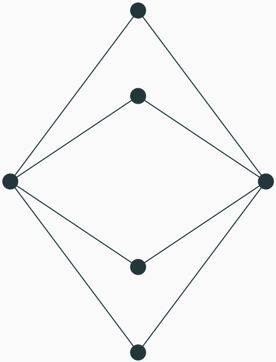

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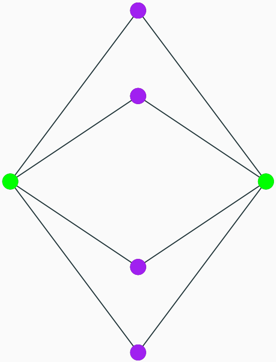

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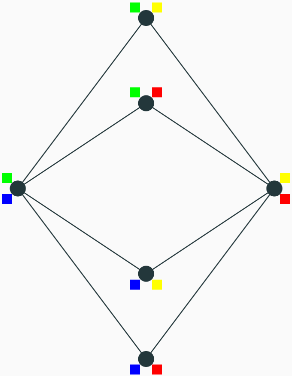

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Nonrepetitive list coloring

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Remark

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For every graph G, let $\Delta(G)$ be the maximal degree of G.

Theorem (Alon et al., 2002)

For every graph G,

$$\pi_{\mathrm{ch}}(G) \leq O(\Delta(G)^2)\,.$$

The proof uses the Lovász Local Lemma.

Bounds on $\pi_{\text{ch}}(P)$

Theorem (Grytczuk)

For any path P, $\pi_{\text{ch}}(P) \leq 4$.

- A first proof using LLL (Grytczuk, 2011)

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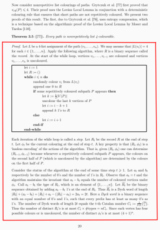

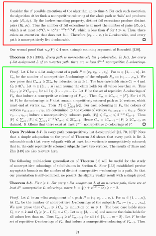

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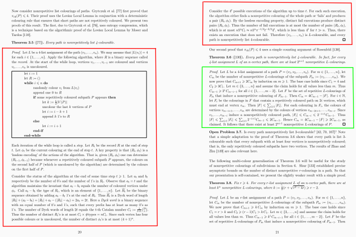

Open question: $\pi_{\mathrm{ch}}(\mathrm{P}) = 3$ for any path P ?

Graphs of bounded degree

Theorem

For any graph G,

- $\pi_{\mathrm{ch}}(\mathrm{G}) \leq 2^{16}\Delta^2$ (Alon et al.,2002)
- $\pi_{\mathrm{ch}}(\mathrm{G}) \leq 36\Delta^2$ (Grytczuk, 2007)
- $\pi_{\mathrm{ch}}(\mathrm{G}) \leq 16\Delta^2$ (Grytczuk, 2007)
- $\pi_{\mathrm{ch}}(\mathrm{G}) \leq (12.2 + \mathrm{o}(1))\Delta^2$ (Haranta and Jendro, 2012)
- $\pi_{\mathrm{ch}}(\mathrm{G}) \leq 10.4\Delta^2$ (Kolipaka, Szegedy, and Xu, 2012)
- $\pi_{\mathrm{ch}}(\mathrm{G}) \leq \Delta^2 + \mathrm{O}(\Delta^{5/3})$ (Dujmovic et al., 2016)
- $\pi_{\mathrm{ch}}(\mathrm{G}) \leq \Delta^2 + \mathrm{O}(\Delta^{5/3})$ (many other authors)

LLL, entropy compression, local cut lemma of Bernshteyn, cluster-expansion...

One more proof

Theorem (R. (Wood version))

For any graph G,

$$\pi_{\mathrm{ch}}(\mathrm{G}) \leq \Delta^2 + 3 \cdot 2^{-2/3}\Delta^{5/3} + 2^{2/3}\Delta^{4/3} - \Delta - 2^{4/3}\Delta^{2/3} + 2\,.$$

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For a list assignment L of a graph G, let $\Pi(G, L)$ be the set of nonrepetitive L-colorings of G.

Lemma

Fix an integer $\Delta > 2$ and a real number $r \in (0, 1)$. Let $\beta = \frac{(\Delta-1)^2}{r}$ and $c = \left\lceil \beta + \frac{\Delta}{(1-r)^2} \right\rceil$. Then for every graph G, every c-list assignment L of G and every vertex v of G,

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Proof of the Thm: chose $r = (1 + 2^{1/3}\Delta^{-1/3})^{-1}$.

The useful observation

Lemma

For every graph G with maximum degree Δ, for every vertex v of G, and for every $s \in \mathbb{N}$, there are at most $s\Delta(\Delta - 1)^{2s-2}$ paths on $2s$ vertices that contain v (where we consider a path to be a subgraph of G, so that a path and its reverse are counted once).

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v in position $i \in \{0, \ldots, s-1\}$

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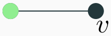
First neighbor of v : Δ choices



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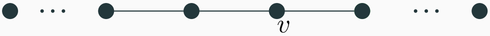

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Hence we get at most $s\Delta(\Delta-1)^{2s-2}$ such paths.

Proof of the lemma - Part I: Induction, F_i and $F_{i,p}$

Lemma

Let $\beta = \frac{(\Delta-1)^2}{r}$ and $c = \left\lceil \beta + \frac{\Delta}{(1-r)^2} \right\rceil$. Then for every graph G, every c-list assignment L of G and every vertex v of G,

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$$|\Pi(G, L)| = c \cdot |\Pi(G - v, L)| - |F|$$

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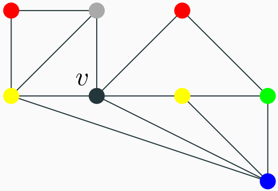
Proof of the lemma - Part II: Bounding $|F_{i,p}|$ and $|F_i|$

Fix p, let p' be the half of p that contains v. Then any coloring of $F_{i,p}$ is uniquely determined by the colors of $G - p$.



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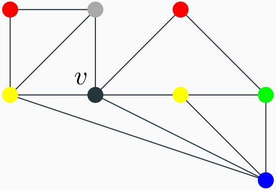

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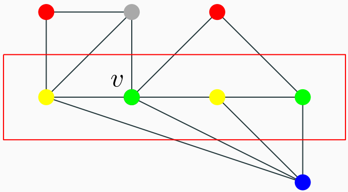

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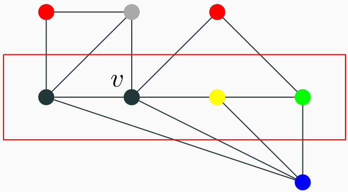

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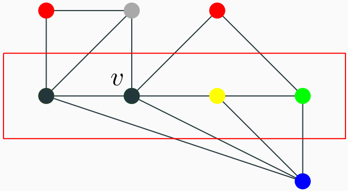

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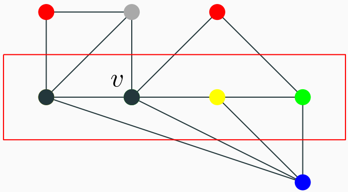

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By summing over every paths,

$$\begin{aligned} |F_i| &\leq \beta^{1-i} |\Pi(G - v, L)| i\Delta(\Delta - 1)^{2s-2} \\ &\leq \left(\frac{(\Delta - 1)^2}{r}\right)^{1-i} |\Pi(G - v, L)| i\Delta(\Delta - 1)^{2s-2} \\ &\leq r^{1-i} |\Pi(G - v, L)| i\Delta \end{aligned}$$

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By definition, $c = \left\lceil \beta + \frac{\Delta}{(1-r)^2} \right\rceil$ and we get

$$\Pi(G, L) \geq \beta \cdot \Pi(G - v, L) . \quad \square$$

Other applications - Hypergraph colorings, Sat...

Theorem (I.M. Wanless and D.R. Wood, 2020)

For all integers $r \geq 3$, and $\Delta \geq 1$, and for every r-uniform hypergraph G with maximum degree Δ

$$\chi_{ch}(H) \leq c := \left\lceil \frac{r-1}{r-2} \left((r-2)\Delta\right)^{1/(r-1)} \right\rceil .$$

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Star coloring (W. and W.), Frugal coloring (W. and W.), (weak) total nonrepetitive colorings (R.), avoidability of fractional powers (R.), other nonrepetitive questions (D. R. Wood, 2020), ...

Conclusion

We successfully applied a simple proof technique from combinatorics on words to graph theory.

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This proof technique seems to be equivalent to entropy compression. It is much simpler and provide exponential lower bounds on the number of solutions. Entropy compression provides algorithmic information.

- Find other applications,
- use computer assisted proof to improve the coefficients a_i.

Thanks !