

A new Approach to Nonrepetitive Colorings of Graphs of Bounded Degree

Matthieu Rosenfeld

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Theorem

There are arbitrarily long square-free words over $\{1, 2, 3, 4\}$.

Avoiding squares over 4 letters

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We show instead a stronger result.

Let C_n be the set of square-free words of length n over $\{1, 2, 3, 4\}$.

Lemma

For any integer n ,

$$|C_{n+1}| \geq 2|C_n|.$$

The proof by induction that $|C_{n+1}| \geq 2|C_n|$ for all n

Suppose that for all $i < n$, $|C_{i+1}| \geq 2|C_i|$.

Then for all $i \in \{0, \dots, n\}$,

$$|C_{n-i}| \leq \frac{|C_n|}{2^i}.$$

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Let $F = \{ua : u \in C_n, a \in \{1, 2, 3, 4\}\} \setminus C_{n+1}$, then

$$|C_{n+1}| = 4|C_n| - |F|.$$

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$$|C_{n+1}| = 4|C_n| - |F|.$$

For all i , let F_i be the set of words from F that end with a square of period i . Then $|F| \leq \sum_{i \geq 1} |F_i|$ and

$$|C_{n+1}| \geq 4|C_n| - \sum_{i \geq 1} |F_i|$$

Let $w \in F_i$. The last i letters of w are uniquely determined by the prefix v of length $n + 1 - i$. Since $v \in C_{n+1-i}$,

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As desired,

$$|C_{n+1}| \geq 2|C_n| \quad \square$$

The starting point of combinatorics on words.

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Many generalizations or variations were studied:

- Cubes, 4th powers, **fractional powers**,
- patterns, formulas (ABABA),
- k -abelian powers, k -binomial powers, additive powers, antipowers,
- **nonrepetitive colorings of graphs** (or other objects).
- ...

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Theorem (Dejean's Conjecture)

For any $k > 5$, there exists a $\frac{k}{k-1}^+$ -free word over k letters.

Growth rate over large alphabets

Let L be a language and L_n be the set of words of length n of L . The *growth* of L is the quantity

$$g(L) = \lim_{n \rightarrow \infty} |L_n|^{1/n} .$$

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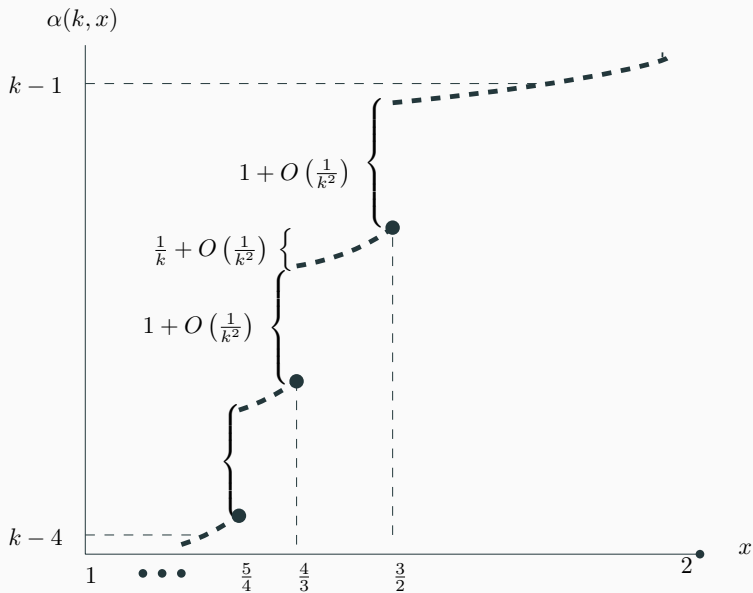
Conjecture (Shur)

For any fixed integer $n \geq 3$ and arbitrarily large integer k the following holds

$$\alpha\left(k, \frac{n}{n-1}\right) = k + 1 - n - \frac{n-1}{k} + O\left(\frac{1}{k^2}\right)$$

$$\alpha\left(k, \frac{n}{n-1}^+\right) = k + 2 - n - \frac{n-1}{k} + O\left(\frac{1}{k^2}\right)$$

The gaps



The Lemma

Lemma (R.)

Let k and n be two integers with $k > n > 1$. For all i , let C_i be the set of $\frac{n}{n-1}$ -free words of length i over the alphabet $\{1, 2, \dots, k\}$. If $\gamma > 1$ is such that $k - (n - 1)\frac{\gamma}{\gamma - 1} \geq \gamma$, then for any integer i ,

$$|C_{i+1}| \geq \gamma |C_i|.$$

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With the right value of γ , it implies:

Corollary (R.)

For any fixed integer $n \geq 3$ and arbitrarily large integer k the following holds

$$\alpha\left(k, \frac{n}{n-1}\right) \geq k + 1 - n - \frac{n-1}{k} + O\left(\frac{1}{k^2}\right)$$

The proof - Part I: Induction and definition of F_p

Lemma (R.)

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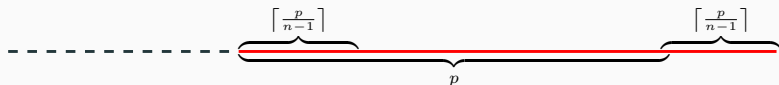
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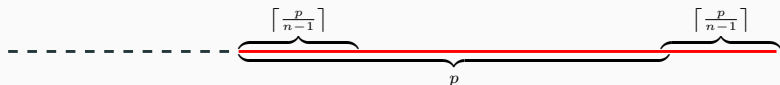
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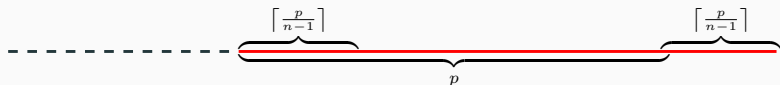
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By assumption $k - (n-1) \frac{\gamma}{\gamma-1} \geq \gamma$. This implies

$$|C_{i+1}| \geq \gamma |C_i| \quad \square$$

The conjecture

Theorem (We showed)

For any fixed integer $n \geq 3$ and arbitrarily large integer k the following holds

$$\alpha\left(k, \frac{n}{n-1}\right) \geq k + 1 - n - \frac{n-1}{k} + O\left(\frac{1}{k^2}\right)$$

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Conjecture (What is left to do)

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The idea of the proof

Let Π_n be the number of **nice** colorings of some objects of size n using c colors (an infinite word is a coloring of the integers).

Suppose that there are $(a_i)_{i \geq 1}$ such that

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One can choose

$$c = \min_{\beta} \beta + \sum_{i \geq 1} a_i \beta^{1-i}$$

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- ? The rest of this talk.

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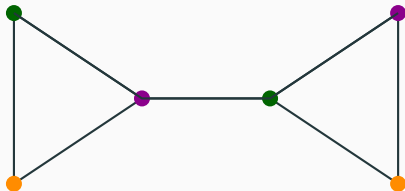
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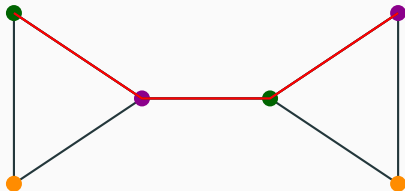
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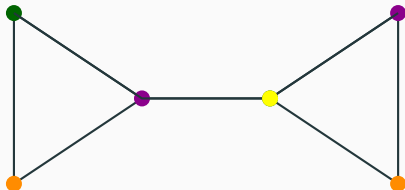
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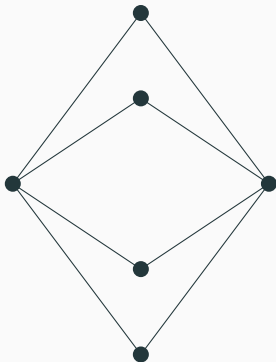
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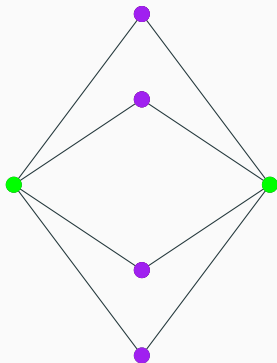
A *k*-list assignment of a graph is map that maps every vertex of the graph to a list of *k*-colors. A graph is *k*-list-colorable if for any *k*-list assignment L there is a proper coloring of G in which every vertex is colored by a color of its list.

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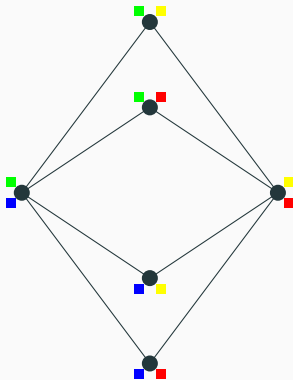
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$\pi_{\text{ch}}(G)$ is the Nonrepetitive choice number of G .

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For every graph G , let $\Delta(G)$ be the maximal degree of G .

Theorem (Alon et al., 2002)

For every graph G ,

$$\pi_{\text{ch}}(G) \leq O(\Delta(G)^2).$$

The proof uses the Lovász Local Lemma.

Bounds on $\pi_{\text{ch}}(P)$

Theorem (Grytczuk)

For any path P , $\pi_{\text{ch}}(P) \leq 4$.

- A first proof using LLL (Grytczuk, 2011)

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Now consider *isomorphism* for the subgraphs of paths. Geyer [10] et al. [9] first proved that $n_{\text{iso}}(P_k) \leq k$. Their proof uses the Linear Local Lemma in conjunction with a determinist is subgraph rule that ensures that short paths are not repetitively selected. The present work proves this result. The fact, due to Ono [14] et al. [8], that unique-composition, which is a technique based on the algorithmic proof of the Linear Local Lemma by Moser and Tardos [13], is

Theorem 3.3 ([7]). Every path is asymptotically δ -valuable.

Proof. Let C be a list assignment of the problem $(\gamma_1, \dots, \gamma_n)$. The map assumes that $|L(\gamma_i)| = 1$ for each $i \in \{1, \dots, n\}$. Apply the following algorithm, where R is a binary sequence called the *record*. At the start of the while loop, vertices v_1, \dots, v_{r-1} are evaluated and vertices v_r, \dots, v_n are unvisited.

```

for  $i \leftarrow 1$ 
  do  $R_i \leftarrow \emptyset$ 
  while  $i \leq n$  do
    randomly colour  $u_i$  from  $K(u_i)$ 
    append  $u_i$  to  $R_i$ 
    if some representative coloured output  $P^*$  appears then
      for  $k \in \mathcal{K}(P^*)$ 
        uncolour the last  $k$  vertices of  $P$ 
      for  $i \leftarrow i - k + 1$ 
        append  $U_i$  to  $R_i$ 
    else
      for  $i \leftarrow i + 1$ 
        break

```

Each iteration of the while-loop is called a *step*. Let R_i be the record R at the end of step i . Let α_i be the current subquery at the end of step i . It has property $\alpha_i = \{R_i, \alpha_{i-1}\}$ (a "Kleene encoding" of the set-union of R_i and α_{i-1}). That is, given $\{R_i, \alpha_{i-1}\}$ one can determine $\{R_{i+1}, \alpha_i\}$ because whenever a repeatedly selected subquery P appears, the column in the second half of P' (which is unselected by the algorithm) are determined by the column in the first half of P' .

Consider the choice of the sign (plus or the minus) of some line i (2.1). Let a_i and b_i represent the number of the lines of T which share $a_i = 1$ and $b_i = 0$ with the line i , respectively. Let the line i be a 0 -line. Let n_i be the number of lines which share $a_i = 0$ and $b_i = 1$ with the line i . Call $n_i = n_i$ the type of R_i , which is a element of $\{1, \dots, n\}$. Let R_i be the binary sequence obtained by adding n_i to T at the line of i and 0. Then R_i is a 0 -line of length $|R_i| = |a_i| + |b_i| + |n_i| + 1 = |a_i| + |b_i| + 2n_i + 1 = 2n_i + 2$. Before going on to a binary sequence with an equal number of 0's and 1's, call that binary pattern has at least as many 0's as 1's. The number of 0 -lines of length $2n$ equals the 1 -th Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$. The number of 0 -lines of length $2n$ is equal to 2 times C_n . Since each vertex has two possible values to be associated, the number of 0 -lines $q(n)$ is at most $2 \times 2 \times C_n = 4 \times \frac{1}{n+1} \binom{2n}{n}$.

Consider the d possible executions of the algorithm up to time t . For each such execution, the algorithm at any fixed step attempts to subdivide the whole path or 'fold' and produces a pair (R, A) . By the random-walk property, distinct fold executions produce distinct pairs (R, A) . Thus the number of fold executions is at most the number of pairs (R, A) , which is at most $2^{2n} = 2^{2n-1/2-1/2}$, which is less than $d^2/4$ for $d \geq 4$. Thus, there exists an execution that does not fail. Therefore $\{x_n - 1/2\} \in \mathcal{L}$ -dominable, and this

get it is an overpriced luxury item. It's ridiculous.

THEOREM 9.8 ([21]). *Every path is competitively but ρ -colorable. In fact, for every ρ -fat assignment λ of an n -vertex path, there are at least ρ^{-2} competitive ρ -colorings.*

Proof. Let C be a finite subsequence of a path $P = [p_1, p_2, \dots, p_n]$ for $n \in \mathbb{N}$, $n \geq 1$, and let C_n be the number of nonempty Linkings of the subpath P_n . Then $C_n \in \mathbb{N}$, and the sequence of paths $C_n \geq 2C_{n-1}$ by induction on $n \geq 1$. The base case holds since $C_1 = 1$ and $C_2 = 2$ in \mathbb{N} (cf. [10, 11]). Let $C_n \geq 2C_{n-1}$ for $n \geq 1$. Then $C_{n+1} \geq 2C_n$ for $n \geq 1$. Since $p_n \neq p_{n+1}$, let $q_1 \in [p_n, p_{n+1}]$. Let q_2 be the set of sequences Linkings of P_n that induce a nonempty linking of P_{n+1} . Then $C_{n+1} = |q_1 \cup q_2|$. For $i \in \mathbb{N}$, let F_i be the number of F -flat, containing a negatively oriented path on 20 vertices, with i end of vertex c_{20} . Then $|F_i| \leq |C_i|$ (cf. [10]). For each coloring in F_i , the scheme of vertices c_1, c_2, \dots, c_{20} is determined by the colors of vertices c_1, c_2, \dots, c_{19} . Since $|F_i| \leq |C_i|$, F_i induce a nonempty linking of P_{n+1} , $|q_2| \leq |C_n|$, and $|q_1| \geq 2|C_n|$. Then $|F_i| \leq |C_n|$ and $|F_i| \leq |C_n| + |C_n| = 2|C_n|$. Hence $C_{n+1} = |q_1 \cup q_2| \geq 2|C_n|$. It follows that there exist at least 2^n nonempty Linkings of P . \square

The following multi-color generalization of Theorem 3.6 will be useful for the study of noncompetitive colorings of subdivisions in Section 6. The [34] established the asymptotic bounds on the number of distinct noncompetitive colorings in a subk. In the

Theorem 4.8. For $r \geq 6$, for every r -bit assignment k of an n -active path, there are at least 2^r nonisomorphic \mathcal{L} -subalgebras, where $k := \frac{1}{2}(r + n^{\frac{r-1}{2}} - 2) \geq r - 2$.

Proof. Let ℓ be an ν -bit assignment of a path $P = (x_0, x_1, \dots, x_\ell)$. For $m \in \{0, \dots, \ell\}$ let C_m be the number of unoccupied 1-vertices of the subpath $P_m = (x_0, \dots, x_m)$. We now prove that $C_{m+1} \geq C_m$ by induction on $m \geq 1$. The base case holds since $C_1 \geq 0 = 1$ and $C_1 \geq 0 = 1 - |E_1| + |C_1|$. Let $m \in \{1, \dots, \ell\}$ and assume the claim holds for all values less than m . Thus $C_m \geq 0 \geq |E_m|$ for all $m \in \{1, \dots, m-1\}$. Let P be the set of unoccupied 1-vertices of P , then indeed a nonempty subpath of P .

Theorem

For any graph G ,

- $\pi_{\text{ch}}(G) \leq 2^{16}\Delta^2$ (Alon et al., 2002)
- $\pi_{\text{ch}}(G) \leq 36\Delta^2$ (Grytczuk, 2007)
- $\pi_{\text{ch}}(G) \leq 16\Delta^2$ (Grytczuk, 2007)
- $\pi_{\text{ch}}(G) \leq (12.2 + o(1))\Delta^2$ (Haranta and Jendro, 2012)
- $\pi_{\text{ch}}(G) \leq 10.4\Delta^2$ (Kolipaka, Szegedy, and Xu, 2012)
- $\pi_{\text{ch}}(G) \leq \Delta^2 + O(\Delta^{5/3})$ (Dujmovic et al., 2016)
- $\pi_{\text{ch}}(G) \leq \Delta^2 + O(\Delta^{5/3})$ (many other authors)

LLL, entropy compression, local cut lemma of Bernshteyn,
cluster-expansion...

One more proof

Theorem (R. (Wood version))

For any graph G ,

$$\pi_{\text{ch}}(G) \leq \Delta^2 + 3 \cdot 2^{-2/3} \Delta^{5/3} + 2^{2/3} \Delta^{4/3} - \Delta - 2^{4/3} \Delta^{2/3} + 2.$$

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For a list assignment L of a graph G , let $\Pi(G, L)$ be the set of nonrepetitive L -colorings of G .

Lemma

Fix an integer $\Delta > 2$ and a real number $r \in (0, 1)$. Let $\beta = \frac{(\Delta-1)^2}{r}$ and $c = \left\lceil \beta + \frac{\Delta}{(1-r)^2} \right\rceil$. Then for every graph G , every c -list assignment L of G and every vertex v of G ,

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Proof of the Thm: chose $r = (1 + 2^{1/3} \Delta^{-1/3})^{-1}$.

The useful observation

Lemma

For every graph G with maximum degree Δ , for every vertex v of G , and for every $s \in \mathbb{N}$, there are at most $s\Delta(\Delta - 1)^{2s-2}$ paths on $2s$ vertices that contain v (where we consider a path to be a subgraph of G , so that a path and its reverse are counted once).

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v in position $i \in \{0, \dots, s-1\}$



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First neighbor of v : Δ choices



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- v is at distance $\{0, \dots, s - 1\}$ from the closest end of the path.
- When choosing the first vertex of the short side, there are at most Δ choices.
- When choosing any other $2s - 2$ vertices there are at most $\Delta - 1$ possible choices for each.

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Hence we get at most $s\Delta(\Delta - 1)^{2s-2}$ such paths.

Lemma

Let $\beta = \frac{(\Delta-1)^2}{r}$ and $c = \left\lceil \beta + \frac{\Delta}{(1-r)^2} \right\rceil$. Then for every graph G , every c -list assignment L of G and every vertex v of G ,

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$$|\Pi(G, L)| = c \cdot |\Pi(G - v, L)| - |F|$$

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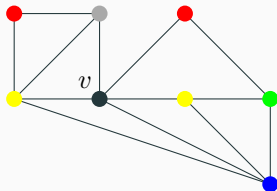
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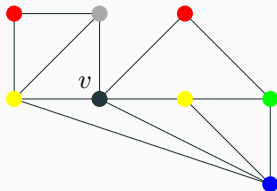
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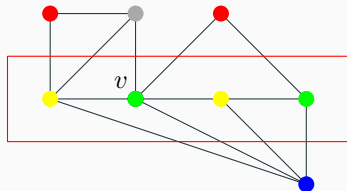
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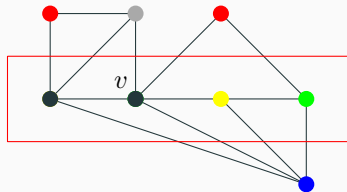
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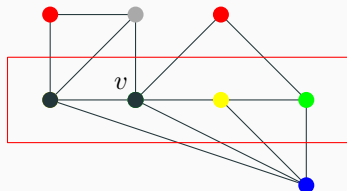
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Thus, $|F_{i,p}| \leq |\Pi(G - p', L)|$

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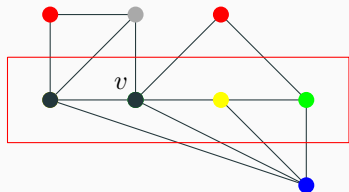
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Thus, $|F_{i,p}| \leq |\Pi(G - p', L)| \leq \beta^{1-i} |\Pi(G - v, L)|$.

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Thus, $|F_{i,p}| \leq |\Pi(G - p', L)| \leq \beta^{1-i} |\Pi(G - v, L)|$.

By summing over every paths,

$$\begin{aligned}
 |F_i| &\leq \beta^{1-i} |\Pi(G - v, L)| i \Delta (\Delta - 1)^{2s-2} \\
 &\leq \left(\frac{(\Delta - 1)^2}{r} \right)^{1-i} |\Pi(G - v, L)| i \Delta (\Delta - 1)^{2s-2} \\
 &\leq r^{1-i} |\Pi(G - v, L)| i \Delta
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By definition, $c = \left\lceil \beta + \frac{\Delta}{(1-r)^2} \right\rceil$ and we get

$$\Pi(G, L) \geq \beta \cdot \Pi(G - v, L) . \quad \square$$

Other applications - Hypergraph colorings, Sat...

Theorem (I.M. Wanless and D.R. Wood, 2020)

For all integers $r \geq 3$, and $\Delta \geq 1$, and for every r -uniform hypergraph G with maximum degree Δ

$$\chi_{\text{ch}}(H) \leq c := \left\lceil \frac{r-1}{r-2} ((r-2)\Delta)^{1/(r-1)} \right\rceil .$$

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Let ϕ be a Boolean formula in conjunctive normal form, with variables v_1, \dots, v_n and clauses c_1, \dots, c_m , each with exactly k literals. Assume that each variable is in at most $\Delta := \frac{2^k}{k-1} \left(\frac{k-1}{k}\right)^{k-1}$ clauses. Then there exists a satisfying truth assignment for ϕ . In fact, there are at least $(2^{\frac{2}{k}})^n$ such truth assignments.

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Theorem (I.M. Wanless and D.R. Wood, 2020)

Let ϕ be a Boolean formula in conjunctive normal form, with variables v_1, \dots, v_n and clauses c_1, \dots, c_m , each with exactly k literals. Assume that each variable is in at most $\Delta := \frac{2^k}{k-1} \left(\frac{k-1}{k}\right)^{k-1}$ clauses. Then there exists a satisfying truth assignment for ϕ . In fact, there are at least $(2^{\frac{2}{k}})^n$ such truth assignments.

Star coloring (W. and W.), Frugal coloring (W. and W.), (weak) total nonrepetitive colorings (R.), avoidability of fractional powers (R.), other nonrepetitive questions (D. R. Wood, 2020), ...

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This proof technique seems to be equivalent to entropy compression. It is much simpler and provide exponential lower bounds on the number of solutions. Entropy compression provides algorithmic information.

- Find other applications,
- use computer assisted proof to improve the coefficients a_i .

Thanks !