Two applications of the composition of a 2-tape automaton and a weighted automaton

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Day of Short Talks on Combinatorics on Words

Introduction

Consider two alphabets A and B and a symbol $\$ \notin A \cup B$.

We denote

$$A_\$ = A \cup \{\$\} \text{ and } B_\$ = B \cup \{\$\}.$$

For all $u \in A^*$ and $v \in B^*$, the \$-padding of $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ is defined by

$$\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]^\$ = \begin{cases} \left[\begin{smallmatrix} \$^{|v|-|u|}u \\ v \end{smallmatrix}\right] & \text{if } |u| \leq |v| \\ \left[\begin{smallmatrix} u \\ \$^{|u|-|v|}v \end{smallmatrix}\right] & \text{if } |u| > |v| \end{cases}$$

Two-tape automata

Consider a DFA

$$\mathcal{A} = (Q, i, T, A_\$ \times B_\$, \delta)$$

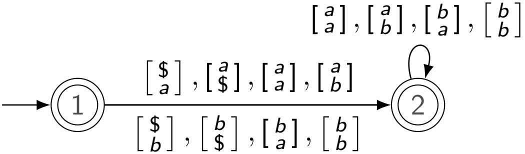
- Q: set of states
- i: initial state
- T: set of final states
- A and B: alphabets
- δ: $Q \times (A_\$ \times B_\$) \to Q$: (partial) function

An **image** $u \in A^*$ by $\mathcal{A}$ is a word $v \in B^*$ such that

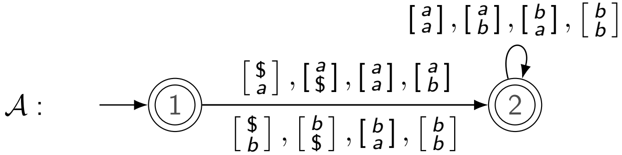
$$\delta(i, \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]^\$) \in T.$$

Example

$\mathcal{A}$:



Example



For all $u \in \{a, b\}^*$, then

$$\text{im}(u) = \{v \in \{a, b\}^* \colon ||u| - |v|| \leq 1\}.$$

The 2-tape automaton $\mathcal{A}$ accepts

$$\begin{bmatrix} \$u \\ v \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} u \\ \$v \end{bmatrix}$$

with $v \in \{a, b\}^{|u|+1}$, $v \in \{a, b\}^{|u|}$ and $v \in \{a, b\}^{|u|-1}$ respectively.

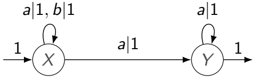
Weighted automata

Let $\mathbb{K}$ be a semiring and consider a $\mathbb{K}$-**automaton**

$$\mathcal{B} = (Q, I, T, B, E)$$

- Q: set of states
- B: alphabet
- $I\colon Q \to \mathbb{K}$, a state q is **initial** if $I(q) \neq 0$
- $T\colon Q \to \mathbb{K}$, a state q is **final** if $T(q) \neq 0$
- $E\colon Q \times B \times Q \to \mathbb{K}$.

A triple $(p, b, q) \in Q \times B \times Q$ is called a *transition*. The *label* of a transition (p, b, q) is the letter b and its **weight** is $E(p, b, q)$.



A **path** in $\mathcal{B}$ is a sequence

$$c = (q_0, b_1, q_1)(q_1, b_2, q_2) \cdots (q_{n-1}, b_n, q_n)$$

of transitions. The **weight** of the path c is the product

$$E(c) = E(q_0, b_1, q_1)E(q_1, b_2, q_2) \cdots E(q_{n-1}, b_n, q_n).$$

Its **label** is the word $b_1 b_2 \cdots b_n$.

The path c is **initial** if q_0 is initial and **final** if q_n is final.

For $w \in B^*$, we let $C_{\mathcal{B}}(w)$ denote the set of paths in $\mathcal{B}$ of label w that are both initial and final. The **weight** of w in $\mathcal{B}$ is the quantity

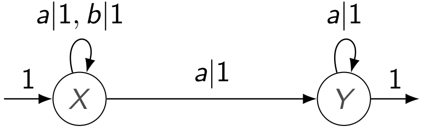
$$\sum_{c \in C_{\mathcal{B}}(w)} I(i_c)E(c)T(t_c).$$

Example

The weight of $v \in \{a, b\}^*$ in $\mathcal{B}$ equals

$$\max |\mathrm{Suff}(v) \cap \{a\}^*|.$$

$\mathcal{B}$:



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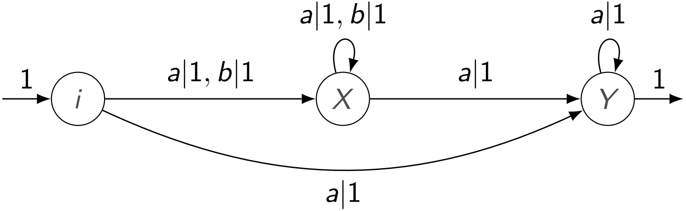

Without loss of generality, $\mathcal{B}$ has a unique initial state with no incoming transition.

Example

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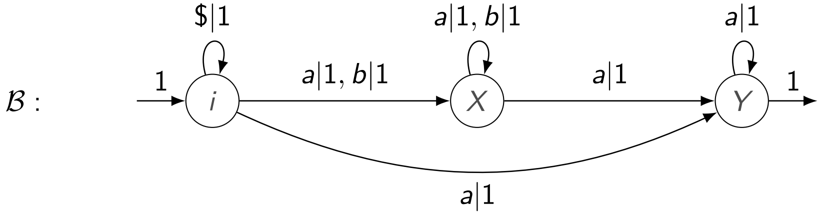

Without loss of generality, $\mathcal{B}$ has a unique initial state with no incoming transition. We add a loop on this initial state of label \$ and weight 1. For all $v \in B^*$ and $k \in \mathbb{N}$, the weight of $\$^k v$ in $\mathcal{B}$ equals the weight of v.

Question:
Considering a 2-tape DFA

$$\mathcal{A} = (Q_{\mathcal{A}}, i_{\mathcal{A}}, T_{\mathcal{A}}, A_{\$} \times B_{\$}, \delta_{\mathcal{A}})$$

and a (modified) $\mathbb{K}$-automaton

$$\mathcal{B} = (Q_{\mathcal{B}}, I_{\mathcal{B}}, T_{\mathcal{B}}, B_{\$}, E_{\mathcal{B}}),$$

can we compute a $\mathbb{K}$-automaton on the alphabet A in which the weight of $u \in A^*$ is the sum of the weights of its images by $\mathcal{A}$ in $\mathcal{B}$?

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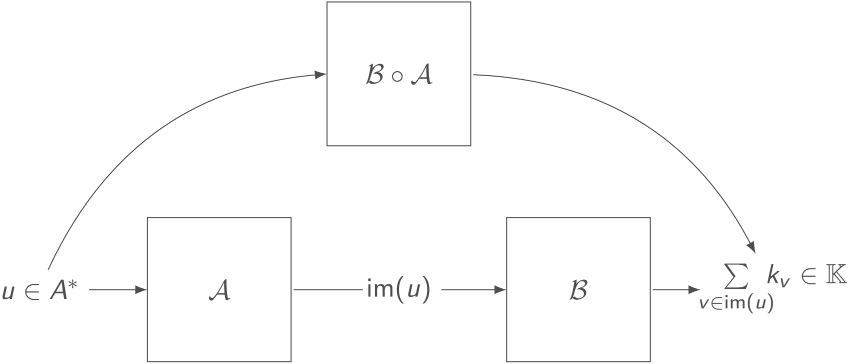
Example: We have

$$\mathrm{im}(a) = \{\varepsilon, a, b, aa, ab, ba, bb\}$$

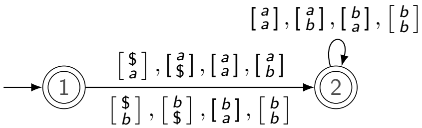
so we want the weight of a to be $0 + 1 + 0 + 2 + 0 + 1 + 0 = 4$.

Idea: Define the "composition" $\mathcal{B} \circ \mathcal{A}$.

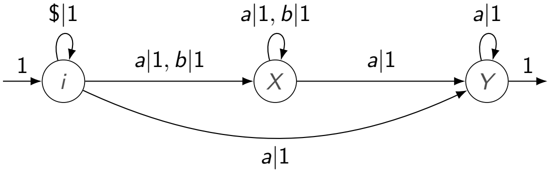


Automata composition

$\mathcal{A}$:

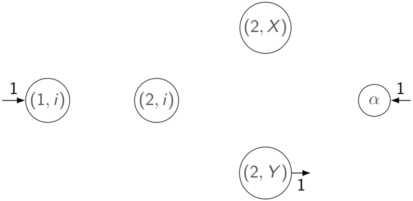


$\mathcal{B}$:



We define a new $\mathbb{K}$-automaton $\mathcal{B} \circ \mathcal{A} = (Q, I, T, A_\$, E)$ as follows.

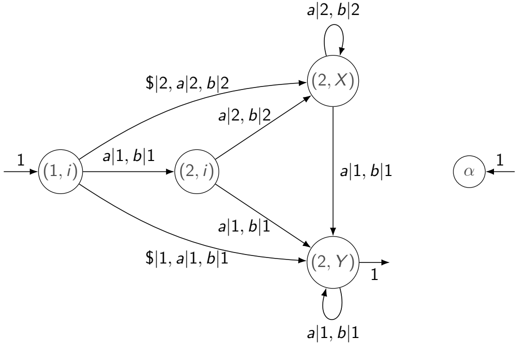
1. $Q = (Q_\mathcal{A} \times Q_\mathcal{B}) \cup \{\alpha\}$.
2. $I \colon Q \to \mathbb{K}$ is defined by
   - $I(i_\mathcal{A}, i_\mathcal{B}) = I_\mathcal{B}(i_\mathcal{B})$
   - For $(q, q') \in (Q_\mathcal{A} \times Q_\mathcal{B}) \setminus \{(i_\mathcal{A}, i_\mathcal{B})\}$, $I(q, q') = 0$
   - $I(\alpha) = 1$.
3. $T \colon Q \to \mathbb{K}$ is defined by
   - For $(q, q') \in T_\mathcal{A} \times Q_\mathcal{B}$, $T(q, q') = T_\mathcal{B}(q')$
   - For $(q, q') \in (Q_\mathcal{A} \setminus T_\mathcal{A}) \times Q_\mathcal{B}$, $T(q, q') = 0$
   - $T(\alpha) = 0$.



4. $E\colon Q \times A_\$ \times Q \to \mathbb{K}$ is defined by
   - For $(q_1, q_1'), (q_2, q_2') \in Q_\mathcal{A} \times Q_\mathcal{B}$ and $a \in A_\$$,

$$E((q_1, q_1'), a, (q_2, q_2')) = \sum_{\substack{b \in B_\$ \\ \delta_\mathcal{A}(q_1, \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]) = q_2}} E_\mathcal{B}(q_1', b, q_2')$$

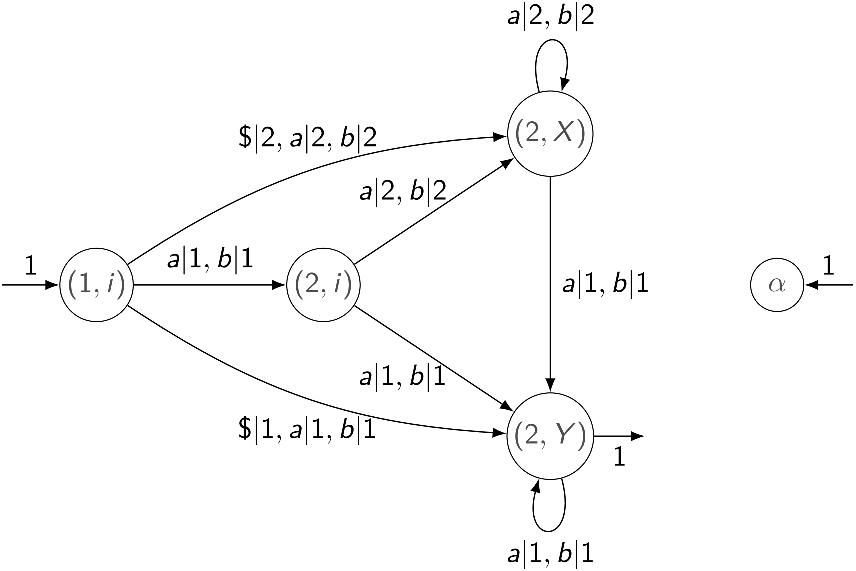
   - For $a \in A_\$$, $E(\alpha, a, \alpha) = 0$
   - For $(q, q') \in Q_\mathcal{A} \times Q_\mathcal{B}$ and $a \in A_\$$, $E((q, q'), a, \alpha) = 0$



- For $(q, q') \in Q_{\mathcal{A}} \times Q_{\mathcal{B}}$ and $a \in A_{\$}$,

$$E(\alpha, a, (q, q')) = \begin{cases} I(i_{\mathcal{A}}, i_{\mathcal{B}}) \sum_{\ell \geq 1} \sum_{c \in C_{q,q',a,\ell}} E(c) & \text{if } (q, q') \text{ is co-accessible} \\ 0 & \text{else} \end{cases}$$

where $C_{q,q',a,\ell}$ denotes the set of non-zero weight paths from $(i_{\mathcal{A}}, i_{\mathcal{B}})$ to (q, q') labeled by $\$^{\ell} a$.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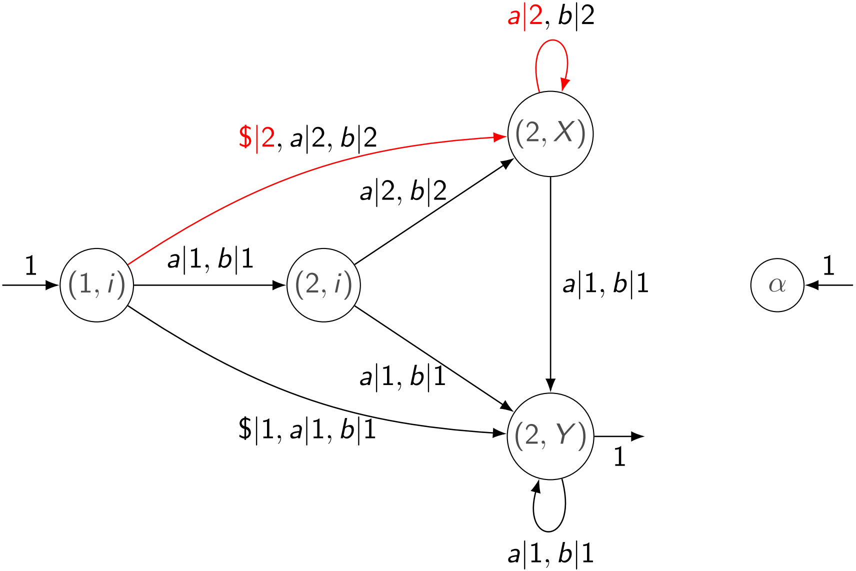
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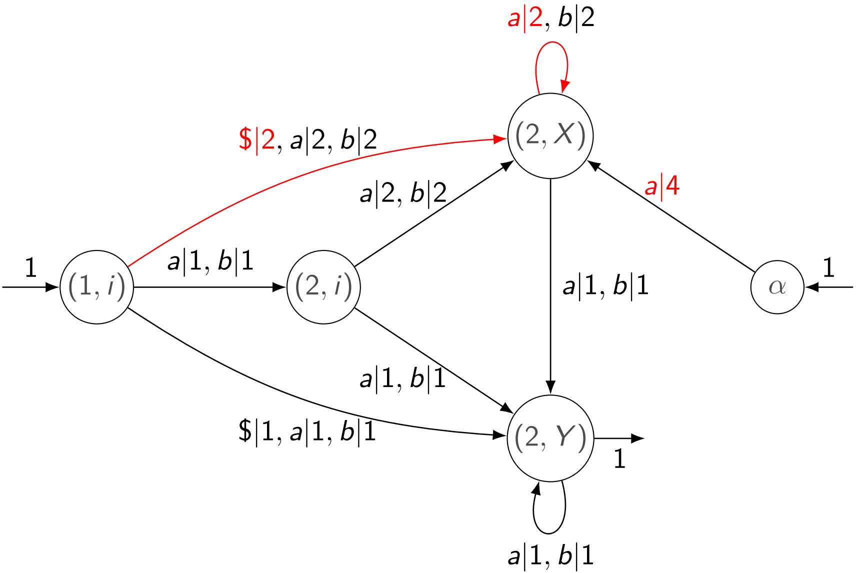

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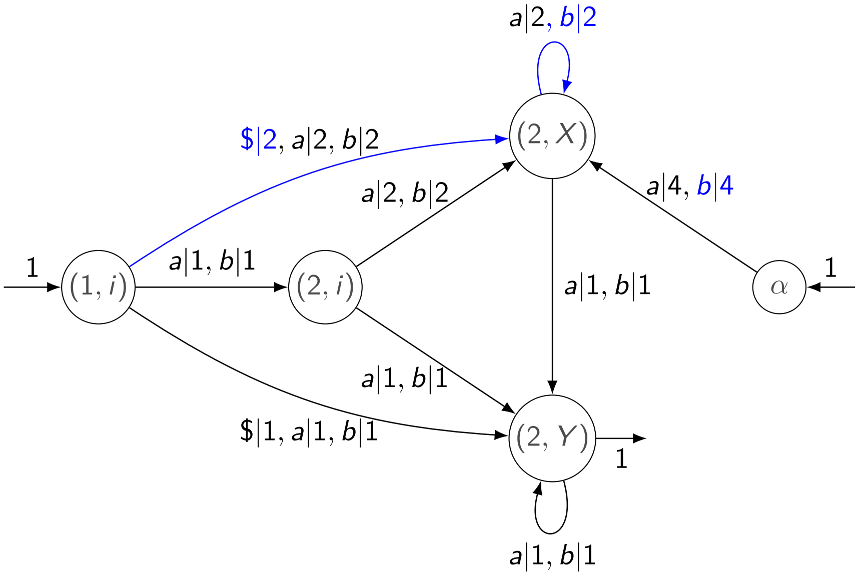
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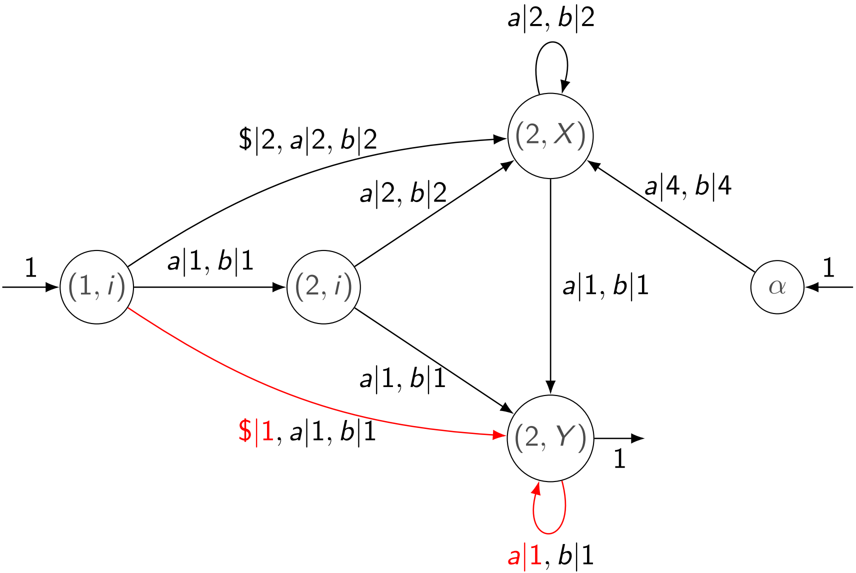
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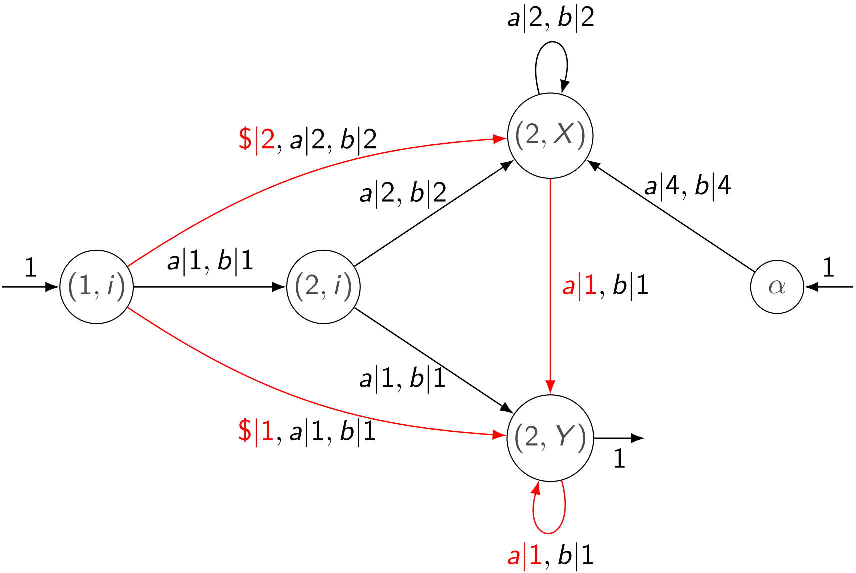
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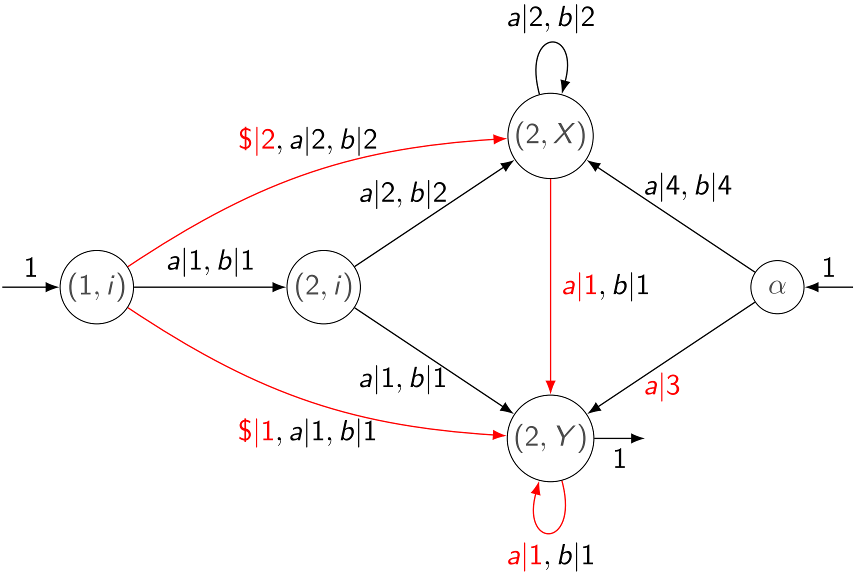
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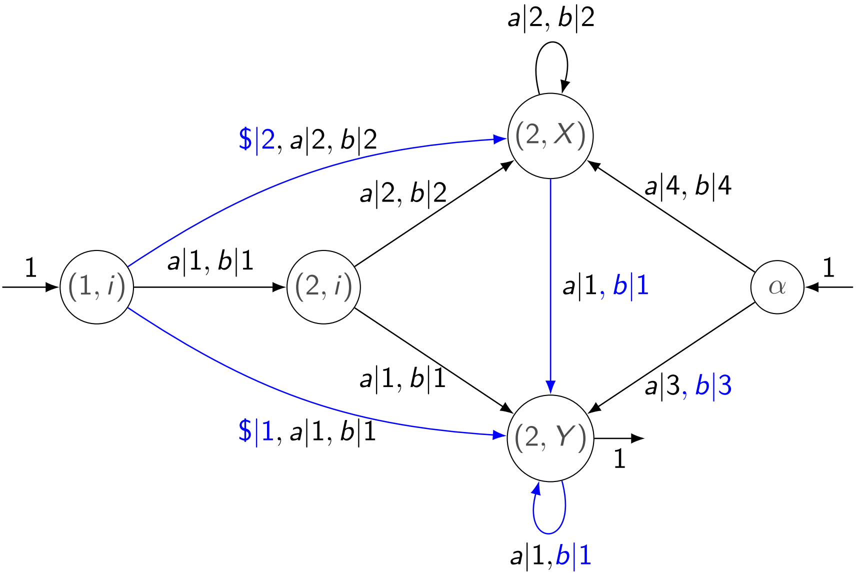
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a|1, b|1
α
a|1, b|1
a|3, b|3
$|1, a|1, b|1
(2, Y)
1
a|1,b|1

Intuition:

- The state α *bypasses* the leading \$ for the images greater than u since
$$\begin{bmatrix} \$^{|v|-|u|}u \\ v \end{bmatrix}$$
is accepted in $\mathcal{A}$. In fact, without α, $\$^{|v|-|u|}u$ (instead of u) is the label of the path in $\mathcal{B} \circ \mathcal{A}$.
- Le loop $\$|1$ on $i_{\mathcal{B}}$ is for the images smaller than u since
$$\begin{bmatrix} u \\ \$^{|u|-|v|}v \end{bmatrix}$$
is accepted in $\mathcal{A}$.

Synchronized relations and 2-tape automata

The relation $R\colon A^* \to B^*$ is **synchronized** if there exists a 2-tape automaton accepting the language

$$\{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]^{\$} : uRv\}.$$

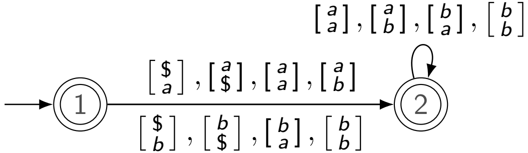
Example:

The relation $R : \{a, b\}^* \to \{a, b\}^*$ defined by

$$uRv \Leftrightarrow ||u| - |v|| \leq 1$$

is synchronized.

$\mathcal{A}$:



Formal series and $\mathbb{K}$-automata

A **(formal) series** is a function

$$S\colon A^* \to \mathbb{K},\ w \mapsto (S, w)$$

A series $S\colon A^* \to \mathbb{K}$ is **$\mathbb{K}$-recognizable** if there exist $r \in \mathbb{N}_{\geq 1}$, a morphism $\mu\colon A^* \to \mathbb{K}^{r\times r}$ and two matrices $\lambda \in \mathbb{K}^{1\times r}$ and $\gamma \in \mathbb{K}^{r\times 1}$ such that for all $w \in A^*$,

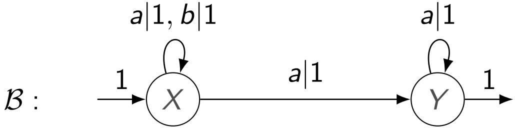
$$(S, w) = \lambda\mu(w)\gamma.$$

Proposition

A series is recognized by a $\mathbb{K}$-automaton if and only if it is $\mathbb{K}$-recognizable.

Example: The following $\mathbb{N}$-automaton recognizes the series

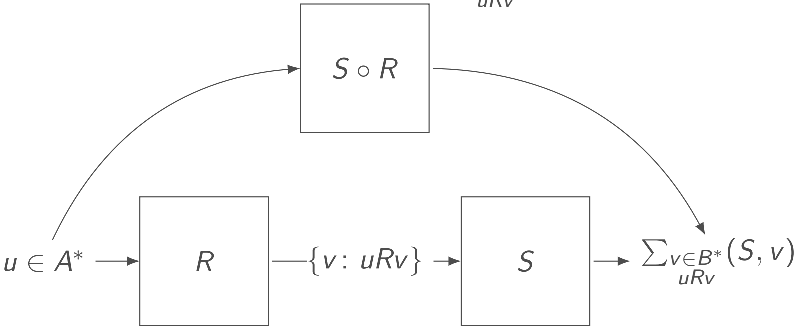
$$S\colon \{a,b\}^* \to \mathbb{N},\ v \mapsto \max|\mathrm{Suff}(v) \cap \{a\}^*|$$



Composition of a relation and a series

For a relation $R\colon A^* \to B^*$ and a series $S\colon B^* \to \mathbb{K}$ such that for all $u \in A^*$, the language $\{v \in B^* \colon uRv\}$ is finite, we define the *composition* of R and S as the series

$$S \circ R\colon A^* \to \mathbb{K},\ u \mapsto \sum_{\substack{v \in B^* \\ uRv}} (S, v).$$



Theorem (Charlier, C. & Stipulanti)

Let $R\colon A^* \to B^*$ be a synchronized relation, let $S\colon B^* \to \mathbb{K}$ be a $\mathbb{K}$-recognizable series, and suppose that for all $u \in A^*$, the language $\{v \in B^*\colon uRv\}$ is finite. Then $S \circ R$ is a $\mathbb{K}$-recognizable series.

Sketch of the proof:

- Let $\mathcal{A}$ be a DFA recognizing $\{[\begin{smallmatrix} u \\ v \end{smallmatrix}]^\$: uRv\}$.
- Let $\mathcal{B}$ be a $\mathbb{K}$-automaton recognizing the series S.
- Modify $\mathcal{B}$: unique initial state with no incoming edge and loop $\$|1$.
- Construct the $\mathbb{K}$-automaton $\mathcal{B} \circ \mathcal{A}$.
- Project on A and change the final weight of the initial state $(i_{\mathcal{A}}, i_{\mathcal{B}})$ from $T_{\mathcal{B}}(i_{\mathcal{B}})$ to $\frac{1}{I_{\mathcal{B}}(i_{\mathcal{B}})}\Big(\sum_{\substack{v \in B^* \\ \varepsilon R v}} (S, v)\Big)$

This automaton recognizes the series $S \circ R$.

First application

An **abstract numeration system** is a triple $\mathcal{S} = (L, A, <)$ where

- $(A, <)$ is a totally ordered alphabet
- L is an infinite regular language over A

The words in L are ordered with respect to the **radix** order $<_{\mathrm{rad}}$ induced by the order $<$ on A: for $u, v \in A^*$, $u <_{\mathrm{rad}} v$ either if $|u| < |v|$, or if $|u| = |v|$ and u is lexicographically less than v.

The $\mathcal{S}$-**representation function** $\mathrm{rep}_{\mathcal{S}} \colon \mathbb{N} \to L$ maps any non-negative integer n onto the nth word in L.
The $\mathcal{S}$-**value function** $\mathrm{val}_{\mathcal{S}} \colon L \to \mathbb{N}$ is the reciprocal function of $\mathrm{rep}_{\mathcal{S}}$.

Example: Let $\mathcal{S} = (a^* b^*, a < b)$ then $\mathrm{rep}_{\mathcal{S}}(7) = aab$ and $\mathrm{val}_{\mathcal{S}}(aaa) = 6$.

$\varepsilon <_{\mathrm{rad}} a <_{\mathrm{rad}} b <_{\mathrm{rad}} aa <_{\mathrm{rad}} ab <_{\mathrm{rad}} bb <_{\mathrm{rad}} aaa <_{\mathrm{rad}} aab <_{\mathrm{rad}} \cdots$

A sequence $f\colon \mathbb{N} \to \mathbb{K}$ is called $(\mathcal{S}, \mathbb{K})$-**regular** if the formal series

$$\sum_{n\in\mathbb{N}} f(n)\,\mathrm{rep}_{\mathcal{S}}(n)$$

is $\mathbb{K}$-recognizable.
A sequence $f\colon \mathbb{N} \to \mathbb{N}$ is $(\mathcal{S}, \mathcal{S}')$-**synchronized** if

$$\left\{ \begin{bmatrix} \mathrm{rep}_{\mathcal{S}}(n) \\ \mathrm{rep}_{\mathcal{S}'}(f(n)) \end{bmatrix}^{\$} : n \in \mathbb{N} \right\}$$

is regular.

Theorem (Charlier, C. & Stipulanti)

If $f\colon \mathbb{N} \to \mathbb{N}$ is an $(\mathcal{S}, \mathcal{S}')$-synchronized sequence and $g\colon \mathbb{N} \to \mathbb{K}$ is an $(\mathcal{S}', \mathbb{K})$-regular sequence, then the sequence $g \circ f\colon \mathbb{N} \to \mathbb{K}$ is $(\mathcal{S}, \mathbb{K})$-regular.

Second application

Let U be a (positional) **numeration system**.
A sequence $f\colon \mathbb{N} \to \mathbb{K}$ is called $(U, \mathbb{K})$-**regular** if the series

$$\sum_{n \in \mathbb{N}} f(n)\, \mathrm{rep}_U(n)$$

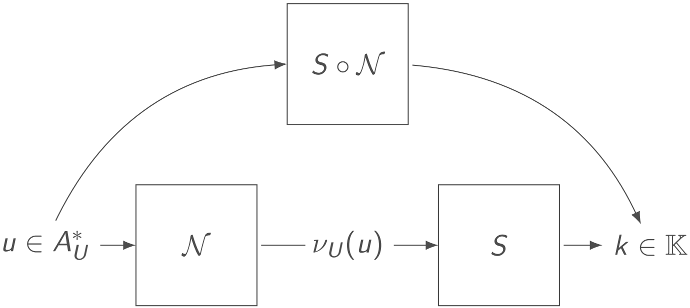
is $\mathbb{K}$-recognizable.
In numeration systems, the U-value function can be extended over all words over the numeration alphabet A_U.
An alternative definition: f is $(U, \mathbb{K})$-regular if the series

$$\sum_{w \in A_U^*} f(\mathrm{val}_U(w))\, w$$

is $\mathbb{K}$-recognizable.

Proposition

For any Pisot numeration system U, the normalization is effectively computable.



Theorem (Charlier, C. & Stipulanti 2020)

For $f : \mathbb{N} \to \mathbb{K}$ and a Pisot numeration system U, the following assertions are equivalent.

1. The sequence f is $(U, \mathbb{K})$-regular: that is, the series

$$\sum_{n \in \mathbb{N}} f(n) \, \mathrm{rep}_U(n)$$

   is $\mathbb{K}$-recognizable.

2. The series

$$\sum_{w \in A_U^*} f(\mathrm{val}_U(w)) \, w$$

   is $\mathbb{K}$-recognizable.

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Thank you!