On Balanced Sequences with the Minimal Asymptotic Critical Exponent

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Introduction & Motivation

- **Dejean's theorem** (conjecture), 1972–2011:
  (proved by Currie, Rampersad; Rao)
  the least critical exponent of sequences over an alphabet of size d equals $\frac{d}{d-1}$ for $d \geq 5$
- the least critical exponent for particular classes of sequences
  - **Carpi and de Luca**, 2000: Sturmian sequences
  - **Currie, Mol, and Rampersad**, 2020: binary rich sequences
  - **Baranwal, Rampersad, Shallit, Vandomme**, 2019:
    balanced sequences over alphabets of size 3 to 8
    conjecture for alphabets of size d : $\frac{d-2}{d-3}$

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Program

1. Calculation of asymptotic critical exponent of balanced sequences

2. Balanced sequences with the least asymptotic critical exponent

Definitions CoW

- *sequence* $\mathbf{u} = u_0 u_1 u_2 \ldots$ over $\mathcal{A}$
- *bispecial factor* of $\mathbf{u}$
- *power* $z = u^e$ if z is a prefix of u^ω and $e = \frac{|z|}{|u|}$
- *Parikh vector* $\vec{V}(v)$ of a word v

Example

$\mathbf{u}_F = \mathtt{abaababaabaababaa}\ldots$, $\mathcal{A} = \{\mathtt{a}, \mathtt{b}\}$
$\mathbf{u}_F = \varphi(\mathbf{u}_F)$, *where* $\varphi : \mathtt{a} \to \mathtt{ab}, \quad \mathtt{b} \to \mathtt{a}$
aba *is a bispecial factor since* aaba, baba *and* abab, abaa *are factors of* $\mathbf{u}_F$
$z = \mathtt{ababa} = (\mathtt{ab})^{5/2}$
$\vec{V}(ababa) = \binom{3}{2}$

Definitions CoW

- *return word* to a factor v of $\mathbf{u}$
- *derived sequence* $\mathbf{d_u}(v)$ to a factor v of $\mathbf{u}$

Example

$\mathbf{u}_F = \mathtt{aba\underline{aba}baaba ab\underline{aba}a}\ldots$

$r = \mathtt{aba}$ *and* $s = \mathtt{ab}$ *are return words to the factor* $v = \mathtt{aba}$

$\mathbf{d_u}(v) = \mathtt{abaababaabaababa}\cdots = rsrrsr\ldots$

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(Asymptotic) critical exponent

- *critical exponent*
  $E(\mathbf{u}) = \sup\{e \in \mathbb{Q} \mid u^e \text{ is a non-empty factor of } \mathbf{u}\}$
- *asymptotic critical exponent*
  $E^*(\mathbf{u}) = \lim_{n\to\infty} \sup\{e \in \mathbb{Q} \mid u^e \text{ is a factor of } \mathbf{u} \text{ and } |u| \geq n\}$

Evidently, $E^*(\mathbf{u}) \leq E(\mathbf{u})$.

Proposition

Let $\mathbf{u}$ be a uniformly recurrent aperiodic sequence. Let v_n be the n-th bispecial of $\mathbf{u}$ and w_n a shortest return word to v_n. Then $E(\mathbf{u}) = 1 + \sup\{\frac{|v_n|}{|w_n|} \mid n \in \mathbb{N}\}$ *and* $E^*(\mathbf{u}) = 1 + \limsup_{n\to\infty} \frac{|v_n|}{|w_n|}$.

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Example

$|v_n| = F_{n+2} + F_{n+1} - 2$ and $|w_n| = F_{n+1}$ with $F_0 = 0, F_1 = 1$
$E(\mathbf{u}_F) = 2 + \tau = 2 + \frac{1+\sqrt{5}}{2} = E^*(\mathbf{u}_F)$ – minimal for Sturmian

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Return words and bispecials in Sturmian sequences

$\mathbf{u}$ a standard Sturmian sequence of slope $\alpha \in (0,1)$
$\mathbf{u}$ starting in b $\to \theta = \frac{1-\alpha}{\alpha} = [0; a_1, a_2, a_3, \ldots] \in (0,1)$
$\frac{p_N}{q_N}$ the N-th convergent to θ

Proposition

Let b be a bispecial of $\mathbf{u}$, r the more frequent, s the less frequent return word to b. There is a unique (N, m), $0 \leq m < a_{N+1}$

$$\vec{V}(r) = \binom{p_N}{q_N} \quad \vec{V}(s) = m\binom{p_N}{q_N} + \binom{p_{N-1}}{q_{N-1}} \quad |b| = |r| + |s| - 2.$$

Moreover, $\mathbf{d_u}(b)$ is a standard Sturmian sequence with $\theta_b = [0, a_{N+1} - m, a_{N+2}, a_{N+3}, \ldots]$.

b is *primary* for $m = 0$; in this case $|r| > |s|$
b is *secondary* for $m \geq 1$; in this case $|r| < |s|$

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Balanced sequences

Definition

$\mathbf{u}$ *over* $\mathcal{A}$ *balanced if* $(\forall a \in \mathcal{A})(|u| = |v| \Rightarrow ||u|_a - |v|_a| \leq 1)$.

Theorem (Graham 1973, Hubert 2000)

$\mathbf{v}$ *recurrent aperiodic is balanced iff* $\mathbf{v}$ *obtained from a Sturmian sequence* $\mathbf{u}$ *over* $\{\mathtt{a}, \mathtt{b}\}$ *by replacing*

- $\mathtt{a}$ *with a constant gap sequence* $\mathbf{y}$ *over* $\mathcal{A}$,
- $\mathtt{b}$ *with a constant gap sequence* $\mathbf{y}'$ *over* $\mathcal{B}$,

where $\mathcal{A}$ *and* $\mathcal{B}$ *disjoint. We write* $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$.

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Example

$\mathbf{v} = \mathrm{colour}(\mathbf{u}_F, \mathbf{y}, \mathbf{y}')$, where $\mathbf{y} = (0102)^\omega$ and $\mathbf{y}' = (34)^\omega$

$$\begin{aligned} \mathbf{u}_F &= \mathtt{abaababaabaabab}\ldots \\ \mathbf{v} &= 031042301402304\ldots \end{aligned}$$

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Return words and bispecials in balanced sequences

Proposition (Dolce, D., Pelantová, 2020)

Let $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$. *For a sufficiently long bispecial* v *in* $\mathbf{v}$*:* $|v| = |b|$ *for some bispecial* b *in* $\mathbf{u}$. *The shortest return word to* v *is of length* $\min\{k|r| + \ell|s|\}$, *where*

1. $k\vec{V}(r) + \ell\vec{V}(s) = \begin{pmatrix} 0 \bmod \mathrm{Per}(\mathbf{y}) \\ 0 \bmod \mathrm{Per}(\mathbf{y}') \end{pmatrix}$;
2. $\binom{k}{\ell}$ *is the Parikh vector of a factor in* $\mathbf{d_u}(b)$.

Recall $E^*(\mathbf{u}) = 1 + \limsup_{n\to\infty} \frac{|v_n|}{|w_n|}$.

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Program (implemented by D. Opočenská):
Input: α quadratic irrational, $\mathrm{Per}(\mathbf{y})$, $\mathrm{Per}(\mathbf{y}')$.
Output: $E^*(\mathbf{v})$, where $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$.

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Minimal critical exponent

d	α	$\mathbf{y}$	$\mathbf{y}'$	$E(\mathbf{v})$	$E^*(\mathbf{v})$
3	$[0,\overline{2}]$	$(01)^\omega$	2^ω	$2+\frac{1}{\sqrt{2}}$	$2+\frac{1}{\sqrt{2}}$
4	$[0,2,\overline{1}]$	$(01)^\omega$	$(23)^\omega$	$1+\frac{1+\sqrt{5}}{4}$	$1+\frac{1+\sqrt{5}}{4}$
5	$[0,\overline{2}]$	$(0102)^\omega$	$(34)^\omega$	$\frac{3}{2}$	$\frac{3}{2}$
6	$[0,1,2,1,1,\overline{1,1,1,2}]$	0^ω	$(123415321435)^\omega$	$\frac{4}{3}$	$\frac{4}{3}$
7	$[0,1,1,3,\overline{1,2,1}]$	$(01)^\omega$	$(234526432546)^\omega$	$\frac{5}{4}$	$\frac{5}{4}$
8	$[0,1,3,1,\overline{2}]$	$(01)^\omega$	$(23452673254623752643257 6)^\omega$	$\frac{6}{5}=1.2$	$\frac{12+3\sqrt{2}}{14}\doteq 1.16$
9	$[0,1,2,3,\overline{2}]$	$(01)^\omega$	$(2345672843652748632547 68)^\omega$	$\frac{7}{6}\doteq 1.167$	$1+\frac{2\sqrt{2}-1}{14}\doteq 1.13$
10	$[0,1,4,2,\overline{3}]$	$(01)^\omega$	$(23456728496325476829436527 4869)^\omega$	$\frac{8}{7}\doteq 1.14$	$1+\frac{\sqrt{13}}{26}\doteq 1.139$

Table: Baranwal, Rampersad, Shallit, Vandomme: balanced sequences with the least critical exponent over alphabets of size d.

Program

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Graph of admissible tails

Problem 1: Given $\mathrm{Per}(\mathbf{y}), \mathrm{Per}(\mathbf{y}')$ and $\beta \geq 1$, find $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$ with $E^*(\mathbf{v}) \leq 1 + \frac{1}{\beta}$.

Graph of admissible tails:
An oriented graph $\Gamma = \Gamma_{\mathrm{Per}(\mathbf{y}), \mathrm{Per}(\mathbf{y}'), \beta}$ with labeled edges with the following property:
If $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$ and $E^*(\mathbf{v}) \leq 1 + \frac{1}{\beta}$, then there is an infinite path in Γ such that the sequence of its edge labels is a suffix of the continued fraction of θ.

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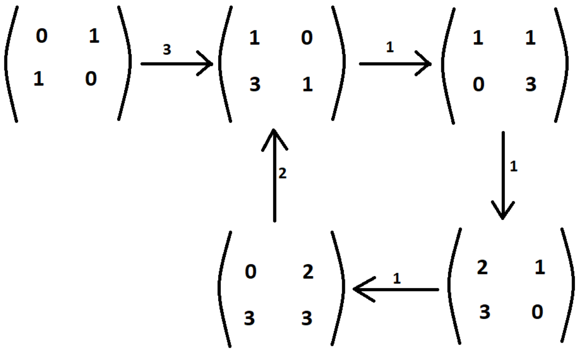
Infinite paths associated to $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$

Given $\mathrm{Per}(\mathbf{y}), \mathrm{Per}(\mathbf{y}')$ and $\theta = [0; a_1, a_2, a_3, \ldots]$.

- Nth primary bispecial $b \to$ matrix $M_b = (\vec{V}(r)\ \vec{V}(s)) \bmod \mathrm{Per}$
- $(N+1)$st primary bispecial has the matrix $M_b \left(\begin{smallmatrix} a_{N+1} & 1 \\ 1 & 0 \end{smallmatrix}\right)$ mod Per
- A, B unimodular matrices, define $A \sim_{\mathrm{Per}} B$ if $A \left(\begin{smallmatrix} k \\ \ell \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ mod Per iff $B \left(\begin{smallmatrix} k \\ \ell \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ mod Per

Graph associated to $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$

$\mathrm{Per}(\mathbf{y}) = 3, \mathrm{Per}(\mathbf{y}') = 4$ and $\theta = [0; 3, \overline{1, 1, 1, 2}]$



Construction of the graph of admissible tails Γ

Given $\mathrm{Per}(\mathbf{y}), \mathrm{Per}(\mathbf{y}')$.
$A \sim_{\mathrm{Per}} B$ if $A \left(\begin{smallmatrix} k \\ \ell \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ mod Per iff $B \left(\begin{smallmatrix} k \\ \ell \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ mod Per.
Vertices of Γ= classes of equivalence $\sim_{\mathrm{Per}}$ on unimodular matrices.
Edge with label $a \in \mathbb{N}$ from A to B if $B \sim_{\mathrm{Per}} A \left(\begin{smallmatrix} a & 1 \\ 1 & 0 \end{smallmatrix}\right)$ mod Per.
Erasing edges we get $\Gamma_{\mathrm{Per}(\mathbf{y}),\mathrm{Per}(\mathbf{y}'),\beta}$.

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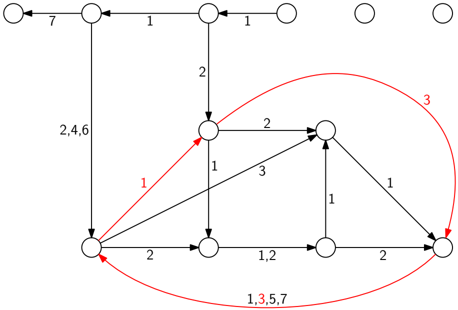
$\Gamma_{\mathrm{Per}(\mathbf{y}),\mathrm{Per}(\mathbf{y}'),\beta} = \Gamma_{2,16,7.5}$



Minimum: $E^*(\mathbf{v}) = \frac{39+2\sqrt{41}}{46} \doteq 1,12622\ldots$, where $\mathbf{v} = \mathrm{colour}(\mathbf{u},\mathbf{y},\mathbf{y}')$ with $\theta = [0;3,\overline{3,3,1}]$.

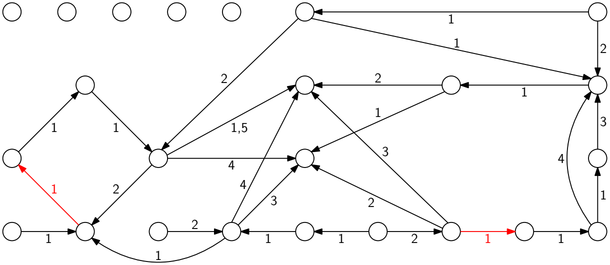
Program (implemented by D. Opočenská):

Input: Period of θ, $\mathrm{Per}(\mathbf{y})$, $\mathrm{Per}(\mathbf{y}')$.

Output: Optimal preperiod of θ guaranteeing minimal $E^*(\mathbf{v})$.

$\Gamma_{\mathrm{Per}(\mathbf{y}),\mathrm{Per}(\mathbf{y}'),\beta} = \Gamma_{3,4,3}$

Minimum: $E^*(\mathbf{v}) = 1 + \frac{1}{3}$, where $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$ with $\theta = [0; 3, \overline{1, 1, 1, 2}]$.



Minimal asymptotic critical exponent

Problem 1: Given $\mathrm{Per}(\mathbf{y}), \mathrm{Per}(\mathbf{y}')$ and $\beta \geq 1$, find $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$ with $E^*(\mathbf{v}) \leq 1 + \frac{1}{\beta}$.

Problem 2: Given $\mathrm{Per}(\mathbf{y}), \mathrm{Per}(\mathbf{y}')$, find $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$ with the minimal asymptotic critical exponent.

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Problem 3: Given $d \in \mathbb{N}$, find $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$ with the minimal asymptotic critical exponent over alphabet of size d.

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4	$[0,\overline{1}]$	2	2	$1+\frac{1+\sqrt{5}}{4}$
5	$[0,1,1,\overline{2}]$	2	4	$\frac{3}{2}$
6	$[0,\overline{1}]$	4	4	$\frac{5}{4} < \frac{4}{3} = \min E$
7	$[0,1,3,\overline{3,3,1}]$	2	16	$1.12622 < \frac{5}{4} = \min E$

Table: Balanced sequences with the least asymptotic critical exponent over alphabets of size d.

Thank you for attention