The semigroup of trimmed morphisms

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Initial question

Consider a morphism φ over an ordered alphabet $\{a, b, c, \ldots\}$, where $a < b < c < \cdots$.

Suppose that u, v are right infinite words with $u < v$ (lexicographically).

What can we say on the order between $\varphi(u)$ and $\varphi(v)$?

Example

If $\varphi_{tm} : a \to ab, b \to ba$ is the Thue-Morse morphism, then

$$u < v \Longrightarrow \varphi_{tm}(u) < \varphi_{tm}(v).$$

Why? Since $u = xa \cdots, v = xb \cdots$, and $\varphi_{tm}(a) < \varphi_{tm}(b)$.

Order-preserving and order-reversing

If

$$u < v \Longrightarrow \varphi(u) < \varphi(v),$$

we say that φ is *order-preserving*.

The Thue-Morse morphism is order preserving, but the period-doubling morphism

$$\varphi_{pd} : \begin{cases} a \to ab \\ b \to aa \end{cases}$$

is not: $\varphi_{pd}(a) > \varphi_{pd}(b)$.

In fact, the latter morphism is *order-reversing*:

$$u < v \Longrightarrow \varphi(u) > \varphi(v).$$

Another example

Example

The Fibonacci morphism

$$\varphi_f : \begin{cases} a \to ab \\ b \to a \end{cases}$$

is order-reversing:

$$\varphi_f(a\cdots) = ab\cdots, \varphi_f(b\cdots) = aa\cdots$$

.

but its square

$$\varphi_f^2 : \begin{cases} a \to aba \\ b \to ab \end{cases}$$

is order-preserving.

Binary case

Theorem (Borchert, Rampersad, 2018)

Every morphism over the binary alphabet $\{a, b\}$ with $a < b$ is either order-preserving or order-reversing. In both cases, its square is order-preserving.

Obvious when no image of a letter is a prefix of the other one; a one-page proof otherwise.

This result allows to extend to all binary morphisms the results of [Andrieu, Frid, 2020], normally valid for order-preserving ones.

What about larger alphabets?

Natural question

Consider u and v, $\varphi(u)$ and $\varphi(v)$, $\varphi^2(u)$ and $\varphi^2(v)$ and so on. Is the sequence of orders between these pairs periodic? Is there a power of φ which is order-preserving?

No to the second question.

$$\psi : \begin{cases} a \to ac \\ b \to ab \\ c \to cb \end{cases} .$$

Here $\psi^k(a) > \psi^k(b)$ for all k.

Goal

How can we describe what happens?

Here is a description for the case when φ-images of letters are not prefixes of one another.

Example

$$\varphi : \begin{cases} a \to adba, \\ b \to aebab, \\ c \to adca, \\ d \to aebac, \\ e \to aebd. \end{cases} \quad \to t_\varphi : \begin{cases} a \to db, \\ b \to eab, \\ c \to dc, \\ d \to eac, \\ e \to ed. \end{cases}$$

Here t_φ is the *trimmed* version of φ.

Properties

- For every φ, its trimmed version t_φ induces the same order;
- $\forall \varphi, \psi$ we have $t_{\varphi\circ\psi} = t_{t_\varphi \circ t_\psi} = t_\varphi * t_\psi$;
- There exists a finite number of trimmed morphisms over a given alphabet;
- They form a $*$-monoid;
- So, the sequence of t_{φ^k}, $k = 0, 1, \ldots$, is ultimately periodic;
- And so is the sequence of orders among φ^k of letters.

Another example

If φ contains a permutation of symbols after a common prefix:

$$\varphi : \begin{cases} a_1 \to p\sigma(a_1)s_1, \\ a_2 \to p\sigma(a_2)s_2, \\ \dots \\ a_n \to p\sigma(a_n)s_n, \end{cases}$$

then t_φ is a permutation:

$$t_\varphi : \begin{cases} a_1 \to \sigma(a_1), \\ a_2 \to \sigma(a_2), \\ \dots \\ a_n \to \sigma(a_n). \end{cases}$$

A long example

The maximal number of symbols in t_φ corresponds to the situation when images of symbols separate one by one. Then we may have something like

$$t_\varphi : \begin{cases} a_1 & \to x_1, \\ a_2 & \to y_1 x_2, \\ \dots & \\ a_{n-1} & \to y_1 \cdots y_{n-2} x_{n-1} \\ a_n & \to y_1 \cdots y_{n-2} y_{n-1}. \end{cases}$$

A typical example

Example

Consider the ternary morphism

$$\varphi : \begin{cases} a \to ac, \\ b \to ab, \\ c \to ba. \end{cases}$$

Then

$$t_\varphi : \begin{cases} a \to ac, \\ b \to ab, \\ c \to b, \end{cases} \quad t_\varphi^2 : \begin{cases} a \to acb, \\ b \to acab, \\ c \to ab, \end{cases} \quad t_\varphi^{(2)} : \begin{cases} a \to cb, \\ b \to ca, \\ c \to b. \end{cases}$$

The sequence continues by

$$t_\varphi^{(3)}:\begin{cases}a\to bb,\\ b\to bc,\\ c\to a,\end{cases}\quad t_\varphi^{(4)}:\begin{cases}a\to ba,\\ b\to bb,\\ c\to c,\end{cases}\quad t_\varphi^{(5)}:\begin{cases}a\to ac,\\ b\to ab,\\ c\to b,\end{cases}\quad t^{(5)}=t.$$

So, the sequence of orders is

$$\sigma_\varphi=\begin{pmatrix}a & b & c\\ 2 & 1 & 3\end{pmatrix},\sigma_{\varphi^2}=\begin{pmatrix}a & b & c\\ 3 & 2 & 1\end{pmatrix},$$

$$\sigma_{\varphi^3}=\begin{pmatrix}a & b & c\\ 2 & 3 & 1\end{pmatrix},\sigma_{\varphi^4}=\begin{pmatrix}a & b & c\\ 1 & 2 & 3\end{pmatrix}=Id,$$

and then there is a cycle of length 4.

Numbers

Lemma

The number of trimmed morphisms over the n-letter alphabet satisfies

$$\frac{n!}{2} n^{n-1}(n-1)^{n-1} \leq T \leq n^{n(n+1)/2-1}.$$

Over 1,2,3,4 letters there are 1, 2, 114=108+6, 28824 trimmed morphisms.

And this sequence is not yet in OEIS.

Orders and periods

What is the longest period of orders among φ^k-images of letters?

At least $g(n)$, where g is the Landau function = the largest order of a permutation of order n.

$$g(n) \leq e^{C\sqrt{n \ln n}}$$

It seems to be greater, but has not been really studied.

Conclusion

- The sequence of how the order between φ^k-images of letters changes with k can be described by $*$-powers of trimmed morphisms (if φ is prefix-free).
- Trimmed morphisms constitute a monoid under the operation $*$ (composition+trimming).
- This monoid is finite for a given alphabet but huge.
- We can study its properties (or not).