Coverable bi-infinite substitution subshifts

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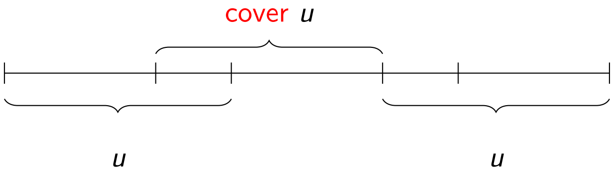
Joint work with Manuel Joseph Loquias and Eden Delight Miro

Day of Short Talks on Combinatorics on Words
March 22, 2021

Coverable words

A finite word w over $\mathcal{A}$ is **coverable** if, for some proper subword u of w, w is formed by overlapping or adjacent occurrences of u.



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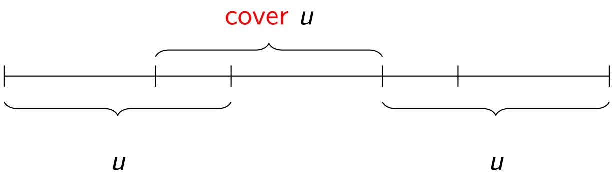
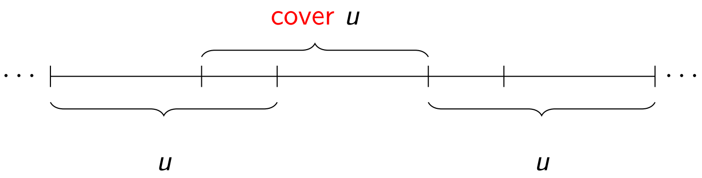


aaba-coverable:

aabaaabaabaabaaaba
aabaaabaabaabaaaba
aabaaabaabaabaaaba
aabaaabaabaabaaaba
aabaaabaabaabaaaba

Coverable words



- Apostolico and Ehrenfeucht (1993): Quasiperiodic finite words
- Gamard and Richomme (2015): Coverability in two dimensions

Example: Coverable bi-infinite sequences

Consider the Fibonacci substitution on $\mathcal{A} = \{a, b\}$ defined by

$$\phi_{Fib} : a \mapsto ab, b \mapsto a.$$

- A word $w \in \mathcal{A}^{\mathbb{Z}}$ such that $\phi_{Fib}^2(w) = w$ is *aba*-coverable.
- If $x \in X_{Fib} = \overline{\{S^n(w) \mid n \in \mathbb{Z}\}}$, then x is *aba*-coverable.

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Theorem. (Barbero, Gamard and Grandjean, 2020)

Each bi-infinite Sturmian word has an infinite number of cover.

Motivation

Problem posed by F. Levé and G. Richomme in 2013:

Given a morphism f prolongable on a letter a,
can we decide whether the word

$$f^{\omega}(a) = \lim_{n\to\infty} f_n(a)$$

is coverable?

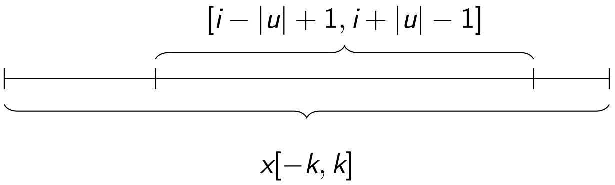
Coverable subshifts

Proposition.

If $w \in \mathcal{A}^{\mathbb{Z}}$ is coverable, then every element of $\mathbb{X}(w) = \overline{\{S^n(w) \mid n \in \mathbb{Z}\}}$ is coverable.

Outline of the proof:

- The shift map S preserves coverability in $\mathcal{A}^{\mathbb{Z}}$.
- Every limit point of $\mathbb{X}(w)$ is coverable.
  Consider $(x^{(n)})_{n \in \mathbb{N}} \subseteq \mathcal{O}(w)$ with $x^{(n)} \to x$.



Definition.

A subshift $X \subset \mathcal{A}^{\mathbb{Z}}$ is **coverable** if there is a coverable $w \in X$ such that $X = \mathbb{X}(w)$.

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Question: Is coverability of subshifts topologically invariant?

Illustration: With $w = {}^{\omega}(aba).a(aba)^{\omega}$, the subshift $\mathbb{X}(w)$ is coverable. However, the image of w under a 4-block code is not coverable.

$$ {}^{\omega}(aba).a(aba)^{\omega} \mapsto {}^{\omega}(541)23.(415)^{\omega} $$

abaa	*baaa*	*aaab*	*aaba*	*baab*
1	2	3	4	5

Non-special cover

Definition.

A coverable subshift $X \subset \mathcal{A}^{\mathbb{Z}}$ is said to have **non-special coverability** if X has a cover that is not left nor right special in $\mathcal{L} = \mathcal{B}(X)$.

For example, *baababaa* is not special in X_{Fib}. Thus, X_{Fib} has a non-special coverability.

Proposition.

Let ϕ and γ be primitive substitutions such that X_ϕ is topologically conjugate to X_γ. The subshift X_ϕ is non-special coverable if and only if X_γ is non-special coverable.

<u>Outline of the proof:</u>

β_i is an i-block code,
f is the 1-block code such that $\beta_{2m+1} \circ g = f \circ \beta_{2n+1}$,
$\phi^{(n)}$ is the n-(double) collared substitution of ϕ, and
$\gamma^{(m)}$ is the m-(double) collared substitution of γ

$$\begin{array}{ccc} X_\phi & \xrightarrow{\quad g \quad} & X_\gamma \\ \Big\downarrow{\scriptstyle \beta_{2n+1}} & & \Big\downarrow{\scriptstyle \beta_{2m+1}} \\ X_{\phi^{(n)}} & \xrightarrow{\quad f \quad} & X_{\gamma^{(m)}} \end{array}$$

1-collared Fibonacci substitution

Consider the Fibonacci substitution ϕ_{Fib} defined by $a \mapsto ab, b \mapsto a$.

- 1-collared substitution $\phi^{(1)}$ on $\mathcal{A}_1 = \{b_{aba}, a_{baa}, a_{aab}, a_{bab}\}$:

$$\begin{array}{lcl} b_{aba} & \mapsto & a_{baa} \\ a_{baa} & \mapsto & a_{aab}b_{aba} \\ a_{aab} & \mapsto & a_{bab}b_{aba} \\ a_{bab} & \mapsto & a_{aab}b_{aba} \end{array}$$

  or $\phi^{(1)} : 0 \mapsto 1, 1 \mapsto 20, 2 \mapsto 30, 3 \mapsto 20.$

- $a \underbrace{baababaa}_{u} b \longrightarrow b_{aba}a_{baa}a_{aab}b_{aba}a_{bab}b_{aba}a_{baa}a_{aab}$
- $X_{\phi^{(1)}}$ is 01203012-coverable.

Quasiperiodic morphism

As defined by Levé and G. Richomme in 2007, a morphism $\varphi : \mathcal{A} \to \mathcal{B}$ is said to be **quasiperiodic** if for any coverable word w, $\varphi(w)$ is also coverable.

strongly quasiperiodic: w non-coverable implies $\varphi(w)$ is coverable (Eq. SQP on $\{a, b\}^{\mathbb{Z}}$: $\phi^2_{Fib} : a \mapsto aba, b \mapsto ab$)

weakly quasiperiodic: for some non-coverable w, $\varphi(w)$ is coverable (Eq. WQP on $\{a, b\}^{\mathbb{Z}}$: $\phi_{Fib} : a \mapsto ab, b \mapsto a$)

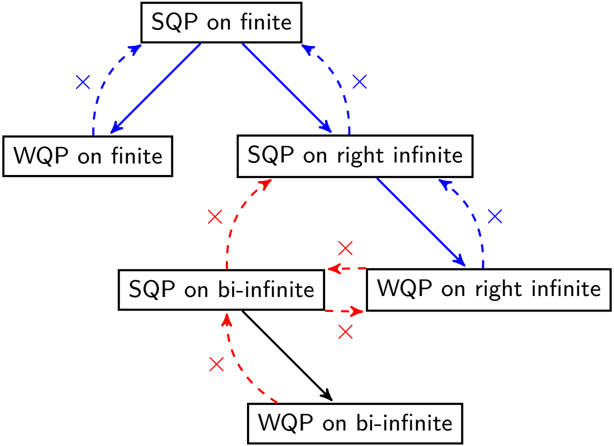
quasiperiod-free: w non-coverable implies $\varphi(w)$ is non-coverable (Eg. $\phi : a \mapsto bababaa, b \mapsto baababa$ is QP-free on $\{a, b\}^{\mathbb{N}}$)

Remarks

- Partial results: Let ϕ be a primitive substitution on $\mathcal{A}$.
  - If ϕ is SQP on $\mathcal{A}^{\mathbb{Z}}$, then X_ϕ is coverable.
  - If $X_\phi \subset \mathcal{A}^{\mathbb{Z}}$ is coverable, then ϕ is not QP-free on $\mathcal{A}^{\mathbb{Z}}$.

- If ϕ is primitive and proper, then X_ϕ has a non-special coverability whenever it is coverable.

- Goal: Relate SQP/WQP substitutions on right infinite sequences to the bi-infinite case



Legend:

Levé and G. Richomme, 2013

Thank you for your attention!